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A Note on the Conservation of Energy and Volume in the Setting of Nonholonomic Mechanical Systems

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This note concerns the analysis of conservation of energy and volume for a series of well known examples of nonholonomic mechanical systems, with linear and non-linear constraints, and aims to make evident some geometric aspects related with them.

Key Words: mechanical systems, energy and volume conservation, nonholonomic constraints.

1. INTRODUCTION

When we study a classical mechanical problem, we start by looking at the set of all their possible configurations, in general represented by a finite dimensional smooth differentiable manifold M, the so called **configuration space**. Subsequently we have to analyze the main internal quantities, as well as the external effects acting on configurations and velocities of the mechanical system.

The kinetic energy is an internal quantity which depends strongly on the geometry and on the velocities of the configurations and is formally given by a real valued smooth function

$$K : TM \to \mathbf{R}$$

such that restricted to each fiber of TM over M is a positive quadratic form; this way K defines a Riemannian metric tensor on M,

$$\mathbf{g} = 2K : TM \to \mathbf{R},$$

and so $(M, \mathbf{g})$ has the ingredients of a smooth Riemannian manifold.

The influence of external effects are represented by a **field of external forces** and is described by a differentiable map

$$\mathcal{F} : TM \to T^*M$$

such that, to each $q \in M$, it takes the fiber T_qM into the fiber T_q^*M.

Any such triple $(M, K, \mathcal{F})$ represents what is called a **classical mechanical system**.

In each concrete case we need to characterize the three elements M, K and $\mathcal{F}$, as we will see in the series of examples of section 2.

The examples of forces we will consider in the sequel are **positional forces**, that is, forces $\mathcal{F}$ such that for any $q \in M$ and any $w_q \in T_qM$, the value of $\mathcal{F}(w_q)$ is a linear form that depends only on q; for instance, if

$$V : M \to \mathbf{R}$$

is a C^2 function, called a **potential**, the corresponding **potential force** $\mathcal{F}_V$ is defined by the condition

$$\mathcal{F}_V(w_q) = \mathrm{d}V(q) \in T_q^*M;$$

$\mathcal{F}_V$ is, in fact, a C^1 positional force.

The next step is to take into account the analysis of nonholonomic aspects of a mechanical system. One needs to describe the fact that the velocities of the motions are not free; they have to satisfy some restrictions that can be reduced to the fact that they belong to a special smooth submanifold $\mathcal{C}$ of TM, the **constraint manifold**. In each particular problem we have to identify the four data $(M, K, \mathcal{F}, \mathcal{C})$ that characterize a classical mechanical system with constraints or simply a **constrained mechanical system**.

As we will see, all the examples considered in the present work have the property that the restriction of the canonical projection $\tau_M : TM \to M$ to $\mathcal{C}$, $\pi_{\mathcal{C}} := \tau_M|_{\mathcal{C}} : \mathcal{C} \to M$, is a submersion, that is, $\tau_M|_{\mathcal{C}}$ as well as its derivative at all points $w_q \in \mathcal{C}$ are surjective maps. Constraints satisfying the latter property are called **regular constraints** (see [22] and [32]) and in the present paper we will only consider regular constraints. When all fibers of $\tau_M|_{\mathcal{C}}$ are linear homogeneous (resp. affine) spaces, $\mathcal{C}$ is said to be a **linear** (resp. **affine**) **constraint**; otherwise $\mathcal{C}$ is called a **non-linear constraint**. E. Cartan considered in [7] a special type of linear constraints, namely, where the constraints are *strongly non-holonomic*; for a recent exposition of E. Cartan's work, we refer the reader to [19] and also [31].

We will state an extension of Liouville's Theorem for constrained mechanical systems proved in [32] (see also [33]), and for this we need some notation and definitions (see [32] for details).

Let $\pi_E : E \to M$ be a vector fibre bundle, the **vertical lift** $\lambda^E : E \oplus_M E \to TE$ is the vector fibre bundle morphism from $\mathsf{pr}_1 : E \oplus_M E \to E$ into $\tau_E : TE \to E$ defined as

$$\lambda^E(u, v) := \left.\frac{T}{dt}\right|_{t=0} (u + tv)$$

for all $u, v \in E$, with $\pi_E(u) = \pi_E(v)$. The image of λ^E is the vertical bundle $\mathrm{Ver}(E) = \ker \mathsf{T}\pi_E$. We will omit the superscript E, when there is no risk of confusion.

Given a constrained mechanical system $(M, K, \mathcal{F}, \mathcal{C})$, since $\pi_{\mathcal{C}}$ is a submersion, $\mathsf{T}\pi_{\mathcal{C}} : T\mathcal{C} \to TM$ is an epimorphism of vector fibre bundles, so that $\ker \mathsf{T}\pi_{\mathcal{C}}$ is a vector sub-bundle of $T\mathcal{C}$, denoted by $\mathrm{Ver}(\mathcal{C})$. In $\mathrm{Ver}(TM)$ consider the metric induced by the metric $\mathbf{g}$ on M such that, for all $v_q \in TM$, $\lambda_{v_q} : T_qM \to \mathrm{Ver}(TM)$ is an isometry. Note that $\mathrm{Ver}(\mathcal{C})$ is also a vector sub-bundle of the pull back $i_{\mathcal{C}}^*\mathrm{Ver}(TM)$, $i_{\mathcal{C}} : \mathcal{C} \to TM$ being the inclusion. Define $W_{v_q} := \left[\mathrm{Ver}_{v_q}(\mathcal{C})\right]^{\perp}$. Then, we have (see [22] and [32])

$$i_{\mathcal{C}}^*\mathrm{Ver}(TM) = \mathrm{Ver}(\mathcal{C}) \oplus_{\mathcal{C}} W.$$

This decomposition defines the projections $P_{\mathcal{C}}$ and P_W into the first and second factors, respectively; with projection $P_{\mathcal{C}}$ we introduce the Gibbs-Maggi-Appell **(GMA) vector field** $X_{\mathcal{C}} := P_{\mathcal{C}}X_0$, where X_0 is the second order vector field associated with the unconstrained mechanical system $(M, K, \mathcal{F})$. It can be shown (see [32]) that $X_{\mathcal{C}}$ is the vector field associated with the d'Alembert-Cheatev principle for constrained mechanical systems. The flow defined by $X_{\mathcal{C}}$ will be called the **GMA flow**.

Let ∇ denote the Levi-Civita connection of $(M, \mathbf{g})$; the **horizontal lifting** is the vector bundle morphism $H : TM \oplus_M TM \to T(TM)$ from $\mathsf{pr}_1 : TM \oplus_M TM \to TM$ into $\tau_{TM} : T(TM) \to TM$ defined as

$$H(u, v) := \left.\frac{T}{dt}\right|_{t=0} \tau^{\gamma}_{0,t}(u)$$

for all $u, v \in TM$, with $\tau_M(u) = \tau_M(v) = p$, where $\tau^{\gamma}_{0,t} : T_pM \to T_{\gamma(t)}M$ is the parallel transport along a curve γ in M such that $\gamma(0) = p$ and $\frac{T\gamma}{dt}\Big|_{t=0} = v$. The image of H is the **horizontal space** $\mathrm{Hor}(TM)$. We denote by $\mathrm{Hor}(\mathcal{C})$ the image of $i^*_{\mathcal{C}}\mathrm{Hor}(TM)$ by $P_{\mathcal{C}}$. We have

$$T\mathcal{C} = \mathrm{Ver}(\mathcal{C}) \oplus_{\mathcal{C}} \mathrm{Hor}(\mathcal{C}),$$

and $\mathrm{Ver}(\mathcal{C})$ is integrable in the sense of Frobenius.

Given a vector bundle $\pi_E : E \to M$ with a connection ∇^E, we denote by $\kappa^E : TE \to E$ the **connector**, that is, given $z_q \in TE$, $\kappa^E(z_q) \in E$ is the unique vector that satisfies

$$z - H^E(\tau_E(z), \mathsf{T}\pi_E(z)) = \lambda(\tau_E(z_q), \kappa^E(z)).$$

We denote by $\mathsf{0}_E$ the null section of E. Again, we omit the superscript E whenever there is no risk of confusion.

Given $v_q \in \mathcal{C}$, we denote by $C_{v_q} \subset T_qM$ the subspace such that $\lambda_{v_q}(C_{v_q}) = \mathrm{Ver}_{v_q}(\mathcal{C})$ and by $\mathcal{P}_{v_q} : T_qM \to C_{v_q}$ the orthogonal projection with respect to the metric $\mathbf{g}$. We denote by $\mathsf{A} : \mathcal{C} \to \mathsf{L}(TM, TM)$ the mapping such that, for any $v_q \in \mathcal{C}$, $A(v_q)(z_q) := \kappa \cdot P_{\mathcal{C}} \cdot H_{v_q}(z_q)$, for all $z_q \in TM$.

One calls **Sasaki's metric tensor** in $\mathcal{C}$ the unique metric tensor $\mathbf{g}_{\mathcal{C}}$ in $\mathcal{C}$ such that for all $v_q \in \mathcal{C}$, $\lambda^{\mathcal{C}}_{v_q} := \lambda_{v_q} \circ \mathcal{P}_{v_q} : C_{v_q} \to \mathrm{Ver}_{v_q}(\mathcal{C})$ and $H^{\mathcal{C}}_{v_q} := P_{\mathcal{C}} \circ H_{v_q} : T_qM \to \mathrm{Hor}_{v_q}(\mathcal{C})$ are linear isometries.

Finally, let $\pi_E : E \to M$ be a vector fibre bundle and $f : TM \to E$ be a smooth fibre preserving mapping, that is, for all $q \in M$, $f(T_qM) \subset E_q$. The **fibre derivative** $\mathsf{F}f : TM \to \mathsf{L}(TM, E)$ is the mapping such that, for any $v_q \in T_qM$, $\mathsf{F}f(v_q)z_q : \kappa \cdot \mathsf{T}_{v_q}f \cdot \lambda^{\mathcal{C}}_{v_q}(z_q)$, for all $z_q \in TM$.

THEOREM 1. *The Lebesgue measure in $\mathcal{C}$ induced by the metric tensor $\mathbf{g}_{\mathcal{C}}$ is conserved by the GMA flow of $(M, K, \mathcal{F}_V, \mathcal{C})$ if and only if, the next condition is satisfied:*

$$\mathrm{tr}\,\mathsf{A}(v_q) + \left\langle \mathrm{tr}\ \mathsf{F}^*\mathcal{P}(v_q)|_{C_{v_q}\times C_{v_q}}, R^A_V(v_q)\right\rangle = 0, \tag{1}$$

for all $v_q \in \mathcal{C}$, where R^A_V is the reaction field determined by the d'Alembert-Chetaev principle, that is, R^A_V is characterized by

$$\lambda_{v_q}(R^A_V(v_q)) := -P_W\left(X_0(v_q)\right)$$

for all $v_q \in \mathcal{C}$.

A consequence of the latter theorem is the following important result.

THEOREM 2. *The Lebesgue measure in $\mathcal{C}$ induced by the metric tensor $\mathbf{g}_{\mathcal{C}}$ is conserved by the GMA flow of $(M, K, \mathcal{F}_V, \mathcal{C})$, for any potential function V, if and only if, the next two conditions are satisfied:*

1. $\operatorname{tr} \mathsf{A}(v_q) = 0$.
2. $\operatorname{tr} \mathsf{F}^* \mathcal{P}(v_q)|_{C_{v_q} \times C_{v_q}} = 0$.

These results appear in [32] and are generalizations of the results in [20] and [17], where the case of linear and affine constraints, respectively, are considered. The next corollary also in [32], which we will use in the sequel, gives a geometrical obstruction on the constraint manifold for conservation of volume by the GMA flow for all potential functions V.

Corollary 3. *Suppose that the Lebesgue measure on $\mathcal{C}$ induced by the metric tensor $\mathbf{g}_{\mathcal{C}}$ is preserved by the flow of the GMA flow of the constrained mechanical system $(M, K, V, \mathcal{C})$, for all potentials $V \in \mathcal{F}(M)$. Then, for all $q \in M$ such that $\mathcal{C}_q \neq \emptyset$, $\mathcal{C}_q$ is a minimal submanifold of $(\mathcal{C}, \mathbf{g}_{\mathcal{C}})$; that is to say, the Riemannian manifold $(\mathcal{C}, \mathbf{g}_{\mathcal{C}})$ admits a regular foliation by minimal leaves.*

Note that, when the constraint is linear or affine, we have,

$$\mathsf{F}^* \mathcal{P}(v_q)|_{C_{v_q} \times C_{v_q}} = 0,$$

so that the condition for conservation of volume is independent of the potential function, that is, if the volume is conserved by some potential function then it is conserved for all potential functions. In the general case, the conservation of volume depends on the potential function through the reaction field R_V^A.

The main purpose of the present paper is to check conservation of the Lebesgue measure (Riemannian volume) under the action of the GMA flow, in a series of examples, including linear, affine and non-linear nonholonomic constraints. General conditions for the existence of an invariant volume for the GMA is still not known for the general case and would be itself an interesting subject for future research. In this respect it is worth mentioning the work of Blackall [5] providing general conditions for invariance of volumes, in the case of linear constraints.

Nonholonomic systems with external forces given only by a potential function and their constraints being linear conserve the mechanical energy, that is, the difference between the kinetic energy and the potential function is a constant along each motion (see [25]). In the affine case, the result that $dE/dt\lambda.b$, where λ is the Lagrange multiplier and $A\dot{q} = b$ is the constraint is well known (which gives conservation in the linear case). Otherwise, when the constraints are non-linear it can be shown (see [32]) that a necessary and sufficient condition in order that the mechanical energy be conserved by the GMA vector field, for any potential function, is that the Liouville vector field be tangent to the constraint manifold. In particular, if the constraint is given by homogeneous functions, the Liouville vector field is tangent to the constraint manifold. Also, when the constraint is a closed submanifold of TM, conservation of mechanical energy for any potential function is equivalent to $\mathcal{D}$ being linear (see [32]). In general, for a particular potential function V, the Liouville vector field being tangent to $\mathcal{C}$ it is a sufficient condition, and not necessary, as can be seen from the isokinetic example 2.3.4, which conserves energy (the kinetic energy) and the Liouville vector field is not tangent to the constraint.

2. EXAMPLES OF NONHOLONOMIC SYSTEMS

In this section, we consider several classical mechanical systems with constraints. The nonholonomic constraints include: (i) linear constraints, (ii) affine constraints and (iii) non-linear constraints. Let us start with linear constraints.

2.1. Linear constraints

The case of a linear constraint, that is, when $\mathcal{C}$ corresponds to a distribution $\mathcal{D}$ over M, the equations defining the constraint distribution are given by local one-forms, which can be expressed by linear equations in the velocities. Also, in the case of linear constraints the conditions in Theorem 2 for the conservation of volume simplify to (see [32] and [20]):

$$\operatorname{tr}\left(B_{\mathcal{D}^\perp}|_{\mathcal{D}^\perp \times_M \mathcal{D}^\perp}\right) = 0, \tag{2}$$

where $B_{\mathcal{D}^\perp} : TM \times_M \mathcal{D}^\perp \to \mathcal{D}$ is the **total second fundamental form** of $\mathcal{D}^\perp$, that is, $\langle B_{\mathcal{D}^\perp}(u_q, v_q), w_q\rangle : \langle \nabla_X Y, Z\rangle$, for all $u_q, v_q \in \mathcal{D}^\perp$ and $w_q \in \mathcal{D}$, with X a germ of vector field on M and Y and Z germs of sections of $\mathcal{D}^\perp$ and $\mathcal{D}$, respectively, such that $X_q = u_q$, $Y_q = v_q$ and $Z_q = w_q$.

Remark 4. In order to compute condition (2) in the examples that follow, we developed an algorithm based on the following result, whose proof only involves simple computations: let $\left\{\omega^i\right\}_{i=1,\ldots,n-m}$ be one-forms defining the constraint $\mathcal{D}$, that is, $u \in \mathcal{D} \Leftrightarrow \omega^i(u) = 0$, $i = 1, \ldots, n-m$; then, for all $u \in \mathcal{D}$,

$$\langle \operatorname{tr} B_{\mathcal{D}^\perp}, u\rangle = \nabla_{\xi_i}\omega^i \cdot u,$$

where $\{\xi_i\}_{i=1,\ldots,n-m}$ is the dual basis of $\left\{\omega^i\right\}_{i=1,\ldots,n-m}$ (hereafter we use the summation convention).

The algorithm is then:

Algorithm 1.

1. Input the local coordinates (x^α), $\alpha = 1, \ldots, n$.

2. Input the one-forms ω^i, $i = 1, \ldots, n-m$ defining the constraint.

3. Input the kinetic energy $K = \frac{1}{2}\mathbf{g}$.

4. Compute the metric matrix, $[g_{\alpha\beta}]$ with entries $g_{\alpha\beta} = \left\langle \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right\rangle$, $\alpha, \beta = 1, \ldots, n$ and its inverse $\left[g^{\alpha\beta}\right]$.

5. Compute $\xi^i = \mathbf{g}^\sharp(\omega^i)$, the index rising of ω^i, $i = 1, \ldots, n-m$.
6. Compute the matrix $\left[h^{ij}\right]$, with entries $h^{ij} = \left\langle \xi^i, \xi^j \right\rangle$ and the matrix $[h_{ij}] = \left[h^{ij}\right]^{-1}$, $i, j = 1, \ldots, n-m$.
7. Compute $\xi_i = h_{ij}\xi^j$, $i, j = 1, \ldots, n-m$.
8. Compute $\nabla_{\xi_i}\omega^i = \omega^i_{\alpha;\beta}\xi^\beta_i \mathrm{d}x^\alpha$, where $\omega^i_{\alpha;\beta} := \frac{\partial \omega^i_\alpha}{\partial x^\beta} - \Gamma^\gamma_{\alpha\beta}\omega^i_\gamma$, $i = 1, \ldots, n-m$, $\alpha, \beta, \gamma = 1, \ldots, n$ ($\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols associated to (x^α), $\alpha = 1, \ldots, n$).
9. Compute $\langle \operatorname{tr} B_{\mathcal{D}^\perp}, u\rangle = \nabla_{\xi_i}\omega^i \cdot u$ for an arbitrary $u \in \mathcal{D}$.

The algorithm was coded in Mathematica and it is listed in the appendix for the convenience of the reader.

Remark 5. The condition $\operatorname{tr}\left(B_{\mathcal{D}^\perp}|_{\mathcal{D}^\perp \times \mathcal{D}^\perp}\right) = 0$ for the conservation of volume by the GMA flow does not depend on the integrability of $\mathcal{D}$. On the other hand, it is well known that when the constraint is integrable, the foliation is invariant under the GMA flow and on each leaf it is Hamiltonian—so conserves the phase volume of the leaf (see, for example, [25]). This apparent discrepancy can be understood by observing that the Riemannian volume corresponding to the previous condition is the volume on $\mathcal{C}$ associated to Sasaki's metric tensor $\mathbf{g}_{\mathcal{C}}$, whereas the volume conserved by the flow restricted to the leaves is the phase volume of each leaf (see the swell stabilizer §2.2.2). The next simple example show the difference between the two volumes. Consider $M = \mathbf{R}^2 - \{0\}$ and the integrable linear constraint given by $\mathrm{d}r = 0$, where (r, θ) denotes the polar coordinates. The leaves are circles so that the (local) Riemannian volume of the leaves is $r^2\, \mathrm{d}\theta \wedge \mathrm{d}\dot{\theta}$, whereas the (local) Sasaki's volume of the distribution is $r^2\, \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\dot{\theta}$.

Remark 6. Note that when all data are analytic it is sufficient to prove $\operatorname{tr}\left(B_{\mathcal{D}^\perp}|_{\mathcal{D}^\perp \times \mathcal{D}^\perp}\right) = 0$ in an open set of M.

Remark 7. Once again, we would like to point out that in a linear constraint the mechanical energy $T - V$ is always conserved; so that, only volume conservation will be computed in this section.

2.1.1. A Routh-Chaplygin sphere rolling without slipping on a surface of revolution.

We will consider a Routh-Chaplygin sphere that has three distinct moments of inertia I_1, I_2 and I_3, and that its center of mass may be off its geometrical center but lies along the principal axis associated with I_3. Let δ be the radius of the sphere and S be the surface of revolution with vertical axis (O, e_z) of a fixed orthonormal system (O, e_x, e_y, e_z). Let us denote by H another surface of revolution of axis (O, e_z), described by all the positions of the center of the sphere; H is characterized as the set of all points that have distance δ from S. Assume from now on that H is analytic.

Let $z = \Phi(r)$, $r = \sqrt{x^2 + y^2}$ be the equation of H. The configuration space M is a bundle on H with fibers $SO(3)$. Using Euler angles (see [2], Ch. 6, p. 149, figure 126 and also [25] exercise 5.6.20, figure 5.4) and from equations 5.74 and 5.76 of [25] the kinetic energy is given by

$$K = \frac{1}{2} \int_B |\dot{q}(t, \xi)|^2 \, \mathrm{d}m(\xi),$$

with B the reference sphere centered at O, $q(t, \xi) M_t^* \xi + C(t)$, so $\dot{q}(t, \xi) = \omega(t) \times (q(t, \xi) - C(t)) + \dot{C}(t)$, $C = (x, y, \Phi(r))$ and $\dot{C} = (\dot{x}, \dot{y}, \Phi'(r)\dot{r})$. Then

$$K = \frac{1}{2} m \left|\dot{C}\right|^2 + \frac{1}{2} \left(I_1 A_1^2 + I_2 A_2^2 + I_3 A_3^2\right) + \int_B \left\langle \dot{C}, \omega \times M_t^* \xi \right\rangle \mathrm{d}m(\xi),$$

where $\omega \times = \dot{M}_t^* \left(M_t^*\right)^{-1}$,

$$\begin{aligned} A_1 &= \dot{\phi} \sin\psi \sin\theta + \dot{\theta} \cos\psi \\ A_2 &= \dot{\phi} \cos\psi \sin\theta - \dot{\theta} \sin\psi \\ A_3 &= \dot{\phi} \cos\theta + \dot{\psi}, \end{aligned}$$

and m is the mass of the sphere B.

Recall that M_t^* is the rotation that takes the fixed frame (e_x, e_y, e_z) into the frame attached to the moving sphere (e_1, e_2, e_3); thus

$$K = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + (\Phi'(r)\dot{r})^2\right) + \frac{1}{2} \left(I_1 A_1^2 + I_2 A_2^2 + I_3 A_3^2\right) + \left\langle \dot{C}, \omega \times (a e_3) \right\rangle,$$

where $|a|$ is the distance between the center of mass G of B and O: $G = O + a e_z$,

$$\omega = \left(\dot{\theta} \cos\phi + \dot{\psi} \sin\theta \sin\phi\right) e_x + \left(\dot{\theta} \sin\phi - \dot{\psi} \cos\phi \sin\theta\right) e_y + \left(\dot{\phi} + \dot{\psi} \cos\theta\right) e_z$$

and

$$e_3 = M_t^* e_z = \sin\phi \sin\theta e_x - \sin\theta \cos\phi e_y + \cos\theta e_z. \tag{3}$$

After easy computations we get

$$
\begin{aligned}
K = {} & \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2+(\Phi'(r)\dot{r})^2\right)+\\
& \tfrac{1}{2}\Big[I_1(\dot{\phi}\sin\psi\sin\theta+\dot{\theta}\cos\psi)^2+I_2(\dot{\phi}\cos\psi\sin\theta-\dot{\theta}\sin\psi)^2+\\
& \quad I_3(\dot{\phi}\cos\theta+\dot{\psi})^2\Big]+ma\left[\dot{x}\left(\sin\theta\cos\phi\dot{\phi}+\sin\phi\cos\theta\dot{\theta}\right)+\right.\\
& \quad \left.\dot{y}\left(\sin\theta\sin\phi\dot{\phi}-\cos\theta\cos\phi\dot{\theta}-\sin\theta\dot{\theta}\Phi'(r)\dot{r}\right)\right].
\end{aligned}
\tag{4}
$$

The case $a \neq 0$ and $I_1 = I_2 \neq I_3$ corresponds to the so called **Routh sphere** and if $a = 0$ with three distinct principal moments of inertia one obtains the **Chaplygin sphere**. In the case of a **homogeneous sphere** we have $a = 0$ and $I_1 = I_2 = I_3$. Finally, if $\Phi(r) = \delta$ the sphere rolls on a horizontal plane.

Let us compute, now, the equations for the constraints. The condition of rolling without slipping is obtained by making $\dot{q}(t,\xi) = 0$ at the point of contact $q(t,\xi)$ of the sphere with the surface S. Then, if $n = \left(\frac{\partial\Phi}{\partial x},\frac{\partial\Phi}{\partial y},-1\right) = \left(\frac{\Phi'(r)}{r}x,\frac{\Phi'(r)}{r}y,-1\right)$ represents the normal at $C(t)$ to H, we have the condition:

$$
0=\dot{q}(t,\xi)=\omega(t)\times\frac{\delta}{|n|}\left(\frac{\Phi'(r)}{r}x,\frac{\Phi'(r)}{r}y,-1\right)+\dot{C}(t).
$$

So we obtain the two equations for the constraints:

$$
\begin{aligned}
\dot{x} &= \tfrac{\delta}{|n|}\left[-\dot{\psi}\sin\theta\cos\phi+\dot{\theta}\sin\phi+\tfrac{\Phi'(r)}{r}y\left(\dot{\phi}+\dot{\psi}\cos\theta\right)\right]\\
\dot{y} &= \tfrac{\delta}{|n|}\left[-\dot{\psi}\sin\theta\sin\phi-\dot{\theta}\cos\phi-\tfrac{\Phi'(r)}{r}x\left(\dot{\phi}+\dot{\psi}\cos\theta\right)\right],
\end{aligned}
\tag{5}
$$

because the third equation

$$
\begin{aligned}
\Phi'(r)\dot{r}=\tfrac{\delta}{|n|}\ & \Big[-\tfrac{\Phi'(r)}{r}y\left(\dot{\psi}\sin\theta\sin\phi+\dot{\theta}\cos\phi\right)+\\
& \tfrac{\Phi'(r)}{r}x\left(-\dot{\psi}\sin\theta\cos\phi+\dot{\theta}\sin\phi\right)\Big],
\end{aligned}
$$

follows from the the two equations (5) above (see also [4], private communication). The case in which H is a plane (and so is S), the equation $\Phi'(r) = 0$ leads (5) to the classical equations (see [24], p. 167):

$$
\begin{aligned}
\dot{x} &= \delta\left[-\dot{\psi}\sin\theta\cos\phi+\dot{\theta}\sin\phi\right] = \delta\omega_y\\
\dot{y} &= \delta\left[-\dot{\psi}\sin\theta\sin\phi-\dot{\theta}\cos\phi\right] = -\delta\omega_x.
\end{aligned}
\tag{6}
$$

The potential function is obtained using (3):

$$V = -mg \langle e_z, C(t) + ae_3 \rangle = -mg (\Phi(r) + a\cos\theta). \tag{7}$$

Conservation of volume of three distinct Routh-Chaplygin spheres has been checked by using algorithm 1 in the coordinate neighborhood of M given by the domain of the local coordinates $(x, y, \phi, \theta, \psi)$. Remark 6 ensures that the result is global. The problems studied are: (i) the homogeneous sphere rolling on a general surface of revolution, which conserves volume; (ii) the Chaplygin sphere rolling on a plane, which also conserves volume (Koiller, in a private communication, has previously informed us about this result). The study of conservation of volume for the particular case of the Chaplygin sphere has been the subject of important studies: [9], [11] and [30]. However, to the best of our knowledge, the present work is the first to show conservation of the Riemann-Sasaki volume; and (iii) the Routh sphere rolling on a plane, which does not conserve volume. To prove that the GMA flow does not conserve volume, it is enough to prove that there is a point $q_0 \in M$ where tr $\left(B_{\mathcal{D}^\perp}|_{\mathcal{D}^\perp\times\mathcal{D}^\perp}\right)(q_0) \neq 0$. We choose q_0 given by the local coordinates $x_0 = 0$, $y_0 0$, $\phi_0 = 0$, $\theta_0 = \pi/2$, $\phi_0 = 0$ and $\psi_0 = 0$, then

$$\langle \text{tr } B_{\mathcal{D}^\perp}(q_0), \eta\rangle - \frac{a\delta}{k_1^2 + \delta^2} \neq 0,$$

where $\eta - \delta\sin(\beta-\phi)\frac{\partial}{\partial r} - \frac{\delta\cos(\beta-\phi)}{r}\frac{\partial}{\partial\beta} + \frac{\partial}{\partial\theta}$, with (r, β) the polar coordinates, $x = r\cos\beta$ and $y = r\sin\beta$.

The two next examples appear in [18] and are due to Hamel [14].

2.1.2. The sleigh of Chaplygin and Carathéodory.

Koiller described an idealized sleigh (see figure 1a in [18]) with the three points of contact with the plane: point A and two others sliding freely. The constraint does not allow transversal velocities, i.e. acts against the runners, laterally at A, so the η component of the velocity is zero. The configuration space $M = \mathbf{R}^2 \times S^1$ has local coordinates (x, y, ϕ) and the kinetic energy is given by

$$K = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}J\dot{\phi}^2$$

where m is the total mass of the sleigh and J is its moment of inertia about a vertical axis through the center of mass C. One assumes the potential force $\mathcal{F} = 0$ and the constraint manifold is locally given by

$$-a\dot{\phi} + \dot{y}\cos\phi - \dot{x}\sin\phi = 0, \; a > 0,$$

where $-a\dot{\phi}$ represents the lateral effect of the constraint.

Again, conservation of volume has been checked using algorithm 1 and the sleigh does not conserve volume. Indeed, in a point with coordinates (x, y, ϕ) we have

$$\langle \mathrm{tr}\; B_{\mathcal{D}^\perp}, \eta \rangle = \frac{am\cos\phi}{J + a^2 m},$$

where $\eta = \frac{\partial}{\partial x} - \frac{\sin\phi}{a}\frac{\partial}{\partial\phi}$; so, $\langle \mathrm{tr}\; B_{\mathcal{D}^\perp}, \eta \rangle \neq 0$ at (x, y, ϕ) if $\phi \neq \pi/2$.

A particular case of the Chaplygin-Carathéodory sleigh corresponds to the vertical knife free to slip along itself on a horizontal plane and also free to twist about the vertical line passing through a point A of the knife: $C = A$, $a = 0$: the configuration space is $\mathbf{R}^2 \times S^1$, the kinetic energy is

$$K = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}J\dot{\phi}^2,$$

the equation for the constraint is

$$\dot{y}\cos\phi - \dot{x}\sin\phi = 0,$$

and the external force $\mathcal{F}$ is assumed to be zero.

The Riemannian volume is conserved since $\mathcal{D}^\perp$ is integrable (it is generated by $\frac{\partial}{\partial\phi}$) and admits a regular foliation of minimal leaves. We used this problem to verify the code and the computations confirm the result.

2.1.3. The two-wheeled carriage.

The idealized system (see figure 2 in [18]) has configuration space

$$M = \mathbf{R}^2 \times S^1 \times S^1 \times S^1$$

with local coordinates (x, y, ϕ, q_1, q_2). Let $2r$ be the lateral lenght, a the radius of the wheels, C_0 the center of mass and l the distance between C_0 and (x, y). Imposing the constraint of no lateral sliding, as well as no sliding on both wheels, one obtains the equations

$$\begin{aligned} \dot{x}\sin\phi - \dot{y}\cos\phi &= 0 \\ \dot{x}\cos\phi + \dot{y}\sin\phi + r\dot{\phi} + a\dot{q}_1 &= 0 \\ \dot{x}\cos\phi + \dot{y}\sin\phi - r\dot{\phi} + a\dot{q}_2 &= 0 \end{aligned}$$

defining a 2-dimensional subspace of T_pM at each point $p \in M$ (here q_1 and q_2 denote the rolling angles of the wheels). Following [18], let m_0 be the mass of the body without wheels and k_0 the radius of gyration about the vertical axis through (x, y); let m_1 be the mass of a wheel, C its axial moment of inertia and A its moment of inertia about a diameter. So $m = m_0 + 2m_1$ is the total mass and let us set $J = m_0k_0^2 + 2m_1r^2 + 2A$. The kinetic energy is then given by (see [18])

$$K = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + m_0l\dot{\phi}\left(\dot{y}\cos\phi - \dot{x}\sin\phi\right) + \frac{1}{2}J\dot{\phi}^2 + \frac{1}{2}C\left(\dot{q}_1^2 + \dot{q}_2^2\right).$$

As in the previous example one assumes the potential force $\mathcal{F}0$.

Using algorithm 1 we conclude that the two-wheeled carriage does not conserve volume. Indeed, in a point with coordinates (x, y, ϕ, q_1, q_2) we have

$$\langle \text{tr } B_{\mathcal{D}^\perp}, \eta\rangle = -\frac{1}{2}\frac{a^3m_0l}{a^2J + 2Cr^2},$$

where $\eta = -\frac{1}{2}a\cos\phi\frac{\partial}{\partial x} - \frac{1}{2}a\sin\phi\frac{\partial}{\partial y} - \frac{a}{2r}\frac{\partial}{\partial\phi} + \frac{\partial}{\partial q_1}$; so, $\langle \text{tr } B_{\mathcal{D}^\perp}, \eta\rangle \neq 0$ for any (x, y, ϕ, q_1, q_2).

2.1.4. Wheeled top constrained inside a sphere.

Let us consider a top B constrained to move in such a way that the angular velocity is always orthogonal to a straight line l fixed in B (see [12], where a possible realization of this constraint is sketched in figure 1A, p. 369).

The configuration space is $M = SO(3)$ and choosing Oz in the reference configuration along l, then, with respect to the Euler angles previously defined, the constraint can be written as

$$\dot{\psi} + \dot{\phi}\cos\theta = 0.$$

The kinetic energy K is

$$K = \frac{1}{2}\Big[I_1(\dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi)^2 + I_2(\dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi)^2 + I_3(\dot{\phi}\cos\theta + \dot{\psi})^2\Big],$$

where I_1, I_2 and I_3 are the principal moments of inertia of the top about the fixed point. The potential function is $V - mg\langle e_z, l_G\rangle$, where l_G is the vector from the fixed point to the center of mass of the top.

The Riemannian volume is conserved since $\mathcal{D}^\perp$ is integrable and its leaves are minimal (see corollary 3): $\mathcal{D}^\perp$ is generated by $\frac{\partial}{\partial\psi}$. For the purpose of verifying the code, we also run this case with the program which confirmed the geometrical result.

2.1.5. The rolling homogeneous disc (the falling penny).

In this case (see figure 2.1 p. 229 in [8]) $M = \mathbf{R}^2 \times SO(3)$ with local coordinates $(x, y, \phi, \theta, \psi)$, where (ϕ, θ, ψ) are the Euler angles previously mentioned. The disc is homogeneous with mass m and radius R. We easily see that the equations for constraints are

$$\begin{aligned} \dot{x} &= -R\dot{\psi}\cos\phi \\ \dot{y} &= -R\dot{\psi}\sin\phi, \end{aligned}$$

the kinetic energy is

$$\begin{aligned} K = \; & \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \tfrac{1}{16}mR\Big[R\left(10\dot{\theta}^2 + 7\dot{\phi}^2 + 4\dot{\psi}^2\right) + 5R\dot{\phi}^2\cos(2\theta) + \\ & 16\dot{\theta}\sin\theta\left(\dot{x}\sin\phi - \dot{y}\cos\phi\right) + 8\dot{\phi}\cos\theta\left(R\dot{\psi} - 2\dot{x}\cos\phi - 2\dot{y}\sin\phi\right)\Big]. \end{aligned}$$

The potential function is $V = -mgR\cos\theta$.

The Riemannian volume is not conserved. In a point with coordinates $(x, y, \phi, \theta, \psi)$ we have

$$\left\langle \text{tr } B_{\mathcal{D}^\perp}, \frac{\partial}{\partial\theta} \right\rangle = \frac{2\sin(2\theta)}{3 + 2\cos(2\theta)}.$$

So, $\langle \text{tr } B_{\mathcal{D}^\perp}, \eta\rangle \neq 0$ for $\theta \neq \pi/2$ (note that $\theta \in]0, \pi[$).

2.1.6. Vertical rolling disc

A special case of the rolling disc corresponds to the vertical rolling disc $(\theta = \pi/2)$. The configuration space is reduced to $M = \mathbf{R}^2 \times S^1 \times S^1$. In this case we consider a rolling disc with center of mass coincident with the geometrical center, but with possible distinct central principal moments of inertia. The kinetic energy is then,

$$\begin{aligned} K = \; & \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \\ & \tfrac{1}{2}\left(I_1(\dot{\phi}\sin\psi)^2 + I_2(\dot{\phi}\cos\psi)^2 + I_3\dot{\psi}^2\right). \end{aligned}$$

and constraint equations remain the same as for the rolling homogeneous disc:

$$\begin{aligned} \dot{x} &= -R\dot{\psi}\cos\phi \\ \dot{y} &= -R\dot{\psi}\sin\phi. \end{aligned}$$

The external force $\mathcal{F}$ is assumed to be zero.

Using algorithm 1 and taking into account remark 6, we conclude that the Riemannian volume is conserved.

2.1.7. A nonholonomically constrained particle

In this example, due to Rosenberg [28], a free particle moves in $M = \mathbf{R}^3$ under the nonholonomic constraint given by

$$\dot{z} = y\dot{x},$$

where (x, y, z) are normal Cartesian coordinates of $\mathbf{R}^3$. It is interesting to note that the associated one-form of the constraint $\omega = \mathrm{d}z - y\mathrm{d}x$ is a contact form.

The kinetic energy is

$$K = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right),$$

and $\mathcal{F} = 0$.

Because $\mathcal{D}^\perp$ is integrable with leaves given by straight lines, we conclude by remark 3 that the Riemannian volume is conserved. Once again, we used this problem to verify the code and the computations confirm the result.

2.1.8. The snakeboard

The snakeboard (see [21]) consists of a rigid body (the board) with two sets of independently actuated wheels at each end of the board (see figure 8.3, p. 90 in [8]). The configuration space is $M = SE(2) \times S^1 \times S^1 \times S^1$ and the kinetic energy is

$$K = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_0\left(\dot{\theta} + \dot{\psi}\right)^2 + \frac{1}{2}J_1\left(\dot{\theta} + \dot{\phi}_1\right)^2 + \frac{1}{2}J_2\left(\dot{\theta} + \dot{\phi}_2\right)^2.$$

The snakeboard is constrained not to slide sideways, so that

$$\begin{aligned} -\sin(\theta + \phi_1)\dot{x} + \cos(\theta + \phi_1)\dot{y} - r\cos\phi_1\dot{\theta} &= 0 \\ -\sin(\theta + \phi_2)\dot{x} + \cos(\theta + \phi_2)\dot{y} + r\cos\phi_2\dot{\theta} &= 0; \end{aligned}$$

the external force $\mathcal{F}$ is assumed to be zero.

Using algorithm 1, we compute $\langle \mathrm{tr}\ B_{\mathcal{D}^\perp}, \eta\rangle$ in a arbitrary point

$$(x, y, \theta, \psi, \phi_1, \phi_2)$$

and arbitrary $\eta \in \mathcal{D}$:

$$\langle \text{tr } B_{\mathcal{D}^\perp}, \eta \rangle = \frac{2\, m\, r^2\, \eta_1\, (\sin(2\,\phi_1)+\sin(2\,\phi_2))}{J+3\, m\, r^2+2\, m\, r^2\, \cos(2\,\phi_1)+(-J+m\, r^2)\, \cos(2\,(\phi_1-\phi_2))+2\, m\, r^2\, \cos(2\,\phi_2)}$$

where $\eta = \eta_1 Y_1 + \eta_2 Y_2 + \eta_3 Y_3 + \eta_4 Y_4$ and

$$\begin{aligned} Y_1 &= -\frac{(r\, (\cos(\theta+\phi_1-\phi_2)+\cos(\theta-\phi_1+\phi_2)+2\, \cos(\theta+\phi_1+\phi_2))\, \csc(\phi_1-\phi_2))}{2}\frac{\partial}{\partial x} - \\ &\quad (r\, \csc(\phi_1-\phi_2)\, (\cos(\phi_2)\, \sin(\theta+\phi_1)+\cos(\phi_1)\, \sin(\theta+\phi_2)))\frac{\partial}{\partial y} + \frac{\partial}{\partial \theta} \\ Y_2 &= \frac{\partial}{\partial \theta} \\ Y_3 &= \frac{\partial}{\partial \phi_1} \\ Y_4 &= \frac{\partial}{\partial \phi_2}. \end{aligned}$$

So, the Riemannian volume is conserved if and only if $\phi_1 = (-1)^{k+1}\phi_2 + \frac{k}{2}\pi$, $k \in \mathbf{Z}$ such that $\phi_1, \phi_2 \in]-\pi, \pi[$ and $J_1 = J_2$. The case $\phi_1 = -\phi_2$ has been studied by Ostrowski in [26] (see also [6]).

2.1.9. Lie groups with left invariant metric and constraint

In the case where the configuration space M is a Lie group, we are able to write the conditions in Theorem 2 as algebraic equations involving the structural constants of the corresponding Lie algebra.

Let $(\mathsf{G}, \langle , \rangle)$ be a Lie group with a left invariant metric, and $\mathcal{D}$ a left invariant m dimensional distribution on G (see [35] and [36]). Note that $\mathcal{D}$ is completely determined by a m dimensional linear subspace of its Lie algebra g.

Let us choose an orthonormal basis $\{\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_n\}$ of g such that $\{\xi_1, \ldots, \xi_m\}$ is a basis of $\mathcal{D}_e$ ($e := \mathsf{id}_G$) and $\{\xi_{m+1}, \ldots, \xi_n\}$ is a basis of $\mathcal{D}_e^\perp$. The left invariant vector fields corresponding to these elements of g will be denoted by the same letters. We denote, as usual, the Christoffel symbols corresponding to these elements by $\Gamma^i_{jk} := \left\langle \nabla_{\xi_j} \xi_k, \xi_i \right\rangle$.

The condition (2) can then be written as the following equality:

$$\sum_{\alpha=m+1}^{n} \Gamma^i_{\alpha\alpha} = 0,\ i = 1, \ldots, m.$$

To bring the structure constants of the Lie algebra g, $c^i_{jk} := \langle [\xi_j, \xi_k], \xi_i \rangle$, into evidence we recall that (see, for instance, [34] and [19])

$$\Gamma^i_{jk} = \frac{1}{2}\left(c^i_{jk} + c^j_{ik} + c^k_{ij} \right).$$

So the GMA flow associated with the left invariant constraint $\mathcal{D}$ conserves the Riemannian volume (for any potential function), if and only if, the following algebraic equation holds

$$\sum_{\alpha=m+1}^{n} c_{i\alpha}^{\alpha} = 0, \quad i = 1, \dots, n$$

2.1.10. Semi-simple Lie groups under Cartan decomposition

Here we present a class of examples of nonholonomic constrained mechanical systems defined by the so called Cartan decomposition of semisimple Lie groups.

Let us start by recalling the following definitions and results for semisimple Lie algebras (see [15]):

1. A Lie algebra g is called *semisimple* if the Killing form $\kappa(X, Y) = \mathrm{tr}\,(\mathsf{ad}\, X \mathsf{ad}\, Y)$ on $\mathsf{g} \times \mathsf{g}$ is non-degenerate. An analytical Lie group is *semisimple* if its Lie algebra is semisimple.

2. Let g be a Lie algebra. Then $\theta \in \mathrm{Aut}(\mathsf{g})$ is an involution if $\theta^2 = 1$.

3. If g is a real semisimple Lie algebra, then an involution θ on g is called a **Cartan involution** if the symmetric bilinear form

$$\kappa_\theta(X, Y) = -\kappa(X, \theta Y)$$

is positive definite, where κ is the so called Killing form of g.

4. Every real semisimple Lie algebra has a Cartan involution. Moreover any two Cartan involutions are conjugate via $\mathrm{Int}(\mathsf{g})$.

5. Any Cartan involution yields a Cartan Decomposition $\mathsf{g} = \mathsf{k} \oplus \mathsf{p}$, with

$$\begin{aligned} \mathsf{k} &= \{X \in \mathsf{g} \mid \theta(X) = X\}, \\ \mathsf{p} &= \{X \in \mathsf{g} \mid \theta(X) = -X\}, \end{aligned}$$

where k is a maximal compactly embedded subalgebra of g.

6. The following properties hold:

(i) $[\mathsf{k}, \mathsf{k}] \subset \mathsf{k}, \quad [\mathsf{k}, \mathsf{p}] \subset \mathsf{p}, \quad [\mathsf{p}, \mathsf{p}] \subset \mathsf{k}$,

(ii) $\kappa_\theta(\mathsf{k}, \mathsf{p}) = \kappa(\mathsf{k}, \mathsf{p}) = 0$,

(iii) $\kappa|_{\mathsf{k}}$ is negative definite, $\kappa|_{\mathsf{p}}$ is positive definite.

On a semisimple analytical Lie group G with Lie algebra g, let us consider the left invariant distribution defined by $\mathcal{D}_e = \mathsf{p}$ and the left invariant metric associated with an arbitrary metric on g such that p and k are orthogonal, for instance, $\langle X, Y\rangle = \kappa_\theta(X, Y)$, for all $X, Y \in \mathsf{g}$. One assumes that there are no external forces acting on the system.

As concrete examples, we mention the so called pseudo-rigid bodies (see [25]), whose configuration space is $\mathsf{SL}(n)$. Also, for any $X \in \mathsf{sl}(n)$, $\theta(X) = -X^\dagger$, $X^\dagger$ being the transpose of X.

Remark 8. A totally geodesic distribution is a distribution invariant under the geodesic spray of the Levi-Civita connection. So, if $\mathcal{D}^\perp$ is totally geodesic we have $B^s_{\mathcal{D}^\perp}\big|_{\mathcal{D}^\perp\times\mathcal{D}^\perp} = 0$ and then the GMA flow conserves volume. Semi-simple Lie groups with the Cartan decomposition satisfy this latter property.

2.2. Affine constraints

In this section we consider two examples of affine constraints. In this case $\mathcal{C}$ corresponds to an affine bundle $\mathcal{A}$ characterized by a pair $\mathcal{A} = (\mathcal{D}, X_a)$, where $\mathcal{D}$ is a smooth distribution and $X_a \in \Gamma^\infty(\mathcal{D}^\perp)$ is a smooth section of $\mathcal{D}^\perp$. Then a vector $v_q \in T_qM$ is compatible with the constraint if and only if $v_q - X_a(q) \in \mathcal{D}_q$. The conditions for the conservation of volume (when $\mathcal{A}$ is oriented as a manifold) are equivalent to ([17]):

$$\mathrm{tr}\,\left(B_{\mathcal{D}^\perp}|_{\mathcal{D}^\perp\times\mathcal{D}^\perp}\right) = 0,$$

and

$$\mathcal{L}_{X_a}\Theta = 0,$$

where Θ is the nowhere vanishing global section of $(\mathcal{D}^\perp)^* \wedge \cdots \wedge (\mathcal{D}^\perp)^*$ ($\dim(\mathcal{D}^\perp)$ times) compatible with the metric, in the sense that for a positively oriented orthonormed basis of $\mathcal{D}^\perp$, $\{\xi_i\}_{i=m+1,\dots,n}$, we have

$$\Theta(\xi_{m+1}, \dots, \xi_n) = 1.$$

2.2.1. A homogeneous ball on a rotating plate.

Following [6] let us consider a model of a homogeneous sphere that rolls without slipping on a horizontal plate (see also [27]); assume also that the plate rotates with a constant angular velocity Ω about a vertical axis passing through the origin of the coordinates (x, y) (see figure 8.2, p. 87 in [6]). The configuration space is $M = \mathbf{R}^2 \times SO(3)$. Let m be the mass of the sphere and a its radius; call mk^2 its moment of inertia about any diameter. Then the kinetic energy is

$$K = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}mk^2(\omega_x^2 + \omega_y^2 + \omega_z^2)$$

where

$$\begin{aligned}\omega_x &= \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi\\ \omega_y &= \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi\\ \omega_z &= \dot{\phi} + \dot{\psi}\cos\theta.\end{aligned}$$

The affine constraints are given by

$$\begin{aligned}\dot{x} - a\omega_y &= -\Omega y\\ \dot{y} + a\omega_x &= \Omega x\end{aligned} \tag{8}$$

that is

$$\begin{aligned}\dot{x} - a\left(\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi\right) &= -\Omega y\\ \dot{y} + a\left(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi\right) &= \Omega x.\end{aligned}$$

From (8), we see that the vector field X_a defining the constraint

$$\mathcal{A} = \mathcal{D} + X_a$$

can be written as

$$\begin{aligned}X_a &= -\Omega\, y\, \xi_1 - \Omega\, x\, \xi_2\\ &= -\frac{k^2\, y\, \Omega}{k^2 + R^2}\frac{\partial}{\partial x} + \frac{k^2\, x\, \Omega}{k^2 + R^2}\frac{\partial}{\partial y} +\\ &\quad \frac{R\,\Omega\, \cot(\theta)\, \left(y\, \cos(\phi) - x\, \sin(\phi)\right)}{k^2 + R^2}\frac{\partial}{\partial\phi} +\\ &\quad \frac{R\,\Omega\, \left(x\, \cos(\phi) + y\, \sin(\phi)\right)}{k^2 + R^2}\frac{\partial}{\partial\theta} +\\ &\quad \frac{R\,\Omega\, \csc(\theta)\, \left(-\left(y\, \cos(\phi)\right) + x\, \sin(\phi)\right)}{k^2 + R^2}\frac{\partial}{\partial\psi}\end{aligned}$$

where $\{\xi_1, \xi_2\}$ is the dual basis of the one-forms defining the constraints $\{\omega^1, \omega^2\}$:

$$\begin{aligned}\omega^1 &:= \mathrm{d}x - a\left(\sin\phi\, \mathrm{d}\theta - \sin\theta\cos\phi\, \mathrm{d}\psi\right)\\ \omega^2 &:= \mathrm{d}y + a\left(\cos\phi\, \mathrm{d}\theta + \sin\theta\sin\phi\, \mathrm{d}\psi\right).\end{aligned}$$

The section Θ needed for the verification of volume conservation is

$$\Theta = \frac{k^2 m}{k^2+R^2}\,\omega^1\wedge\omega^2$$

and the external force is $\mathcal{F}=0$.

The distribution $\mathcal{D}$ is the same as the one for a sphere rolling on a plane, so we obtain $\operatorname{tr}\left(B_{\mathcal{D}^\perp}|_{\mathcal{D}^\perp\times\mathcal{D}^\perp}\right)=0$. Finally, in order to check if there is conservation of the Riemannian volume we only need to verify if $\mathcal{L}_{X_a}\Theta=0$. But,

$$\mathcal{L}_{X_a}\Theta = \frac{k^2 m}{k^2+R^2}\left[(\mathcal{L}_{X_a}\omega^1)\wedge\omega^2+\omega^1\wedge\mathcal{L}_{X_a}\omega^2\right],$$

and

$$\begin{aligned}\mathcal{L}_{X_a}\omega^1 &= -\Omega\,\mathrm{d}y+\frac{R^2x\Omega}{k^2+R^2}\,\mathrm{d}\phi+\frac{R^2x\Omega\cos\theta}{k^2+R^2}\,\mathrm{d}\psi\\ \mathcal{L}_{X_a}\omega^2 &= \Omega\,\mathrm{d}x+\frac{R^2y\Omega}{k^2+R^2}\,\mathrm{d}\phi+\frac{R^2y\Omega\cos\theta}{k^2+R^2}\,\mathrm{d}\psi.\end{aligned}$$

So,

$$\mathcal{L}_{X_a}\Theta|_{x\theta} = \frac{k^2 m}{k^2+R^2}\Omega a\sin\phi\neq 0,$$

and the Riemannian volume is not conserved by the GMA flow.

In order to check conservation of energy, we use the local solution $z = x+\mathrm{i}y$ given by (see [27]):

$$z(t) \;=\; z_0+\frac{\mathrm{i}w_0}{\kappa}(1-e^{\mathrm{i}\kappa t})$$

where $\kappa=\frac{2}{7}\Omega$, $z_0:=z(0)$ and $w_0=\dot{x}_0+\mathrm{i}\dot{y}_0:=\dot{z}(0)$ and $\omega_z=\text{const}$ (ω_x and ω_y can be determined from the constraint equations (8)).

A simple computation then shows that:

$$\begin{aligned}\tfrac{dK}{dt} = {}& \tfrac{1}{14R^2}\big[5\,k^2\,m\,\Omega\,\big(2\,(x_0\,\dot{x}_0+y_0\,\dot{y}_0)\,\Omega\,\cos(\tfrac{2\,t\,\Omega}{7})+\\ &\big(7\,\dot{x}_0^2+2\,\dot{x}_0\,y_0\,\Omega+\dot{y}_0\,(7\,\dot{y}_0-2\,x_0\,\Omega)\big)\,\sin(\tfrac{2\,t\,\Omega}{7})\big)\big]\neq 0,\end{aligned}$$

and the mechanical (kinetic) energy is not conserved. However, it is interesting to note that in average $\frac{dK}{dt}=0$.

2.2.2. Swell Stabilizer

In [10] a swell stabilizer (*stabilisateur de houle*) is considered, where a gyroscope is placed in a ship in order to stabilize the motion due to sea swell. The gyroscope casing is mobile with respect to the bridge of the ship about an axis perpendicular to the longitudinal axis of the ship, the center of mass of the gyroscope being a distance d above the center of mass of the boat. The configuration space is $\mathbf{R}^2 \times S^1$. If θ designate the angle of the mast of the ship with the vertical, α the angle of the casing with the bridge and ϕ the angle of rotation of the gyroscope, then the kinetic energy is locally given by,

$$K\frac{1}{2}\left[A\dot{\alpha}^2 + I(\alpha)\dot{\theta}^2 + C\left(\dot{\phi} + \dot{\theta}\sin\alpha\right)^2\right],$$

where $I(\alpha) = I_1 + md^2 + A\cos^2\alpha$, I_1, (A, C) are the moments of inertia of the ship and gyroscope, respectively, and m is the mass of the gyroscope.

The constraint is then

$$\dot{\alpha} = a\dot{\theta} + b\theta,$$

with $a > 0$ chosen big enough to stabilize the ship. Finally, the potential function representing the swell is $V = k\cos\theta$, with $k > 0$.

To verify volume conservation we run the code with the previous data to obtain

$$\left\langle \text{tr } B\mathcal{D}^{\perp}, \eta\right\rangle = \frac{a^3\, A^2\, \cos\alpha\, \sin\alpha}{(I_1 + d^2\, m + A\cos\alpha^2)\, (a^2\, A + I_1 + d^2\, m + A\cos\alpha^2)},$$

where $\eta = a\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\theta}$. So, the GMA flow does not conserve the Riemannian volume associated with Sasaki's metric $\mathbf{g}_{\mathcal{A}}$.

It is interesting to note that for $b = 0$ the constraint is linear and integrable, so that the GMA flow restricted to the leaves is Hamiltonian and conserves the phase volume of the leaf. But, as we have seen from the previous computation, the GMA does not conserve the Riemannian volume associated with $\mathbf{g}_{\mathcal{A}}$ (see remark 5).

Conservation of energy can be checked by showing that

$$\frac{\partial E}{\partial x^i}\dot{x}^i + \frac{\partial E}{\partial \dot{x}^i}\ddot{x}^i = b\theta\lambda,$$

where λ is the (non-zero) Lagrange multiplier; so we conclude that the GMA flow does not conserve mechanical energy $E = T - V$, except when $b = 0$, case in which the constraint is linear.

2.3. Non-linear constraints

In this section we consider non-linear constraints in the velocities. After Appell's machine ([1]), the interest in this kind of constraint never faded and nowadays it remains an active area of research. In most of the examples that follow the constraint $\mathcal{C}$ can be written as the inverse image of the null section of a fibre bundle morphism. More specifically, let $\pi_E : E \to M$ be a vector bundle morphism, with a connection ∇^E. Let $f : TM \to E$ be a smooth fibre preserving mapping. Then, if $f^{-1}(0_E) \subset TM$ is regular in the sense defined above, $\mathcal{C} := f^{-1}(0_E)$ defines a regular constraint. In this case, the conditions for conservation of volume for any potential function simplify to:

1. $\mathrm{tr}\ \left[\left(\mathsf{F}f(v_q)|_{\ker \mathsf{F}(v_q)^\perp}\right)^{-1} \cdot \mathsf{P}f(v_q)\right] = 0.$
2. $\mathrm{tr}\ \mathsf{F}^2 f(v_q)\big|_{\ker \mathsf{F}f(v_q) \times \ker \mathsf{F}f(v_q)} = 0,$

where $\mathsf{P}f : TM \to \mathsf{L}(TM, S)$ is the **parallel derivative** of f,

$$\mathsf{P}f(z_q) = \kappa_S \cdot \mathsf{T}f \cdot H_{v_q}(z_q)$$

for all $v_q, z_q \in TM$.

On the other hand, if we want to check conservation of volume for a given potential function V, we need to use Theorem 1. The case in which the constraint is given by the inverse image of the null section under a fibre bundle morphism f as above, the condition for conservation of volume in the latter theorem becomes

$$\begin{aligned}
&\mathrm{tr}\ \left[\left(\mathsf{F}f(v_q)|_{\ker \mathsf{F}(v_q)^\perp}\right)^{-1} \cdot \mathsf{P}f(v_q)\right] + \\
&\Big\langle \mathrm{tr}\ \mathsf{F}^2 f(v_q)\big|_{\ker \mathsf{F}f(v_q) \times \ker \mathsf{F}f(v_q)}, \\
&\left(\mathsf{F}f(v_q)|_{\ker \mathsf{F}(v_q)^\perp}\right)^{-1*} \left(\left(\mathsf{F}f(v_q)|_{\ker \mathsf{F}(v_q)^\perp}\right)^{-1} \mathsf{P}f(v_q) \cdot v_q - \mathcal{P}^\perp_{v_q} \cdot \mathsf{grad} V(q)\right)\Big\rangle \\
&= 0.
\end{aligned} \tag{9}$$

2.3.1. Appell's machine

This is the first known example of a mechanical system with a non-linear constraint. It was proposed by Appell in [1] and consists of a heavy material point moving in space under the constraint

$$\dot{x}^2 + \dot{y}^2 = a^2 \dot{z}^2,$$

where $a \in \mathbf{R}$ is a constant (see a possible realization as the limit of the machine shown in figure 4.9, p. 223 in [24]) . The configuration space is

$M = \mathbf{R}^3$ and the kinetic energy is $K = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$, where m is the mass of the particle. The external force is assumed to be zero.

Since the external force and $\mathsf{P}f$ are zero, equation (9) is trivially satisfied then the GMA flow conserves the Riemannian volume.

On the other hand, a straightforward computation shows that

$$\mathrm{tr}\ \mathsf{F}^2 f(v_q)\big|_{\ker \mathsf{F}f(v_q)\times\ker \mathsf{F}f(v_q)} = 2 \neq 0,$$

so there are potential functions for which the Appell's machine does not converse volume.

For the conservation of energy we just observe that the constraint manifold is given by a homogeneous function and so the GMA flow conserves energy for all potential functions.

2.3.2. Benenti's mechanism

Benenti proposed a mechanical system with a non-linear constraint, which may be realizable (see figure in p. 211 of [3]). It consists of two masses in the plane subject to the constraint that their instantaneous velocity must be parallel at all times, that is,

$$\begin{vmatrix} \dot{x}_1 & \dot{y}_1 \\ \dot{x}_2 & \dot{y}_2 \end{vmatrix} = \dot{x}_1\dot{y}_2 - \dot{x}_2\dot{y}_1 = 0.$$

We consider a modified version of the former in order to avoid the degeneration of the constraint at the null section of TM. So, the exclusion of the latter from the constraint equations leads to the constraint manifold

$$\begin{aligned}\mathcal{C}:\ &\big\{(x_1, x_2, y_1, y_2, \dot{x}_1, \dot{x}_2, \dot{y}_1, \dot{y}_2) \in T\mathbf{R}^4 \mid \\ &\dot{x}_1\dot{y}_2 - \dot{x}_2\dot{y}_1 = 0\ \&\ \dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2 \neq 0\big\}\end{aligned}$$

The configuration space is $M = \mathbf{R}^4$ and the kinetic energy is

$$K\frac{1}{2}\left[m_1(\dot{x}_1^2 + \dot{y}_1^2) + m_2(\dot{x}_2^2 + \dot{y}_2^2)\right]$$

where m_1, m_2 are the masses of the particles and (x_1, y_1), (x_2, y_2), the normal Cartesian coordinates of the particles. One assumes also that $\mathcal{F} = 0$.

Conservation of volume has been proved in [32]. Now we remark that the constraint is homogeneous and so the GMA flow conserves energy for all potential functions (see introduction).

2.3.3. Marles's servomechanism

In [23] Marle consider a servomechanism that can sustain a "straight rod in equilibrium on the tip of one finger". He takes Ox to be the line along the finger (in horizontal motion) and Oy the vertical axis (in [32] p. 45 a sketch is proposed). The configuration space is $M = \mathbf{R} \times S^1$, with coordinates (x, θ), where $x \in \mathbf{R}$ is the abscissa of the point of contact of the rod with the horizontal axis and $\theta \in S^1$ the angle made by the rod with that axis. The constraint is

$$\dot{x} = h(x, \theta, \dot{\theta}),$$

where h is some smooth known function and the kinetic energy is

$$K = \frac{1}{2} m \left(\dot{x}^2 - 2l\dot{x}\dot{\theta} \sin\theta \right) + \frac{1}{2} J \dot{\theta}^2$$

where m is the mass of the rod, l is the distance of the center of mass of the rod (may be non homogeneous) to the point of contact and J is the moment of inertia of the rod about the point of contact. The potential energy is $V = -mgl \sin\theta$.

Condition 2. above for conservation of volume for any potential function means

$$\operatorname{tr} \mathsf{F}^2 f\big|_{\ker \mathsf{F} f \times \ker \mathsf{F} f} \, \partial_{\dot{\theta}}^2 h (ml \sin\theta \, \partial_{\dot{\theta}} h - J) = 0.$$

So, for the conservation of volume by the GMA flow for all potential functions it is necessary that the constraint be affine. When we try to check the first condition 1. we obtain a first order partial differential equation on the function h. Computing the derivative of the mechanical energy along the constrained motions, we obtain another condition on the function h for the conservation of energy.

2.3.4. The isokinetic dynamics

This example, first proposed by Hoover [16], finds many interesting applications in statistical mechanics—see for example [16], [29], [13]. Other studies of this constraint can be found in [37] and [38].

Let $e > 0$ and we define $f : TM \to \mathbf{R}_M$ ($\mathbf{R}_M := M \oplus_M \mathbf{R}$) by:

$$f(v_q) = \langle v_q, v_q \rangle - e,$$

for all $v_q \in TM$, so $\mathcal{C} = f^{-1}[0_{\mathbf{R}_M}]$ is a fibre bundle of spheres. We assume the field of forces $\mathcal{F} = 0$.

We have $\mathsf{P} f \equiv 0$ so the GMA flow conserves the Riemannian volume for $V = 0$ and since $\mathsf{F}^2 f(v_q)(u_q, w_q) \, \langle u_q, w_q \rangle$, for all $v_q \in \mathcal{C}$ and all $u_q, w_q \in$

TM, there exist potential functions for which the GMA flow does not conserve Riemannian volume. Conservation of mechanical energy on the level e (kinetic energy) is a trivial consequence of the constraint itself.

APPENDIX

In this appendix, we list the code used for testing the linear constraints and one part of the affine constraints for volume conservation of the respective GMA flow. The code is written in Mathematica and for concreteness we present the test for the Chaplygin sphere rolling on a plane.

```
(* ******************************************** *)
(* Program for the verification of the conservation of volume by the GMA
vector field determined by a linear constraint *)
(* ******************************************** *)
(* Definition of variables *)
ClearAll[dim]; (* dimension of the configuration space *)
ClearAll[dimDo]; (* dimension of 𝒟⊥ *)
ClearAll[xl]; (* local coordinates: position *)
ClearAll[xld]; (* local coordinates: velocity *)
ClearAll[xlf]; (* auxiliary variable *)
ClearAll[xldf]; (* auxiliary variable *)
ClearAll[A1]; (* component along e1 angular velocity in the body frame *)
ClearAll[A2]; (* component along e2 angular velocity in the body frame *)
ClearAll[A3]; (* component along e3 angular velocity in the body frame *)
ClearAll[K]; (* kinetic energy *)
ClearAll[e]; (* identity matrix *)
ClearAll[m]; (* mass of the sphere *)
ClearAll[I1]; (* moment of inertia along e1 *)
ClearAll[I2]; (* moment of inertia along e2 *)
ClearAll[I3]; (* moment of inertia along e3 *)
ClearAll[k1]; (* radius of gyration along e1 *)
ClearAll[k2]; (* radius of gyration along e2 *)
ClearAll[k3]; (* radius of gyration along e3 *)
ClearAll[x]; (* x coordinate *)
ClearAll[y]; (* y coordinate *)
ClearAll[ϕ]; (* Euler angle *)
ClearAll[θ]; (* Euler angle *)
ClearAll[ψ]; (* Euler angle *)
```

```
(* ******************************************** *)
dim = 5;
dimDo = 2;
xl = {x, y, φ, θ, ψ};
xlf = {x_, y_, φ_, θ_, ψ_};
xld = {xd, yd, φd, θd, ψd};
xldf = {xd_, yd_, φd_, θd_, ψd_};
A1 = φd * Sin[ψ] * Sin[θ] + θd * Cos[ψ];
A2 = φd * Cos[ψ] * Sin[θ] − θd * Sin[ψ];
A3 = φd * Cos[θ] + ψd;
I1 = m * k1^2;
I2 = m * k2^2;
I3 = m * k3^2;
Φ[r_] = δ;
K[xlf, xldf] = FullSimplify[m/2(xd^2 + yd^2 + (Φ'[r] * rd)^2) + 1/2(I1 * A1^2 +
I2 * A2^2 + I3 * A3^2)];
e[xlf] = Table[IdentityMatrix[dim][[i]], i, 1, dim];
(* ******************************************** *)
(* Definition of variables *)
ClearAll[g]; (* covariant metric matrix *)
ClearAll[ginv]; (* contravariant metric matrix *)
(* ******************************************** *)
g[xlf]Table[Simplify[1/2(K[xl, e[xl][[i]] + e[xl][[j]]]−
K[xl, e[xl][[i]] − e[xl][[j]]]])], i, 1, dim, j, 1, dim];
ginv[xlf] = Simplify[Inverse[g[xl]]];
(* ******************************************** *)
(* Definition of variables *)
ClearAll[Christoffel]; (* Christoffel symbols *)
(* ******************************************** *)
Christoffel[xlf] = Table[
  Simplify[Sum[ginv[xl][[i, a]]/2 * (D[g[xl][[a, j]], xl[[k]]]+
  D[g[xl][[a, k]], xl[[j]]] − D[g[xl][[k, j]], xl[[a]]]), a, 1, dim]],
  {i, 1, dim}, {j, 1, dim}, {k, 1, dim}];
(* ******************************************** *)
(* Definition of variables *)
ClearAll[ωD]; (* constraint one-forms *)
ClearAll[wD]; (* index rising of constraint one-forms *)
(* ******************************************** *)
ωD[xlf] = {{1, 0, 0, −δ Sin[φ], δ Sin[θ] Cos[φ]},
  {0, 1, 0, δ Cos[φ], δ Sin[θ] Sin[φ]}}; wD[xlf] = ωD[xl].ginv[xl];
```

```
(* Local basis for the constraint distribution *)
xySol = Solve[ωD[xl].{xS, yS, φS, θS, ψS} == {0, 0}, {xS, yS}];
(* ******************************************** *)
(* Definition of variables *)
ClearAll[YD1] (* basic vector *)
ClearAll[YD2] (* basic vector *)
ClearAll[YD3] (* basic vector *)
ClearAll[YD] (* arbitrary vector in 𝒟 *)
(* ******************************************** *)
YD1 = Simplify[{xS/.xySol[[1, 1]], yS/.xySol[[1, 2]], φS, θS, ψS}/.{φS →
1, θS → 0, ψS → 0}];
YD2 = Simplify[{xS/.xySol[[1, 1]], yS/.xySol[[1, 2]], φS, θS, ψS}/.{φS →
0, θS → 1, ψS → 0}];
YD3 = Simplify[{xS/.xySol[[1, 1]], yS/.xySol[[1, 2]], φS, θS, ψS}/.{φS →
0, θS → 0, ψS → 1}];
YD = Simplify[η1 * YD1 + η2 * YD2 + η3 * YD3];
(* ******************************************** *)
(* Definition of variables *)
ClearAll[h]; (* contravariant metric matrix restricted to 𝒟⊥ *)
ClearAll[hinv]; (* covariant metric matrix restricted to 𝒟⊥ *)
(* ******************************************** *)
h[xlf] = Simplify[wD[xl].g[xl].Transpose[wD[xl]]];
hinv[xlf] = Simplify[Inverse[h[xl]]];
(* ******************************************** *)
(* Definition of variables *)
ClearAll[ξD]; (* dual to the constraint one-forms *)
ξD[xlf]Transpose[Table[Sum[hinv[xl][[i, j]] * wD[xl][[j]],
   {j, 1, dimDo}], {i, 1, dimDo}]];
(* ******************************************** *)
(* Definition of variables *)
ClearAll[Traco]; (* ∇_{ξ_i} ω^i *)
(* ******************************************** *)
Traco[xlf] = Table[Sum[Sum[ξD[xl][[χ, a]] * (D[ωD[xl][[a, α]], xl[[χ]]]−
   Sum[Christoffel[xl][[γ, α, χ]] * ωD[xl][[a, γ]], {γ, 1, dim}]),
   {χ, 1, dim}], {a, 1, dimDo}], {α, 1, dim}];
(* ******************************************** *)
(* Test for volume conservation *)
VolumeTest = Simplify[Traco[xl].YD];
Print["Test for volume conservation: Tr(B_𝒟⊥)= ", VolumeTest];
```

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