QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **4**, 311–327 (2004)
ARTICLE NO. 65

Vector Fields on Manifolds with Boundary and Reversibility - An Expository Account -*

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Submitted: March 2, 2003 *Accepted*: March 9, 2004

In this paper an expository account on singularities of reversible vector fields on manifolds and boundary singularities is presented. Also we present the bifurcation diagram of a boundary cusp of codimension three, i.e, a Bogdanov-Takens singular point in the boundary of the semi plane $\{(x, y) \in \mathbf{R}^2 : x \geq 0\}$ whose topological unfolding is given by the quadratic three parameter family $y\frac{\partial}{\partial x} + (x^2 + ax + c + \alpha y(x + b))\frac{\partial}{\partial y}$, $\alpha = \pm 1$. This study can be applied to the analysis of the behavior of singularity of the germ of vector field $X_0(x, y) = (y, 2x(x^4 + x^2 y))$ in the class of reversible vector fields.

Key Words: reversible vector field, boundary singularity, cusp of codimension three

1. INTRODUCTION

The geometric-qualitative study of flows and general dynamical systems on surfaces has been during many decades object of a growing interest

* Dedicated to Prof. Jorge Sotomayor on the occasion of his 60th birthday.

in many branches of pure and applied mathematics. After the works of Poincaré, Lyapunov and Bendixson this has become a well-established subject in mathematics and focus of considerable attention. Moreover, nowadays it is fairly accessible for a broad scientific audience. From various sides, attention has been paid to the structural stability concept and specially to the results of Peixoto (mainly those published in An. Ac. Bras. Sci , 1959 and Topology, 1962) and higher dimensional extensions (due mainly to Smale and Anosov).

We present an elementary discussion of two aspects: classification problems arising for vector fields defined in manifolds with boundary and reversible systems.

The main point treated here concerns the contact between a general vector field and the boundary of a manifold. More specifically, a tangency point between the vector field and the boundary is a distinguished singularity- an important object to be analyzed when one studies reversible systems. We still point out that there is a natural mathematical approach to studying such a phenomenon by means of singularity mappings theory; see for instance [27], [31], [33].

1.1. Historical remarks

A brief historical outline of the qualitative study of bidimensional systems follows:

In 1881 H. Poincaré performed the qualitative study of singular points in the plane (focus, node and saddle), [18].

In 1937 A. Andronov and S. Pontrjagin [1] announced the characterization of the structural stability of a class of vector fields defined on a compact region in the plane.

In 1959 , M. C. Peixoto and M. M. Peixoto [20] generalized this result to a larger class of systems, still defined on a planar region homeomorphic to a disk.

In 1962 this last theorem was extended by Peixoto [21] to flows on 2-dimensional manifolds.

In 1964 J. Sotomayor obtained results of first order structural stability of vector fields on surfaces (saddle node, homoclinic loop, connections of separatrices of hyperbolic saddle points, Hopf points, etc.), see [28, 29, 30] and also the classical book by Andronov at. al. [3].

In 1973 results on classification of flows, without periodic orbits, on orientable surfaces, was obtained by M. Peixoto in [22]. See also [19], [17].

A bibliographical guide of this matter is contained in many works (for example in [2] or in [19]).

1.2. Structural Stability

Andronov and Pontrjagin in [1] introduced the concept of structural stability (via C^0-orbital equivalence) for C^r $(r \geq 1)$ planar vector fields X defined in a neighborhood of a compact region $\mathbf{M}$ in $\mathbf{R}^2$ bounded by a smooth regular curve $\partial\, \mathbf{M}$. They considered the set χ^* of all such vector fields which are transverse to $\partial\, \mathbf{M}$. A general concept of structural stability is the following.

Definition 1. A vector field X in a topological space $\mathcal{X}$ is said to be *structurally stable along a set* $\mathcal{C} \subset \mathcal{X}$ (respectively *relative to* $\mathcal{C}$) if there is a neighborhood $\mathcal{V}$ of X in $\mathcal{X}$, such that, for every Y in the connected component of X in $\mathcal{C}$ (respectively in $\mathcal{V} \cap \mathcal{C}$), there is a homeomorphism $h_Y : \mathbf{M} \to \mathbf{M}$ mapping arcs of orbits of Y on those of X, preserving their orientation.

The following result was stated:

Theorem 2. *A vector field $X \in \chi^*$ is structurally stable in χ^* if, and only if,*

i)all its singular points are hyperbolic,

ii)all its periodic orbits γ are hyperbolic,

iii)there is no connection between separatrices of hyperbolic saddle points.

Remark 3. We recall that a singular point p is hyperbolic if $DX(p)$ has no complex eigenvalues of the form ia, $a \in \mathbf{R}$ and that a periodic orbit is hyperbolic if $\int_\gamma div(X(\gamma(t))dt \neq 0$, or equivalently, $\int_\gamma k^\perp(\gamma(s))ds \neq 0$, where $k^\perp$ denotes the curvature of the integral solutions of $X^\perp$ restricted to γ and ds is arc length of γ.

In 1959, M. Peixoto and M. A. Peixoto (in [20]) generalized Theorem 2, by considering the set χ of all C^r $(r \geq 1)$ vector fields X defined in a neighborhood of a compact region with a Jordan boundary curve $\partial\, \mathbf{M}$ of class C^1 in $\mathbf{R}^2$. The characterization of stable vector fields is given by the following theorem. See also [29].

Theorem 4. *A vector field $X \in \chi$ is structurally stable in χ if and only if,*

i)all its singular points and periodic orbits are hyperbolic;

ii)there is no connection of separatrices of hyperbolic saddles;

iii)all singular points and periodic orbits are in the interior of $\partial\, \mathbf{M}$;

iv)any trajectory of X has at most one point of tangency with $\partial\, \mathbf{M}$ and this contact is quadratic;

v)any saddle separatrix is transverse to $\partial\, \mathbf{M}$;

Moreover the set of structurally stable systems is open and dense in the space χ with C^r-topology of Whitney.

Let $\mathbf{M}$ be a 2-dimensional manifold and $\chi = \chi^r(\mathbf{M})$ be the space of all C^r- vector fields on M with the C^r-topology. We denote by $\Sigma_0 = \Sigma_0(\mathbf{M})$ the set of all structurally stable vector fields in χ. For simplicity, we may call any element of Σ_0 a codimension-zero vector field of χ. When, throughout this work, the treatment is local we use the germ terminology.
When $\mathcal{C} = \mathcal{X}$, X is said *structurally stable* in $\mathcal{X}$.

In 1962, M. Peixoto (in [21]) proved the following result:

Theorem 5. *Let $X \in \chi^r(\mathbf{M})$ $(r \geq 1)$ where $\mathbf{M}$ is a compact orientable surface or compact non-orientable surface of genus $1 \leq g \leq 3$. Then $X \in \Sigma_0(\mathbf{M})$ if and only if it is a Morse-Smale vector field. Moreover $\Sigma_0(\mathbf{M})$ is open and dense in χ.*

2. VECTOR FIELDS IN SURFACES WITH BOUNDARY: LOCAL ASPECTS

In this section we discuss some results concerning the problem of classification of dynamical systems defined on manifolds with boundary under C^0-orbital equivalence. We also recall that, tools in singularity of mappings are fundamental in this approach.
We present here the terminology, concepts and some results introduced in [33].

2.1. Structural stability in manifolds with boundary

For simplicity we assume in this subsection that there exists $h: \mathbf{M} \to \mathbf{R}$ a C^∞ function having 0 as regular value with $S = h^{-1}(0)$ and $h(q) \geq 0$ for all q in $\mathbf{M}$.
Let $X \in \chi$ be as above. Call $S = \partial\, \mathbf{M}$.

Definition 6. We say that $p \in S$ is an $S-$singularity of X if either $X(p) = 0$ or $X(p) \neq 0$ and $Xh(p) = 0$.

Definition 7. We say that $p \in S$ is a fold singularity of X if $X(p) \neq 0$, $Dh(p)(X(p)) = Xh(p) = 0$ and $X(Xh(p)) \neq 0$. In this case we say that the contact between the orbit of X and S at p is quadratic.

Definition 8. Let $X \in \chi_1(p)$. We say that $p \in S$ is a cusp singularity of X if $X(p) \neq 0$, $Xh(p) = XXh(p) = 0$ and $XXXh(p) \neq 0$.

A *separatrix* of X is an orbit which connects either: a) two saddle singular points; b) two tangency points between the vector field and S; or c) a tangency point and a saddle singular point. Any equivalence between two vector fields in χ must preserve such objects.

In [33] this result was generalized for 2-manifolds with boundary in which the techniques and results of Theorem 4 were fundamental. These two results are summarized as follows:

THEOREM 9. *Call* $\chi_1^r = \chi^r \setminus \Sigma_0$ $(r \geq 3)$ *the bifurcation set of* χ. *There exists a* C^{r-1}*-immersed codimension-one submanifold* Σ_1 *of* χ *such that:*

i)Σ_1 *is dense in* χ_1;
ii)for any X *in* Σ_1, *there exists a neighborhood* B *in the intrinsic topology of* Σ_1 *such that any* Y *in* B *is* $C^0 -$ *equivalent to* X;
iii)Σ_1^r, *as well as the part of* Σ_1^r *embedded in* χ *are characterized.*

Following the last theorem we may of course classify the stable one-parameter families X_λ of vector fields in χ by means of the concept of transversality. It is usual to say that X_λ presents a codimension-one bifurcation at $\lambda = 0$ if $X_0 \in \Sigma_1$. This research program attempts the classification of the codimension - k bifurcations in $\chi^r(\mathbf{M})$. It should be mentioned that [34] contains results concerning codimension 2 bifurcations of singularities of vector fields defined on manifolds with boundary. Again the main ideas and techniques come from the former results of M. Peixoto [21] and J. Sotomayor [28].

3. GENERIC BIFURCATION IN MANIFOLDS WITH BOUNDARY (LOCAL SETTING)

In this section we comment briefly the boundary codimension-one singularities.

Let $p \in S$ and $\chi(p)$ be the space of all germs of C^r-vector fields at p. The sets $\Sigma_0(p)$ and $\chi_1(p)$ are defined as above.

DEFINITION 10. A codimension-one $S-$singularity of X is either a *cusp* singularity or an $S-$hyperbolic singular point p in S of the vector field X. In the second case this means that p is a hyperbolic singular point of X with invariant manifolds (stable, unstable and strong stable and strong unstable) transversal to S. The set of elements $X \in \chi_1(p)$ such that p is an $S-$singularity of X will be denoted by $\Sigma_1(p)$.

Remark 11. Given $X \in \chi_1(p)$, the following orbits have to be distinguished: a) an invariant manifold of a saddle singular point $p \in S$; b) a strong invariant manifold of a nodal singular point $p \in S$; c) an orbit of X tangent to S at p. Any C^0 equivalence between two elements of χ must necessarily preserve such objects. We may refer to them as $S-$*separatrices* of X.

The next result was proved in [33].

Proposition 12. *Let $X \in \chi_1(p)$ and $p \in S$. The vector field X is structurally stable (at $p \in S$) relative to $\chi_1(p)$ if and only if $X \in \Sigma_1(p)$. Moreover, $\Sigma_1(p)$ is an embedded codimension-one sub manifold and dense in $\chi_1(p)$.*

The following result is also in [33].

Proposition 13. *(**Normal forms**) **(1)** $X \in \Sigma_0(p)$ if and only if X is equivalent to one of the following normal forms:*

- $X(x, y) = (0, 1)$ *(regular case);*
- $X(x, y) = (1, \delta x)$ *with* $\delta = \pm 1$ *(fold singularity).*

(2) *Any one-parameter family X_λ, ($\lambda \in (-\varepsilon, \varepsilon)$) in χ transverse to $\Sigma_1(p)$ at X_0, has one of the following normal forms:*

- $X_\lambda(x, y) = (1, \lambda + x^2)$ *(cusp singularity);*
- $X_\lambda(x, y) = (ax, x + by + \lambda), a = \pm 1, b = \pm 2$;
- $X_\lambda(x, y) = (x, x - y + \lambda)$;
- $X_\lambda(x, y) = (x + y, -x + y + \lambda)$.

Remark 14. The result above has the correspondent for codimension two boundary singularities. Some of codimension phenomenas are: quartic tangency of a orbit with the boundary, a saddle-node, a Hopf point, a hyperbolic saddle with a separatrix having a quadratic contact with the boundary, a node in the boundary with the strong separatrix having quadratic contact with the boundary.

The local aspects of codimension one and two boundary singularities was studied by Teixeira [33], [34]. Sotomayor in [27], [29] and [30] studied the geometric structure of the systems having boundary singularities of codimension one.

4. REVERSIBLE VECTOR FIELDS

It is generally acknowledged that time-reversal symmetry is one of the fundamental symmetries discussed in many branches of physics. Time-reversible systems share many properties with Hamiltonian systems. In [15] an interesting survey on reversibility in dynamical systems is presented. See also [26].

4.1. Basic concepts and definitions

Let $\mathbf{M}$ be a C^∞ compact orientable two-dimensional manifold and $h : \mathbf{M} \to \mathbf{R}$ be a C^∞ function having 0 as regular value. Call $S = h^{-1}(0)$, $\mathbf{M}^+ = h^{-1}([0,\infty))$, $\mathbf{M}^- = h^{-1}((-\infty, 0])$.

Let $\varphi : \mathbf{M} \to \mathbf{M}$ be a C^∞ diffeomorphism (an involution) from $\mathbf{M}$ onto $\mathbf{M}$ such that $\varphi \circ \varphi = Id$ (φ is an involution) and $Fix(\varphi) = S$.

We say that a vector field X on $\mathbf{M}$ is $\varphi-$ *reversible* (or simply reversible) if

$$D\varphi(p)X(p) = -X(\varphi(p)).$$

Let Ω_R be the space of the C^r $\varphi-$reversible vector fields on $\mathbf{M}$ endowed with the C^r-topology ($r > 2$).

Any singular point of $X \in \Omega_R$ contained in S is called a *symmetric singularity* of the vector field X. Otherwise the singularity is called *asymmetric.*

We shall deal with those involutions which are germs of C^∞ diffeomorphisms (at 0) $\varphi : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$, satisfying $\varphi \circ \varphi = \mathrm{Id}$ and $\det(D\varphi(0)) = -1$. The set $S = \mathrm{Fix}(\varphi)$ is a smooth curve in $(\mathbf{R}^2, 0)$. It is well known (Montgomery-Bochner Theorem in [16]) that such an involution is C^∞ conjugated to $\varphi(x, y) = (-x, y)$ or to $\phi(x, y) = (x, -y)$

Let X be a (germ of) C^∞ vector field on $(\mathbf{R}^2, 0)$ and φ be an involution.

We fix coordinates in $(\mathbf{R}^2, 0)$ in such a way that $\varphi(x, y) = (-x, y)$ and denote by Ω_R the set of all φ-reversible (or just reversible) vector fields on $(\mathbf{R}^2, 0)$. In these coordinates we have that $S = \{x = 0\}$. In the case that we consider ϕ-reversible vector fields the fixed set of the involution is $S = \{y = 0\}$.

Endow Ω_R with the C^r-topology with $r > 3$.

Any singular point of $X \in \Omega_R$ on S (fixed set of φ) is called a *symmetric singularity* (or simply singularity) of X; otherwise it is called an asymmetric singularity.

4.2. Relation between reversibility, quadratic boundary tangency and stable singularities

The object of this subsection is to illustrate the connection between vector fields defined in manifolds with boundary and reversible systems.

Let X be in Ω_R. In the coordinates (x, y) given above, it follows from Division Theorem that X has the following general form:

$$X(x, y) = (\tfrac{1}{2} f(x^2, y), x g(x^2, y))$$

In the half-plane $x > 0$, consider

$$u = x^2 \quad \text{and} \quad v = y.$$

A simple calculation shows that X is transformed into:

$$X_1(u,v) = (\sqrt{v}f(u,v), \sqrt{v}g(u,v)) \quad \text{in} \quad u > 0.$$

It follows that in $x > 0$, the vector field X is topologically equivalent to

$$Y(u,v) = (f(u,v), g(u,v)) \quad \text{for} \quad u > 0.$$

Observe now that Y can be C^r extended to a full neighborhood of 0. Due to the symmetry properties of X (with respect to the canonical involution) we deduce that the behavior of Y at $(\mathbf{M}, 0)$ determines completely the behavior of X at 0. So the problem now is carried out to analyze the phase portrait of Y in $\mathbf{M}$.

At a regular point the trajectory of X is always orthogonal to S. At a singular point of X, the contact between an invariant manifold and S decays by a factor of $1/2$ in comparison with the orbit or invariant manifold of Y passing through the same point. We illustrate this fact by assuming that $\{u = v^k, k > 0\}$ is an invariant manifold of Y on the region $u \geq 0$. This implies that the curve $x = y^{k/2}$ is a an invariant manifold of X on $x \geq 0$.

The following proposition characterizes all stable symmetric singularities of X in Ω_R. See [33].

Proposition 15. *Let $X \in \Omega_R$ be such that $X(0) = 0$ and 0 is a quadratic tangency between Y and the line $\{x = 0\}$. Then the origin is either a saddle singular point (in the case $Yh(0) = Dh(0)Y(0) = 0,\ Y(Yh)(0) > 0$) or a simple center singular point (in the case $Yh(0) = Dh(0)Y(0) = 0,\ Y(Yh)(0) < 0$).*

Remark 16. The techniques used in this the proposition can be used to study codimension-k symmetric singularities in Ω_R. See [35].

4.3. Global stability of reversible vector fields on surfaces

The main result in [36] has a close connection with the results in [4], [21], [20], [27] and [33]. It says that:

Theorem 17. *The set Σ_0 of all vector fields in $\mathbf{M}$ which are structurally stable is open and dense in Ω_R. Moreover $X \in \Sigma_0$ if and only if the following conditions are satisfied:*

i)all asymmetric singular points of X are hyperbolic;
ii)all asymmetric periodic orbits of X are hyperbolic;
iii)X has no saddle connection on $\mathbf{M}^+$ or in $\mathbf{M}^-$;;
iv)all symmetric singularities of X are hyperbolic saddles and simple centers.

v) X does not have nontrivial recurrent orbits;

Call $\Omega_1 = \Omega_R \setminus \Sigma_0$ the bifurcation set of Ω_R. There exists a C^{r-1} immersed codimension-one submanifold Σ_1 of Ω_R such that:

i)Σ_1 is dense in Ω_1 (both with the relative topology);

ii)for any X in Σ_1, there exists a neighborhood B_1 in the intrinsic topology of Σ_1 such that any Y in B_1 is topologically equivalent to X, i. e., Y is structurally stable along to Σ_1;

iii)a part of Σ_1 embedded in Ω_R is also characterized.

In the class of reversible vector fields some persistent phenomena occur which cannot be destroyed by perturbations in Ω_R. Examples are periodic orbits and saddle connections which meet the submanifold S. However, concerning non trivial recurrences its are contained in $\mathbf{M}^+$ and $\varphi(\mathbf{M}^+) = \mathbf{M}^-$. We mention for example that reversible systems on the torus do not admit an irrational flow since $S = Fix(\varphi)$ has two connected components and therefore $\mathbf{M}^+$ is homeomorphic to a cylinder. In the bitorus, $S = Fix(\varphi)$ has one or three connected components. If S has three connected components it follows that $\mathbf{M}^+$ has topological type of a planar region and so no nontrivial recurrence is prohibited. In the other hand, if S has only one connected component $\mathbf{M}^+$ has topological type of torus minus a disk and so taking a Cherry flow on the torus, [19], we can construct a smooth reversible vector field on the bitorus with nontrivial recurrences.

4.4. Local bifurcations of codimension one and two of reversible vector fields

Let Ω_R be the space of the germs of C^r reversible vector fields at 0 on $\mathbf{R}^2$ endowed with the C^r topology, $r > 3$.

In [35] all the symmetric singularities of codimension 0, 1 and 2 are classified. It is presented a technique which enables to classify in a simple manner those singularities. It consists to make a change of coordinates as describe in subsection 4.2 around the point and address the analysis of a boundary singularity. In the theorem below the normal forms of X are in relation to the involution $\phi(x, y) = (x, -y)$.

Theorem 18. **(a)** *The normal forms of codimension 0 singularities in Ω_R are:*

i) $X_0(x, y) = (0, \frac{1}{2})$,

ii) $X_1(x, y) = (y, \frac{1}{2}x)$ *and*

ii) $X_2(x, y) = (-y, \frac{1}{2}x)$.

(b) *(codimension one singularity classification) - In the space of one-parameter families of vector fields in Ω_R, an everywhere dense set is formed by generic families such that their (C^0-) normal forms are:*

i) *The codimension* 0 *normal forms in* Ω_R*;*
ii) $X_\lambda(x,y) = (y, \frac{1}{2}(\lambda + x^2))$*;*
iii) $X_\lambda(x,y) = (\varepsilon xy, \frac{1}{2}(2\varepsilon y^2 + x + \lambda))$ *with* $\varepsilon = \pm 1$*;*
iv) $X_\lambda(x,y) = (xy, \frac{1}{2}(\lambda - y^2 + x))$*;*
v) $X_\lambda(x,y) = (xy + y^3, \frac{1}{2}(\lambda - x + y^2))$.

(c) *(codimension two singularity classification) - In the space of two-parameter families of vector fields in Ω_R, an everywhere dense set is formed by generic families such that their C^0 - normal forms are:*

i) *All the normal forms listed in* **(a)** *and* **(b)** *above.*
ii) $X_{\alpha\beta}(x,y) = (y, \frac{1}{2}(bx^3 + \beta x + \alpha))$, $b = \pm 1$*;*
iii) *a)* $X_{\alpha\beta}(x,y) = (ay(x - y^2) + \beta y(x + y^2), \frac{1}{2}(\alpha + (x + y^2)^2))$ *with* $a = \pm 1$*;*

b) $X_{\alpha\beta}(x,y) = (y(x - y^2) + \beta y(x + y^2), \frac{1}{2}(\alpha + a(x + y^2)^2))$ *with* $a = \pm 1$*;*
iv) $X_{\alpha\beta}(x,y) = (-y^3 + axy(\alpha + x^2 + y^4), \frac{1}{2}(ax - y^2(\alpha + x^2 + y^4) + \beta)$ *with* $a = \pm 1$*;*
v) $X_{\alpha\beta}(x,y) = (axy + \alpha y^3, \frac{1}{2}(x + ay^2 + \beta))$, *with* $a = \pm 1$*;*
vi) $X_{\alpha\beta}(x,y) = (axy, \frac{1}{2}(\alpha x + by^2 + \epsilon x^2 + \beta))$, *with* $ab > 0$, $\epsilon = \pm 3$, $|a| = 1$ *and* $|b| = 3$ *or* $|a| = 3$ *and* $|b| = 1$.

5. A CODIMENSION THREE SINGULAR POINT ON THE BOUNDARY

Our starting point is to consider germs of vector fields on $\mathbf{M} = \{(x,y) \in \mathbf{R}^2 : \ x \geq 0\}$ at $0 \in \partial\, \mathbf{M}$ with nilpotent $1 - jet\ y\partial/\partial x$ and $2 - jet\ C^\infty$ conjugated to:

$$X_0(x,y) = y\frac{\partial}{\partial x} + (x^2 + \alpha xy)\frac{\partial}{\partial y}, \quad \alpha = \pm 1 \tag{1}$$

This codimension two singularity on the plane is known as Takens-Bogdanov, see [5], [6], [8],[25] for the analysis of this point.

For the study of codimension three singularities on the plane see [9] and [10] and also [8].

The following 3-parameter family of vector fields defined in **M**

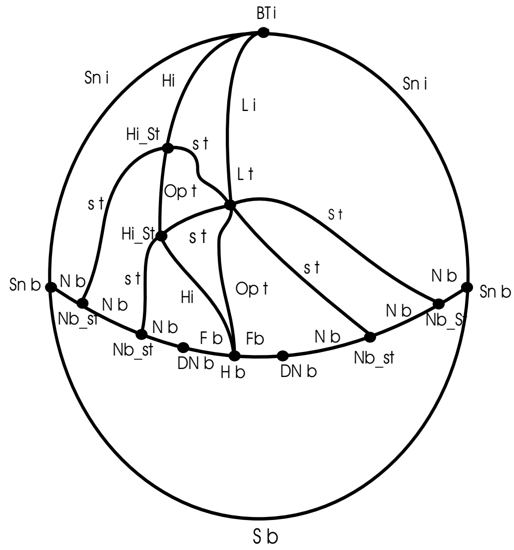
$$y\frac{\partial}{\partial x} + (x^2 + ax + c + \alpha y(x + b))\frac{\partial}{\partial y}, \quad \alpha = \pm 1 \tag{2}$$

generically unfolds X_0; it will be called a *quadratic typical family.*

We state now the main results from [11].

THEOREM 19. *The bifurcation diagram of the quadratic typical family given by equation 2 with* $\alpha = \pm 1$ *is as shown in Fig. 1 below. This diagram is a topological cone with vertex at* 0 *and there are eleven distinct phase portraits which are structurally stable (open regions of the diagram of bifurcation) and thirteen points of codimension two. The lines in the diagram are the bifurcations of codimension one. The phase portrait in each open connected region of the diagram is as shown in Fig. 2 , 3, 4 and 5.*

Remark 20. In Fig. 1 is represented the restriction of the bifurcation set to a hemisphere of $a^2 + b^2 + c^2 = 1$. We observe that outside a topological disk the family has no singular point. In fact there exists a unique non transverse contact between the vector field and the boundary which is an internal quadratic tangency.



BT i - Bogdanov-Takens bifurcation in the interior

DN b- degenerate node in the boundary

Li - interior loop of cod. 1

Lp - loop tangent to the boundary

Op t - hyperbolic periodic orbit tangent to the boundary

St - separatrix of saddle tangent to the boundary

Nb-st - node in the boundary and connectionof strong separatrix of node and saddle separatrix

Hi- interior Hopf point of cod. 1

Sni - interior saddle node of cod. 1

Sb - saddle in the boundary

Fb - focus in the boundary

Nb - node in the boundary

Hb - Hopf in the boundary

Snb- saddle node in the boundary

Hi-st - interior Hopf and separatrix tangent to the boundary

FIG. 1. Bifurcation diagram of the quadratic typical family

Also we consider the following singularity in the reversible context.

$$Y_0(x, y) = (y, 2x(x^4 + \alpha x^2 y)), \quad \alpha = \pm 1. \tag{3}$$

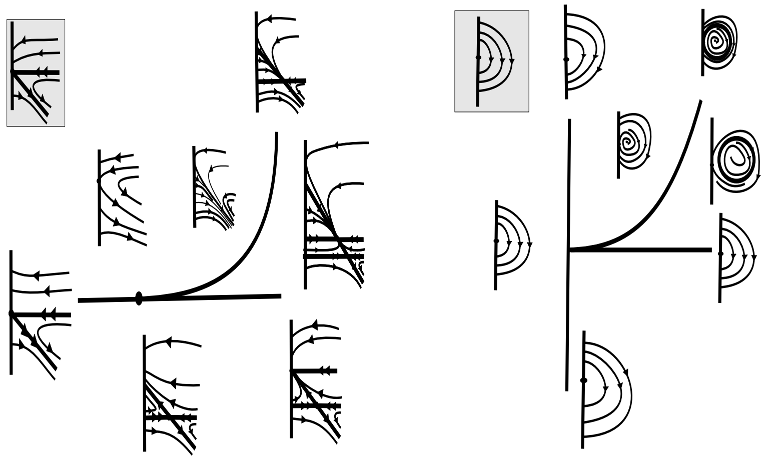


FIG. 2. Bifurcation diagrams of a internal saddle node and of a Hopf singular point in the boundary

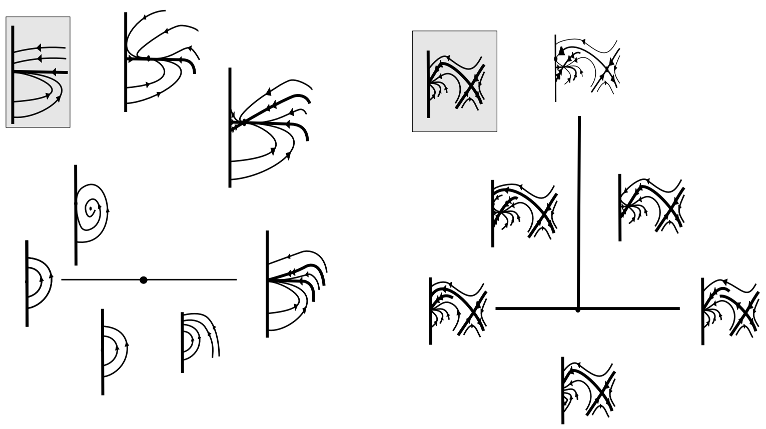


FIG. 3. Bifurcation diagrams of a degenerate hyperbolic node in the boundary and of a node in the boundary with a connection between the separatrices

In the reversible case this non-hyperbolic saddle singularity has codimension 3, whereas in the world of smooth vector fields it has codimension greater than 5.

A 3-parameter family of vector fields defined in $\mathbf{R}^2$ expressed by

$$y\frac{\partial}{\partial x} + 2x[(x^4 + \alpha x^2 y) + ax^2 + by + c]\frac{\partial}{\partial y}. \qquad (4)$$

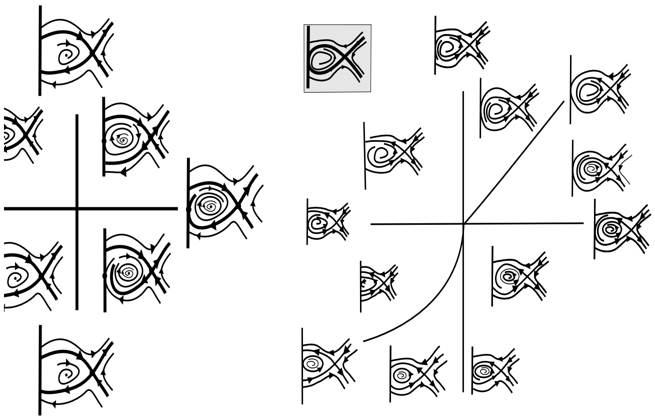


FIG. 4. Bifurcation diagrams of a interior Hopf point and a tangent separatrix of saddle and of a tangent loop of a hyperbolic saddle point

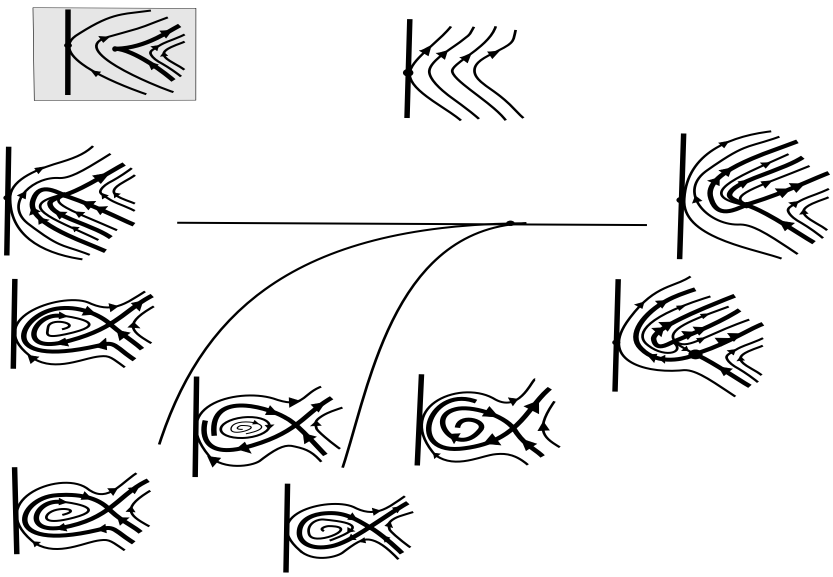


FIG. 5. Bifurcation diagram of a interior Takens-Bogdanov point.

generically unfolds Y_0; it will be called a *reversible polynomial typical family.*

THEOREM 21. *The bifurcation diagram of the reversible typical family given by equation 4 in the parameter space* (a, b, c) *is homeomorphic to that of Fig. 1.*

6. CONCLUDING REMARKS

Bidimensional differential equations is an important source of problems and applications in various areas of science.

Concerning reversible systems on the plane and surfaces, or even higher dimensional manifolds, we can state the following problems:

i) *Period function.* Given a center of a reversible system which are the properties of the associated period function?

The period function has been extensively studied in the class of Hamiltonian systems. In particular, a classification of isochronous centers of polynomial systems is still an open problem.

ii) *Hilbert-Arnold problem.* There exists an uniform bound for the maximum number of limit cycles and cylinders of periodic orbits for analytic reversible systems on surfaces? See [4], [24] and references therein.

iii) *Closing-lemma and recurrences.* For reversible systems on compact surfaces the recurrences can be eliminated by small perturbation (C^r, $r \geq 1$)? See [19] , [22], [23] and [12].

iv) *Second order differential equations.* The second order differential equation $x'' = f(x, x')$, $x \in \mathbf{R}^k$ such that $f(x, x') = f(x, -x')$ defines a reversible vector field $X(x, y) = (y, f(x, y))$ in $\mathbf{R}^k \times \mathbf{R}^k$ with the canonical involution $\varphi(x, y) = (x, -y)$. For $k = 1$ we can ask about the classification of the phase portrait of X and, in particular, the study of problems i) and ii) above. For $k \geq 2$ we can ask if the study of this class have natural applications in problems of differential geometry, for example, in the study of a geodesic flow on manifolds.

7. ACKNOWLEDGEMENTS

The authors are fellows of CNPq. The first author was partially supported by CNPq Grant 476886/2001-5 and FUNAPE/UFG.

This work was done under the project PRONEX/CNPq/MCT - Grant Number 66.2249/1997-6 - Teoria Qualitativa das Equações Diferenciais Ordinárias.

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