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On Local Diffeomorphisms of $\mathbf{R}^n$ that are Injective *

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Dedicated to Jorge Sotomayor on his $60th$ birthday

There are obtained conditions under which maps from $\mathbf{R}^n$ to itself are globally injective. In particular there are proved some partial results related to the Weak Markus-Yamabe Conjecture which states that if a vector field $X : \mathbf{R}^n \to \mathbf{R}^n$ has the property that, for all $p \in \mathbf{R}^n$, all the eigenvalues of $DX(p)$ have negative real part, then X has at most one singularity.

Key Words: Global injectivity, Jacobian Conjecture, Weak Markus-Yamabe Conjecture.

1. INTRODUCTION

Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 map. We denote by Spec(F) the set of (complex) eigenvalues of the derivative DF_p, as p varies in $\mathbf{R}^n$. One of the several equivalent formulations of the famous KELLER JACOBIAN CONJECTURE states that if $F : \mathbf{R}^n \to \mathbf{R}^n$ is a polynomial map having constant non-zero Jacobian, then F is injective. The WEAK MARKUS-YAMABE CONJECTURE states that if $F : \mathbf{R}^n \to \mathbf{R}^n$ is a C^1 map such that Spec$(F) \subset \{z \in \mathbf{C} : \Re(z) < 0\}$, then F is injective. The CHAMBER-

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land Conjecture [5] states that if $F : \mathbf{R}^n \to \mathbf{R}^n$ is a map of class C^1 such that, for some $\epsilon > 0$, $\mathrm{Spec}(F) \cap \{z \in \mathbf{C} : |z| < \epsilon\} = \emptyset$, then F is injective. We shall see that the Chamberland Conjecture implies the Weak Markus-Yamabe one.

As a consequence of the work by H. Bass, E. Conell, D. Wright and A. Yagzhev ([1], [13]) we know that if the Keller Jacobian Conjecture is true for polynomial maps $P : \mathbf{R}^n \to \mathbf{R}^n$ of degree ≤ 3, for all $n \geq 2$, and such that $\mathrm{Spec}(P) = \{-1\}$, then it is also true for general polynomial maps (see [11], Proof of Proposition 8.1.8). As a consequence, if either the Chamberland or the Weak Markus-Yamabe conjectures is true for the set of polynomial maps $P : \mathbf{R}^n \to \mathbf{R}^n$ of degree ≤ 3, for all $n \geq 2$, then the Keller Jacobian Conjecture is also true. For more details about these conjectures, we refer the reader to the book by A. van den Essen [11] and the article by S. Nollet and F. Xavier [7]. In this paper we study cases in which both conjectures are true. The nice survey article of S. Nollet and F. Xavier [7] provides other sufficient conditions for the existence of global diffeomorphism of $\mathbf{R}^n$; in this respect see also the articles [3] and [4] of L. A. Campbell.

In this paper we obtain some partial extensions to $\mathbf{R}^n$ of the following Fernandes, Gutierrez and Rabanal result [6]: If $X : \mathbf{R}^2 \to \mathbf{R}^2$ is a differentiable map (not necessarily of class C^1) and, for some $\epsilon > 0$, $\mathrm{Spec}(X) \cap [0, \epsilon) = \emptyset$, then X is injective. If the assumptions of this bidimensional result are relaxed to $0 \notin \mathrm{Spec}(X)$, then the conclusion, even for polynomials maps X, need no longer be true, as shown by Pinchuck's counterexample [8]. (See also [11], page 241). Also B. Smith and F. Xavier ([10], Theorem 4) proved that there exist integers $n > 2$ and non-injective polynomial maps $P : \mathbf{R}^n \to \mathbf{R}^n$ with $\mathrm{Spec}(P) \cap [0, \infty) = \emptyset$.

A well known result of Hadamard says that if $F : \mathbf{R}^n \to \mathbf{R}^n$ is a locally diffeomorphic proper map of class C^1, then F is a global diffeomorphism. In other direction to Hadamard's Theorem and the Smyth and Xavier examples [10], we prove an injectivity result which is valid for maps which need not be proper:

Theorem 1. *Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a local diffeomorphism of class C^1 such that Spec(F) is disjoint of a sequence $\{t_m\}$ of real numbers which converges to* 0 *as $m \to \infty$. If there exist $R > 0$ and $0 < \alpha < 1$ such that, for all x in $\mathbf{R}^n$ with $|x| > R$, $|F(x)| \leq |x|^\alpha$, then F is injective.*

We will then prove the following results

Theorem 2. *Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 Lipschitz local diffeomorphism. Suppose that for some $\epsilon > 0$,*

$$\liminf_{|x|\to\infty} (|x| \cdot |\det(DF(x))|) > \epsilon.$$

Then F is a global diffeomorphism.

Theorem 3. *Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 Lipschitz local diffeomorphism. Suppose that there exists a sequence $\{D_m\}_{m=1}^{\infty}$ of compact discs of $\mathbf{C}$ (with non-empty interior), centered at points t_m of the real axis, such that $\lim_{m\to\infty} t_m = 0$ and*

$$Spec(F) \cap (\cup_{m=1}^{\infty} D_m) = \emptyset.$$

Then, F is injective.

Corollary 4. *The Chamberland Conjecture and the Weak Markus-Yamabe Conjecture are true for C^1 Lipschitz maps.*

In Section 3 we prove that the Chamberland Conjecture implies the Weak Markus-Yamabe Conjecture in another class of maps. As a consequence we obtain the following results

Theorem 5. *Let $F: \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 map such that if $n \geq 2$, there exist $K > 0$ and a compact subset of $\mathbf{R}^n$ outside of which*

$$||DF(x)|| \leq K|x|^{\frac{1}{n-1}}.$$

(a)If there is $\epsilon > 0$ such that $|\det(DF(x))| > \epsilon$, for all $x \in \mathbf{R}^n$, then F is a global diffeomorphism.
(b)If $Spec(F) \subset \{z \in \mathbf{C} : \Re(z) < 0\}$, then F is injective.

Theorem 6. *Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 map such that there is a compact subset of $\mathbf{R}^n$ outside of which the Jacobian matrix $JF(x)$ commutes with its transpose. If $Spec(F) \subset \{z \in \mathbf{C} : \Re(z) < 0\}$, then F is injective.*

2. ON THE WEAK MARKUS-YAMABE CONJECTURE

Main Lemma. Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 map such that $\det(F'(x)) \neq 0$ for all x in $\mathbf{R}^n$. Given $t \in \mathbf{R}$, let $F_t: \mathbf{R}^n \to \mathbf{R}^n$ denote the map $F_t(x) = F(x) - tx$. If there exists a sequence $\{t_m\}$ of real numbers converging to 0 such that every map $F_{t_m}: \mathbf{R}^n \to \mathbf{R}^n$ is injective, then F is injective.

Proof. Choose $x_1, x_2 \in \mathbf{R}^n$ such that $F(x_1) = y = F(x_2)$. We will prove $x_1 = x_2$. By the Inverse Mapping Theorem, we may find neighborhoods U_1, U_2, V of x_1, x_2, y, respectively, such that, for $i = 1, 2$, $F|_{U_i} : U_i \to V$ is a homeomorphism and $U_1 \cap U_2 = \emptyset$. If m is large enough, then $F_{t_m}(U_1) \cap F_{t_m}(U_2)$ will contain a neighborhood W of y. In this way, for all $w \in W$, $\#(F_{t_m}^{-1}(w)) \geq 2$. This contradiction with the assumptions, proves the lemma. ▌

Remark 7. Even if $n = 1$ and the maps F_{tm} in Main Lemma are diffeomorphisms, we cannot conclude that F is a diffeomorphism. For instance, if $F : \mathbf{R} \to (0, 1)$ is an orientation reversing diffeomorphism, then for every $t > 0$, the map $F_t : \mathbf{R} \to \mathbf{R}$ (defined by $F_t(x) = F(x) - tx$) will be an orientation reversing global diffeomorphism.

The following theorem is in the opposite direction from the Smyth and Xavier example [[10], Theorem 4]

THEOREM 1. *Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a local diffeomorphism of class C^1 such that Spec(F) is disjoint of a sequence $\{t_m\}$ of real numbers which converges to 0 as $m \to \infty$. If there exist $R > 0$ and $0 < \alpha < 1$ such that, for all x in $\mathbf{R}^n$ with $|x| > R$, $|F(x)| \leq |x|^\alpha$, then F is injective.*

Proof. Define $F_{t_m} : \mathbf{R}^n \to \mathbf{R}^n$ by $F_{t_m}(x) = F(x) - t_m x$. Since every t_m is not in Spec(F), we have that every F_{t_m} is a local diffeomorphism. By the assumptions, $F_{t_m}(x) \to \infty$ as $x \to \infty$, which implies that F_{t_m} is proper. It follows from Hadamard Theorem that F_{t_m} is injective, for every t_m. Therefore, we conclude from Main Lemma that F is injective. ▌

We shall need the following result ([2], 5.1.5 or [9], Thm. 4.2).

HADAMARD-PLASTOCK THEOREM. *A local C^1 diffeomorphism $F : \mathbf{R}^n \to \mathbf{R}^n$ is bijective if*

$$\int_0^\infty \inf_{|x|=r} \|DF(x)^{-1}\|^{-1} dr = \infty.$$

We now prove the Chamberland Conjecture for Lipschitz maps.

THEOREM 8. *Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 Lipschitz map. Suppose that for some $\epsilon > 0$, Spec$(F) \cap \{z \in \mathbf{C} : |z| \leq \epsilon\} = \emptyset$. Then F is bijective.*

Proof. Let $K > 0$ be such that, for all $x, y \in \mathbf{R}^n$, $|F(x) - F(y)| \leq K|x - y|$. Since F is a C^1 map, for all $x \in \mathbf{R}^n$,

$$\|DF(x)\| := \sup\{|DF(x)v| \ : \ |v| = 1\} \leq K,$$

where $|\cdot|$ is the Euclidean norm of $\mathbf{R}^n$. Let $\|\cdot\|_M$ be the norm, on the space of real matrices $n \times n$, given by

$$\|A\|_M := \sup\{|a_{ij}| \ : \ 1 \leq i, j \leq n\},$$

where $A = \{a_{ij}\}$. As the norms $\|\cdot\|_M$ and $\|\cdot\|$ are equivalent, there exists $K_1 > 0$ such that, for all $x \in \mathbf{R}^n$, $\|DF(x)\|_M \leq K_1$. Therefore, there exists a positive constant $K_2 > 0$ such that the classical adjoint matrix $A(x)$ of $DF(x)$ satisfies, for all $x \in \mathbf{R}^n$,

$$\|A(x)\|_M \leq K_2.$$

By the assumptions on $\mathrm{Spec}(F)$, we have that for all $x \in \mathbf{R}^n$, $|\det(DF(x))| \geq \epsilon^n$. Therefore, for all $x \in \mathbf{R}^n$,

$$\|DF(x)^{-1}\|_M \leq K_3,$$

where $K_3 = K_2/\epsilon^n > 0$ is constant. Again since the norms $\|\cdot\|_M$ and $\|\cdot\|$ are equivalent, there exists $K_4 > 0$ such that for all $x \in \mathbf{R}^n$,

$$\|DF(x)^{-1}\| \leq K_4.$$

This theorem follows after applying Hadamard-Plastock Theorem. ∎

We are grateful to L. A. Campbell and N. Van Chau who, among other helpful comments, let us know that the proof of Theorem 8 could also be applied to the following case:

Theorem 2. *Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 Lipschitz local diffeomorphism. Suppose that for some $\epsilon > 0$,*

$$\liminf_{|x| \to \infty} (|x| \cdot |\det(DF(x))|) > \epsilon.$$

Then F is a global diffeomorphism.

Proof. Before the last statement involving an inequality in the proof of Theorem 8, this proof proceeds the same way. By the current assumptions, such last statement takes this time the following form: there exists $K_4 > 0$ such that, for all $x \in \mathbf{R}^n$ large enough,

$$\|DF(x)^{-1}\| \leq K_4|x|.$$

This theorem follows, again as above, after applying Hadamard-Plastock Theorem. ∎

We now prove that the Weak Markus-Yamabe Conjecture is true for Lipschitz maps.

THEOREM 3. *Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 Lipschitz local diffeomorphism. Suppose that there exists a sequence $\{D_m\}_{m=1}^{\infty}$ of compact discs of $\mathbf{C}$ (with non-empty interior), centered at points t_m of the real axis, such that $\lim_{m\to\infty} t_m = 0$ and*

$$Spec(F) \cap (\cup_{m=1}^{\infty} D_m) = \emptyset.$$

Then, F is injective.

Proof. Let $Id : \mathbf{R}^n \to \mathbf{R}^n$ denote the Identity Map. Then, for every t_m,

$$\text{Spec}(F - t_m Id) \cap (D_m - t_m) = \emptyset,$$

where $D_m - t_m = \{z \in \mathbf{C} : z + t_m \in D_m\}$ is a compact disc centered at 0. Since $F - t_m Id : \mathbf{R}^n \to \mathbf{R}^n$ is a C^1 Lipschitz map, applying Theorem 8, we obtain that, for all t_m, the map $F - t_m Id$ is a (global) diffeomorphism. Main Lemma allow us to conclude that F is injective ▮

COROLLARY 9. *Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 Lipschitz map. Suppose that $Spec(F) \cap \{z \in \mathbf{C} : \Re(z) \geq 0\} = \emptyset$. Then F is injective.*

Remark 10. Corollary 9 is stronger than Vidossich's Theorem 2,a [12] which has the additional assumption that, for some constant $A > 0$ and for all $x \in \mathbf{R}^n$, $Trace(DF(x)) \leq -A$. The proof of Vidossich fails when he claims that the linear map h is invertible ([12], page 972). Nevertheless, his proof still works if the assumptions are strengthened, for instance, to the following one: "for some real constant $A > 0$, $\text{Spec}(F) \cap \{z \in \mathbf{C} : \Re(z) \geq -A\} = \emptyset$."

3. EQUIVALENT STATEMENTS

It is interesting to know if these conjectures are true for other families of maps. We prove the equivalence of conjectures in another class of maps.

Let $\mathcal{L}$ be a subset of the C^1 self-maps of $\mathbf{R}^n$ which contains the identity map Id of $\mathbf{R}^n$ and such that, for all $(F, s, t) \in \mathcal{L} \times \mathbf{R} \times \mathbf{R}$, $sF + tId \in \mathcal{L}$. Let $A > 0$ be a real constant. We state

(1) A-WEAK MARKUS-YAMABE CONJECTURE FOR $\mathcal{L}$: If $F \in \mathcal{L}$ satisfies $\text{Spec}(F) \subset \{z \in \mathbf{C} : \Re(z) < -A\}$, then F is injective.

(2) STRONG CHAMBERLAND CONJECTURE FOR $\mathcal{L}$: If $F \in \mathcal{L}$ is a local diffeomorphism which has the property that there exists a sequence

$\{D_m\}_{m=1}^{\infty}$ of compact discs of $\mathbf{C}$ (not reduced to points), centered at points t_m of the real axis, such that $\lim_{m\to\infty} t_m = 0$ and

$$\mathrm{Spec}(F) \cap (\cup_{m=1}^{\infty} D_m) = \emptyset.$$

then, F is injective.

PROPOSITION 11. *The Weak Markus-Yamabe Conjecture for $\mathcal{L}$ is true if, and only if, A-Weak Markus-Yamabe Conjecture for $\mathcal{L}$ is true.*

Proof. Suppose that the A-Weak Markus-Yamabe Conjecture for $\mathcal{L}$ is true. Let t be any positive real constant. Choose $F \in \mathcal{L}$ such that $\mathrm{Spec}(F) \subset \{z \in \mathbf{C} : \Re(z) < -t\}$. Then $(A/t)F = G \in \mathcal{L}$ and $\mathrm{Spec}(G) \subset \{z \in \mathbf{C} : \Re(z) < -A\}$. G is injective and so F is injective. This implies the t-Weak Markus-Yamabe Conjecture for $\mathcal{L}$. Since for all $(F,t) \in \mathcal{L} \times \mathbf{R}$, $F + tId \in \mathcal{L}$, we apply Main Lemma to show that the Weak Markus Yamabe Conjecture for $\mathcal{L}$ is also true. The converse is obvious. ∎

PROPOSITION 12. *The Strong Chamberland Conjecture for $\mathcal{L}$ and the Chamberland Conjecture for $\mathcal{L}$ are equivalent*

Proof. It is obvious that the Strong Chamberland Conjecture for $\mathcal{L}$ implies the Chamberland Conjecture for $\mathcal{L}$. The proof of the converse follows the lines of the proof of Theorem 3. ∎

It is obvious that the Chamberland Conjecture for $\mathcal{L}$ implies the A-Weak Markus-Yamabe Conjecture for $\mathcal{L}$. Therefore,

COROLLARY 13. *The Chamberland Conjecture for $\mathcal{L}$ implies the Weak Markus-Yamabe Conjecture for $\mathcal{L}$.*

Let $\mathcal{L}_1$ be the set of C^1 maps $F: \mathbf{R}^n \to \mathbf{R}^n$ such that if $n \geq 2$, there exist $K > 0$ and a compact subset of $\mathbf{R}^n$ outside of which

$$||DF(x)|| \leq K|x|^{\frac{1}{n-1}}.$$

THEOREM 5. *Let $F \in \mathcal{L}_1$.*

(a) If there is $\epsilon > 0$ such that $|\det(DF(x))| > \epsilon$, for all $x \in \mathbf{R}^n$, then F is a global diffeomorphism.
(b) If Spec$(F) \subset \{z \in \mathbf{C} : \Re(z) < 0\}$, then F is injective.

Proof. Item (a) follows by using the same argument as that of the proof of Theorem 2. Item (b) follows immediately from item (a) and Corollary 13 applied to the set $\mathcal{L}_1$. ∎

Observe that, by the Mean Value Theorem, $F: \mathbf{R}^n \to \mathbf{R}^n$ is a C^1 Lipschitz map if, and only if, there is a constant $K > 0$ such that, for all $x \in \mathbf{R}^n$, $||DF(x)|| \leq K$. Therefore, Theorem 5 is stronger than Corollary 9.

Now we state a result which follows at once from [[4], Theorem 3.2] and Corollary 13 applied to the set $\mathcal{L}_2$ of the C^1 maps $F : \mathbf{R}^n \to \mathbf{R}^n$ such that there is a compact subset of $\mathbf{R}^n$ outside of which the Jacobian matrix $JF(x)$ commutes with its transpose

THEOREM 14. *If* $F \in \mathcal{L}_2$ *and* $Spec(F) \subset \{z \in \mathbf{C} : \Re(z) < 0\}$, *then* F *is injective.*

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