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$n-1$ Independent First Integrals for Linear Differential Systems in $\mathbf{R}^n$ and $\mathbf{C}^n$

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We prove that every linear system with constant coefficients on $\mathbf{R}^n$ or $\mathbf{C}^n$ is Darboux integrable by providing a complete explicit list of $n-1$ independent Darboux first integrals.

Key Words: First integral, Darboux integrability, invariant surface

1. INTRODUCTION

Integrability is a topic of great interest in the study of differential equations. We take this notion to mean that a system of differential equations has a sufficient number of independent first integrals belonging to some specified class. But, we note that the term "integrable" is used in several other ways (see [6] and [14]). The purpose of this paper is to show that every n-dimensional linear system with constant coefficients is Darboux integrable in the sense that $n-1$ independent explicit Darboux integrals can be constructed.

Darboux integrals have been studied extensively for polynomial differential systems. Darboux constructed first integrals for planar polynomial systems that have sufficiently many invariant algebraic curves (see [5]). Subsequently, Darboux's methods were generalized in various directions (see [2],[3],[4],[7],[11]).

The integrability of linear systems has been studied by several authors. The existence of a common integral for two coupled linear systems is discussed in [9], where coupled linear systems of dimension three (which satisfy the Frobenius compatibility condition) are proved to have a common first integral. First integrals are constructed using integrating factors in [12]. Rational first integrals are obtained for linear systems in [13] using the Darboux theory related to the existence of sufficiently many invariant curves. The class of linear systems with no rational first integrals is determined in [10].

The existence of first integrals can be important in applications. For example, a quadratic first integral is used in [1] for the stability analysis of a linearized Hamiltonian system with two degrees of freedom that models certain gyroscopic motions. We also mention that the preservation of first integrals for certain linear systems by Runge-Kutta integration is investigated in [8].

The precise statements of our results are given in Section 2; the proofs are given in Section 3. It is perhaps surprising to compare the simplicity of the general solution of a linear system with the complexity of its first integrals, which are listed in Section 2. In this paper we will consider real linear systems. But, as we point out in Section 3, similar results for complex linear systems can be obtained with minor modifications. The special case of Hamiltonian linear systems will be considered in a future publication.

2. MAIN RESULTS

Before to state the main results of this work, we need to introduce the following functions.

$$\begin{aligned} H_1(x) &= x_1, \\ H_l(x) &= \sum_{j=1}^{l} (-1)^{j+1} x_j x_{l+1-j}, \text{ with } l \geq 3 \text{ odd}, \\ H_l(x) &= x_1^{l-2} x_l - \frac{1}{(l-1)!} x_2^{l-1} + \sum_{j=2}^{l-1} \frac{(-1)^j}{(l-j)!} x_1^{j-2} x_2^{l-j} x_j, \ l \geq 4, \text{even.} \end{aligned} \tag{1}$$

Notice that H_l only depends on $x_1, \cdots, x_l$. If we define σ_l, for $l \neq 2$, as

$$\sigma_l = \begin{cases} 1 & \text{if } l = 1, \\ 2 & \text{if } l > 1 \text{ is odd}, \\ l-1 & \text{if } l \text{ is even}, \end{cases} \tag{2}$$

then H_l is a homogeneous polynomial of degree σ_l.

We consider the linear differential system

$$\dot{x} = Ax, \tag{3}$$

where A is a $n-$dimensional real square matrix. In Theorems 1 and 2 it is assumed that the matrix A has been reduced to its real canonical Jordan form and that all its eigenvalues are real. Therefore, the matrix A has α real eigenvalues λ_τ which correspond to diagonal blocks R_τ of the form

$$R_\tau = \begin{pmatrix} \lambda_\tau & 0 & \cdots & 0 \\ 0 & \lambda_\tau & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_\tau \end{pmatrix}, \tag{4}$$

and it has β real eigenvalues μ_j which correspond to blocks S_j of the form

$$S_j = \begin{pmatrix} \mu_j & 0 & . & . & . & 0 \\ 1 & \mu_j & 0 & . & . & . \\ 0 & 1 & . & . & . & . \\ . & 0 & . & . & . & . \\ . & . & . & . & \mu_j & 0 \\ 0 & . & . & 0 & 1 & \mu_j \end{pmatrix}. \tag{5}$$

If all the eigenvalues of A^t are real, then the space $\mathbb{R}^n$ can be decomposed as

$$\mathbb{R}^n = \left(\bigoplus_{\tau=1}^{\alpha} V_\tau \right) \bigoplus \left(\bigoplus_{j=1}^{\beta} U_\tau \right), \tag{6}$$

in such a way that $A\mid_{V_\tau} = R_\tau$ and $A\mid_{U_j} = S_j$. Moreover, the subspaces V_τ and U_τ are invariant under the linear application determined by A. We denote by r_τ and s_j the dimensions of the linear spaces V_τ and U_j, respectively. We define d_r and d_s by $d_r = r_1 + \cdots + r_\alpha$ and $d_s = s_1 + \cdots + s_\beta$. So, we have the equality $n = d_r + d_s$. Also, we will use the following notation: $Y\pi^j y$ is the projection onto the subspace Y_j of $Y = \bigoplus Y_j$ of $y \in Y$, and $Y\pi_l^j y$ is the projection of $Y\pi^j y$ onto the l coordinate of the subspace Y_j. The transpose of matrix A is denoted by A^t.

THEOREM 1. *Suppose that the matrix A of system (3) has all its eigenvalues real and that it has been reduced to its real canonical Jordan form and $\alpha > 0$. Let $a \in V_1$ be an eigenvector of A^t with eigenvalue λ_1. The functions H_l are defined as in (1). Then, system (3) has $n-1$ independent first integrals which can be chosen in the following way.*

(a)There are $r_1 - 1$ first integrals of the form

$$F_1 = \langle a, x\rangle \, / \, \langle b, x\rangle \,, \tag{7}$$

with $b \neq a$ an eigenvector of A^t in V_1.

(b)For each $k = 2, \cdots, \alpha$, there are r_k first integrals of the form

$$F_2 = \langle a, x\rangle^{\lambda_k} \, / \, \langle b, x\rangle^{\lambda_1} \,, \tag{8}$$

with $b \in V_k$ an eigenvector of A^t.

(c)For each $k = 1, \cdots, \beta$, there are $s_k - 1$ first integrals of the form

$$F_3 = \langle a, x\rangle^{\sigma_l \mu_k} \, / \, \left[H_l \left(U\pi^k x\right)^{\lambda_1} \right], \tag{9}$$

with $l = 1, 3, 4, \cdots, s_k$.

(d)There are β first integrals of the form

$$F_4 = \langle a, x\rangle \, / \exp\left(\lambda_1 \left(U\pi_2^k x / U\pi_1^k x\right)\right), \tag{10}$$

with $k = 1, \cdots, \beta$.

THEOREM 2. *Let A be a matrix as in Theorem 1 with $\alpha = 0$. Then system (3) has $n - 1$ independent first integrals which can be constructed in the following way.*

(a)There are $s_1 - 2$ first integrals of the form

$$F_1(x) = H_1 \left(U\pi^1 x\right)^{\sigma_l} / H_l \left(U\pi^1 x\right), \tag{11}$$

with $l = 3, 4, \cdots, s_1$.

(b)For each $k = 2, \cdots, \beta$, there are $s_k - 1$ first integrals of the form

$$F_2(x) = H_1 \left(U\pi^1 x\right)^{\sigma_l \mu_k} / H_l \left(U\pi^k x\right)^{\mu_1} , \tag{12}$$

with $l = 1, 3, 4, \cdots, s_k$.

(c)There are β first integrals of the form

$$F_3(x) = H_1 \left(U\pi^1 x\right) / \exp\left(\mu_1 \left(U\pi_2^k x / U\pi_1^k x\right)\right), \tag{13}$$

with $k = 1, \cdots, \beta$.

If the matrix A has complex eigenvalues we need to add to decomposition (6) an invariant subspace $W = \left(\bigoplus_{k=1}^{j} W_k\right) \bigoplus \left(\bigoplus_{u=1}^{\delta} M_u\right)$ of even dimension ω, such that $A\,|_{W_k}$ has the form

$$P_k = \begin{pmatrix} \Lambda_k & 0 & \cdots & 0 \\ 0 & \Lambda_k & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Lambda_k \end{pmatrix}, \tag{14}$$

where $\Lambda_k = \begin{pmatrix} a_k & -b_k \\ b_k & a_k \end{pmatrix}$ with $b_k \neq 0$. On the other hand, $A\,|_{M_u}$ is of the type

$$Q_u = \begin{pmatrix} \Lambda_u & 0 & \cdots & 0 \\ I & \Lambda_u & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & I & \Lambda_u \end{pmatrix}, \tag{15}$$

with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The eigenvalue corresponding to the block P_k is $\nu_k = a_k + b_k i$ and the one corresponding to the block Q_u is $\xi_u = a_u + b_u i$. Let w_k be the dimension of W_k and let m_u be the dimension of M_u. The numbers w_k and m_u are even, for each $k = 1, \cdots, \gamma$ and $u = 1, \cdots, \delta$. Furthermore, we have the equality $\omega = \left(\sum_{k=1}^{\gamma} w_k\right) + \left(\sum_{u=1}^{\delta} m_u\right) = d_w + d_m$. So, $n = d_r + d_s + d_w + d_m$ and $\mathbb{R}^n$ has the decomposition

$$\mathbb{R}^n = \left(\bigoplus_{\tau=1}^{\alpha} V_\tau\right) \bigoplus \left(\bigoplus_{j=1}^{\beta} U_j\right) \bigoplus \left(\bigoplus_{k=1}^{\gamma} W_k\right) \bigoplus \left(\bigoplus_{u=1}^{\delta} M_u\right). \tag{16}$$

Let $(x_1, x_2, \cdots, x_{w_k})$ be the coordinates of a point x in W_k. We define the complex space W_k^c as the set of all points of the form $C(x) = (x_1 + ix_2, \cdots, x_{w_k-1} + ix_{w_k})$. Analogously, we define the space M_u^c corresponding to M_u. Notice that W_k and M_u are realifications of W_k^c and M_u^c, respectively. Then $W_a^c = C(W_a)$ and $M_a^c = C(M_a)$.

THEOREM 3. *Suppose that the matrix A of system (3) has been reduced to its real canonical Jordan form and that the decomposition of $\mathbb{R}^n$ is given by (16) with $\gamma > 0$. Then, system (3) has $n-1$ independent first integrals which can be constructed in the following way.*

(a) There are $(w_1/2)-1$ *first integrals of the form*

$$F_1(x) = Re\left\{W^c\pi_1^1\left(C\left(W\pi^1 x\right)\right)/W^c\pi_j^1\left(C\left(W\pi^1 x\right)\right)\right\}, \tag{17}$$

with $j = 2, \cdots, w_1/2$*; and there are* $w_1/2$ *first integrals of the form*

$$F_2(x) = Re\left\{\left[W^c\pi_1^1\left(C\left(W\pi^1 x\right)\right)\right]^{\overline{\nu_1}} / \left[\overline{W^c\pi_j^1\left(C\left(W\pi^1 x\right)\right)}\right]^{\nu_1}\right\}, \tag{18}$$

with $j = 1, \cdots, w_1/2$.

(b) For each $k = 2, \cdots, \gamma$*, there are* $w_k/2$ *first integrals of the form*

$$F_3(x) = Re\left\{\left[W^c\pi_1^1\left(C\left(W\pi^1 x\right)\right)\right]^{\nu_k} / \left[W^c\pi_l^k\left(C\left(W\pi^1 x\right)\right)\right]^{\nu_1}\right\}, \tag{19}$$

and $w_k/2$ *first integrals of the form*

$$F_4(x) = Re\left\{\left[W^c\pi_1^1\left(C\left(W\pi^1 x\right)\right)\right]^{\overline{\nu_k}} / \left[\overline{W^c\pi_l^k\left(C\left(W\pi^1 x\right)\right)}\right]^{\nu_1}\right\}, \tag{20}$$

with $l = 1, \cdots, w_k/2$.

(c) For each $u = 1, \cdots, \delta$*, there are* $(m_u/2)-1$ *first integrals of the form*

$$F_5(x) = Re\left\{\left[W^c\pi_1^1\left(C\left(W\pi^1 x\right)\right)\right]^{\sigma_j\xi_u} / \left[H_j\left(C\left(M\pi^u x\right)\right)\right]^{\nu_1}\right\}, \tag{21}$$

and $(m_u/2)-1$ *first integrals of the form*

$$F_6(x) = Re\left\{\left[W^c\pi_1^1\left(C\left(W\pi^1 x\right)\right)\right]^{\sigma_j\overline{\xi_u}} / \left[H_j\left(\overline{C\left(M\pi^u x\right)}\right)\right]^{\nu_1}\right\}, \tag{22}$$

with $j = 1, 3, \cdots, m_u/2$.

(d) For each $u = 1, \cdots, \delta$*, there are two first integrals given by*

$$F_7(x) = Re\left\{\frac{W^c\pi_1^1\left(C\left(W\pi^1 x\right)\right)}{\exp\left(\nu_1\left[M^c\pi_2^u\left(C\left(W\pi^u x\right)\right)/M^c\pi_1^u\left(C\left(W\pi^u x\right)\right)\right]\right)}\right\}, \tag{23}$$

and

$$F_8(x) = Re\left\{\frac{W^c\pi_1^1\left(C\left(W\pi^1 x\right)\right)}{\exp\left(\nu_1\left[\overline{M^c\pi_2^u\left(C\left(W\pi^u x\right)\right)}/\overline{M^c\pi_1^u\left(C\left(W\pi^u x\right)\right)}\right]\right)}\right\}. \tag{24}$$

(e) For all $\tau = 1, \cdots, \alpha$*, there are* r_τ *first integrals of the form*

$$F_9(x) = \left[\left(W\pi_1^1 x\right)^2 + \left(W\pi_2^1 x\right)^2\right]^{\lambda_\tau} / \langle a_j, x\rangle^{2Re\{\nu_1\}}, \tag{25}$$

where a_j *is an eigenvector of* A^t *with eigenvalue* λ_τ, $j = 1, \cdots, r_\tau$.

(f)For all $j=1,\cdots,\beta$, there are s_j-1 first integrals of the form

$$F_{10}(x)=\left[\left(W\pi_1^1x\right)^2+\left(W\pi_2^1x\right)^2\right]^{\sigma_k\mu_j}/\left[H_k\left(U\pi^jx\right)\right]^{2Re\{\nu_1\}}, \tag{26}$$

with $k=1,3,4,\cdots,s_j$.

(g)There are β first integrals of the form

$$F_{11}(x)=\left[\left(W\pi_1^1x\right)^2+\left(W\pi_2^1x\right)^2\right]/\exp\left(2Re\{\nu_1\}\left[U\pi_2^jx/U\pi_1^jx\right]\right), \tag{27}$$

$j=1,\cdots,\beta$.

The last case $\gamma=0$ is described in the following theorem.

THEOREM 4. *Let A be a matrix as in Theorem 3 with $\gamma=0$ and $\delta>0$. Then, system (3) has $n-1$ independent first integrals which can be constructed in the following way.*

(a)There are $m_1/2-2$ first integrals of the form

$$F_1(x)=Re\left\{\left[H_1\left(C\left(M\pi^1x\right)\right)\right]^{\sigma_l}/\left[H_l\left(C\left(M\pi^1x\right)\right)\right]\right\}, \tag{28}$$

with $l=3,\cdots,m_1/2$. Furthermore, there are $m_1/2-1$ first integrals of the form

$$F_2(x)=Re\left\{\left[H_1\left(C\left(M\pi^1x\right)\right)\right]^{\sigma_j\overline{\xi_1}}/\left[H_j\left(\overline{C\left(M\pi^1x\right)}\right)\right]^{\xi_1}\right\}, \tag{29}$$

with $j=1,3,4,\cdots,m_1/2$.

(b)For each $u=2,\cdots,\delta$, there are $m_u/2-1$ first integrals of the form

$$F_3(x)=Re\left\{\left[H_1\left(C\left(M\pi^1x\right)\right)\right]^{\sigma_l\xi_u}/\left[H_j\left(C\left(M\pi^ux\right)\right)\right]^{\xi_1}\right\}, \tag{30}$$

and there are $m_u/2-1$ first integrals of the form

$$F_4(x)=Re\left\{\left[H_1\left(C\left(M\pi^1x\right)\right)\right]^{\sigma_l\overline{\xi_u}}/\left[H_j\left(\overline{C\left(M\pi^ux\right)}\right)\right]^{\xi_1}\right\}, \tag{31}$$

with $l=1,3,\cdots,m_u/2$.

(c)For each $u=1,\cdots,\delta$, there are two first integrals given by

$$F_5(x)=Re\left\{\frac{H_1\left(C\left(M\pi^1x\right)\right)}{\exp\left(\xi_1\left[M^c\pi_2^u\left(C\left(M\pi^1x\right)\right)/M^c\pi_1^u\left(C\left(M\pi^1x\right)\right)\right]\right)}\right\}, \tag{32}$$

and

$$F_6(x) = Re\left\{\frac{H_1\left(C\left(M\pi^1 x\right)\right)}{\exp\left(\xi_1\left[\overline{M^c\pi_2^u\left(C\left(M\pi^1 x\right)\right)/M^c\pi_1^u\left(C\left(M\pi^1 x\right)\right)}\right]\right)}\right\}. \quad (33)$$

(d)For all $\tau = 1, \cdots, \alpha$, there are r_τ first integrals of the form

$$F_7(x) = \left[\left(M\pi_1^1 x\right)^2 + \left(M\pi_2^1 x\right)^2\right]^{\lambda_\tau} / \langle a_j, x\rangle^{2Re\{\xi_1\}}, \quad (34)$$

where a_j is an eigenvector of A^t corresponding to the eigenvalue λ_τ, $j = 1, \cdots, r_\tau$.

(e)For all $j = 1, \cdots, \beta$, there are $s_j - 1$ first integrals of the form

$$F_8(x) = \left[\left(M\pi_1^1 x\right)^2 + \left(M\pi_2^1 x\right)^2\right]^{\sigma_k\mu_j} / \left[H_k\left(U\pi^j x\right)\right]^{2Re\{\xi_1\}}, \quad (35)$$

with $k = 1, 3, 4, \cdots, s_j$.

(f)There are β first integrals of the form

$$F_9(x) = \left[\left(M\pi_1^1 x\right)^2 + \left(M\pi_2^1 x\right)^2\right] / \exp\left(2Re\{\xi_1\}\left[U\pi_2^j x / U\pi_1^j x\right]\right), \quad (36)$$

$j = 1, \cdots, \beta$.

3. PROOFS OF THE RESULTS

Consider the linear differential system

$$\dot{x} = Ax, \quad (37)$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{K}^n$ and A is a $n \times n$ constant matrix in $\mathbb{K}$. The field $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

A hypersurface $f = 0$ is an *invariant (algebraic) hypersurface* of (37) if f is a non–constant polynomial in $\mathbb{C}[x]$ and there exists a constant $k \in \mathbb{C}$ such that the equation

$$\langle Ax, \nabla f\rangle = kf \quad (38)$$

holds. As it is usual the gradient is denoted by $\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_n)$. Note that if $k = 0$, then f is a first integral of system (37).

A function $F(x) = \exp(g(x)/f(x))$ is an *exponential factor* of (37), if $f = 0$ is an invariant algebraic hypersurface, g is a polynomial and there exists a constant $k \in \mathbb{C}$, such that

$$\langle Ax, \nabla F\rangle = kF. \quad (39)$$

The constant k as used in (38) is known as *cofactor* of the invariant algebraic curve $f = 0$. When used as in (39) is called the *cofactor* of the exponential factor F.

Invariant hypersurfaces and exponential factors are useful to construct first integrals. This is proved in the following lemma (see [3]).

LEMMA 5. *Let f_1 and f_2 be a couple of functions which satisfy equations (38) or (39) with cofactors λ_1 and λ_2, respectively. If $\lambda_1\alpha_1 + \lambda_2\alpha_2 = 0$, then $H = f_1^{\alpha_1} f_2^{\alpha_2}$ is a first integral of system (37).*

Proof: We must prove that $\langle Ax, \nabla H\rangle = 0$. From the expression of H follows

$$\begin{aligned}
&\langle Ax, \nabla H\rangle \\
&= \langle Ax, \nabla f_1^{\alpha_1} f_2^{\alpha_2}\rangle = \langle Ax, f_1^{\alpha_1}\nabla f_2^{\alpha_2} + f_2^{\alpha_2}\nabla f_1^{\alpha_1}\rangle \\
&= \left\langle Ax, \alpha_2 f_1^{\alpha_1} f_2^{\alpha_2-1}\nabla f_2 + \alpha_1 f_2^{\alpha_2} f_1^{\alpha_1-1}\nabla f_1\right\rangle \\
&= \alpha_2 f_1^{\alpha_1} f_2^{\alpha_2-1}\langle Ax, \nabla f_2\rangle + \alpha_1 f_2^{\alpha_2} f_1^{\alpha_1-1}\langle Ax, \nabla f_1\rangle \\
&= \alpha_2\lambda_2 f_1^{\alpha_1} f_2^{\alpha_2} + \alpha_1\lambda_1 f_1^{\alpha_1} f_2^{\alpha_2} = (\alpha_1\lambda_1 + \alpha_2\lambda_2)\, f_1^{\alpha_1} f_2^{\alpha_2} = 0.
\end{aligned}$$

Now we analyze some properties of the solutions of equations (38) and (39).

LEMMA 6. *Let $f = 0$ be an invariant hypersurface of system (37) and let g and h be non–constant homogeneous polynomials having different degrees. If $f = g + h$, then $g = 0$ and $h = 0$ are invariant hypersurfaces of system (37).*

Proof: Let d_1 be the degree of g and let d_2 be the degree of h. Note that $d_1 \neq d_2$. Suppose that $f = g + h$, then equation (38) is equivalent to $\langle Ax, \nabla g(x)\rangle + \langle Ax, \nabla h(x)\rangle = k(g(x) + h(x))$. When we compare the degrees in both sides of this equation, we see that it holds if and only if $\langle Ax, \nabla g(x)\rangle = kg(x)$ and $\langle Ax, \nabla h(x)\rangle = kh(x)$. ∎

So, in the following if $f = 0$ is an invariant hypersurface of degree d we can assume that f is a homogeneous polynomial of degree d.

LEMMA 7. *If $F = \exp(f/g)$ is an exponential factor of system (37), then either f and g are polynomials with the same degree or f_2/g is a first integral, where f_2 is the part of f of degree higher than the degree of g.*

Proof: According to the definition of exponential factor, we have

$$\langle Ax, \nabla \exp(f/g)\rangle = k\exp(f/g), \tag{40}$$

or equivalently

$$\left\langle Ax, \exp(f/g)\frac{g\nabla f - f\nabla g}{g^2}\right\rangle = k\exp(f/g).$$

Then it follows that

$$\langle Ax, g\nabla f - f\nabla g\rangle = kg^2.$$

In the following r denotes the degree of g and s is the degree of f. Suppose that $r \geq s$. All the left side terms in the last equation have degree no greater than $r+s$, whereas on the right side there are terms of degree $2r$. Therefore, we necessarily have that $r+s=2r$. As a consequence $r=s$. So, we must consider the case when $s>r$. In this case, $\dfrac{f}{g} = \dfrac{f_1}{g} + \dfrac{f_2}{g}$, where f_2 is the polynomial having all the terms of f whose degree is greater than r. From (40) we obtain the equality

$$k = \left\langle Ax, \nabla\frac{f_1}{g} + \nabla\frac{f_2}{g}\right\rangle = \left\langle Ax, \nabla\frac{f_1}{g}\right\rangle + \left\langle Ax, \nabla\frac{f_2}{g}\right\rangle.$$

It follows that

$$\langle Ax, g\nabla f_2 - f_2\nabla g\rangle + \langle Ax, g\nabla f_1 - f_1\nabla g\rangle = kg^2.$$

Since the degree of the polynomial $\langle Ax, g\nabla f_2 - f_2\nabla g\rangle$ is greater than both of the degrees of $\langle Ax, g\nabla f_1 - f_1\nabla g\rangle$ and kg^2, the last equation implies that $\langle Ax, g\nabla f_2 - f_2\nabla g\rangle = 0$. Hence, f_2/g is a first integral. ▮

Lemma 8. *The hyperplane* $f(x) = \sum\limits_{j=1}^{n} a_j x_j = 0$ *is invariant under system (37) if and only if* $a = (a_{1,\cdots,}a_n)$ *is an eigenvector of* A^t. *Moreover, the cofactor of* $f=0$ *is the eigenvalue of the eigenvector* a.

Proof: We prove the "only if" part, the "if" part follows similarly. Let $f(x) = \sum\limits_{j=1}^{n} a_j x_j = 0$ be an invariant hyperplane of (37) with cofactor k. Then $\nabla f(x) = a$ and equality (38) can be written as $\langle Ax, a\rangle = k\langle x, a\rangle$,which is equivalent to $\langle x, (A^t - kI)\,a\rangle = 0$. Since the above equation holds for all $x \in \mathbb{K}^n$, we obtain that $(A^t - kI)\,a = 0$. This means that a is an eigenvalue of A^t and that the cofactor k is its eigenvalue. ▮

We say that the spectrum of a matrix A is *complete* in $\mathbb{K}^n$ if there is a basis of eigenvectors of A. We have the following proposition as a consequence of Lemma 8.

Proposition 9. *If the spectrum* $\{\lambda_j \mid j = 1,\cdots,n\}$ *of* A *is complete, then system (37) has* $n-1$ *independent first integrals of the form*

$$H_j = \frac{\langle a_1, x\rangle^{\lambda_j}}{\langle a_j, x\rangle^{\lambda_1}},$$

where a_j is an eigenvector of A^t with eigenvalue λ_j, $j = 1, \cdots, n$. The eigenvalues are taking into account according to their multiplicities.

Proof: Let $\{a_{1,}a_2, \cdots, a_n\}$ be a basis of eigenvectors, corresponding to the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ (repeated if necessary). We denote by $k_j = -\dfrac{\lambda_1}{\lambda_{j+1}}$, $j = 1, \cdots, n-1$, and $f_j(x) = \langle a_j, x\rangle$, $j = 1, \cdots, n$. Then, by Lemmas 5 and 8, it follows that $H_j = f_1 f_{j+1}^{k_j}$ is a first integral of system (37) for each $j = 1, \cdots, n-1$. Now, we will prove that these first integrals are independent. The gradient of H_j is given by

$$\nabla H_j = f_{j+1}^{k_j} a_1 + k_j f_1 f_{j+1}^{k_j - 1} a_{j+1}, \quad j = 1, \cdots, n-1.$$

Let x_0 be a point in $\left(\bigcap\limits_{j=1}^{n-1} \{H_j = l_j\}\right) \bigcap \{f_1 \neq 0\}$, such that $\nabla H_j(x_0) \neq 0$ for all $j = 1, \cdots, n-1$. We write $\nabla H_j(x_0) = \alpha_{j+1} a_1 + \beta_{j+1} a_{j+1}$. Suppose that there exist $n-1$ numbers in $\mathbb{C}$, c_{j+1}, $j = 1, \cdots, n-1$, with at least one different from zero such that

$$\sum_{j=1}^{n-1} c_{j+1} \nabla H_j(x_0) = \sum_{j=1}^{n-1} c_{j+1} (\alpha_{j+1} a_1 + \beta_{j+1} a_{j+1}) = 0.$$

From this expression and after a rearrangement of the terms, we obtain an equation of the form

$$\left(\sum_{j=1}^{n-1} c_{j+1} \alpha_{j+1}\right) a_1 + \sum_{j=1}^{n-1} c_{j+1} \beta_{j+1} a_{j+1} = 0.$$

From the linear independence of the vectors a_j, $j = 1, \cdots, n$, we get $\sum\limits_{j=1}^{n-1} c_{j+1}\alpha_{j+1} = 0$, and $c_{j+1}\beta_{j+1} = 0$ for $j = 1, \cdots, n-1$. Without loss of generality, it is assumed that $c_2 \neq 0$, which implies that $\beta_2 = 0$. Since $\beta_2 = k_1 f_1(x_0) f_2^{k_1 - 1}(x_0)$, we have that either $f_1(x_0) = 0$ or $f_2(x_0) = 0$. But $f_2(x_0)$ is different from zero because $\nabla H_1(x_0) = f_2^{k_1}(x_0) a_1 \neq 0$. Hence $f_1(x_0) = 0$. This contradicts the choice of x_0. Therefore, the set $\{\nabla H_j(x_0)\}_{j=1,\cdots,n-1}$ is an independent linear set. ▌

Proposition 9 corresponds to Theorem 1 when $\mathbb{K} = \mathbb{R}$ and $\beta = 0$. So, the problem of the integrability of system (3) when $\mathbb{K} = \mathbb{R}$ reduces to study the case when either the real spectrum is not complete or the spectrum has complex eigenvalues. If the spectrum of a $\mathbb{K}-$matrix A is not complete, then its canonical Jordan form contains blocks of the form

$$\begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 1 & . & 0 & 0 & 0 \\ 0 & 0 & . & . & 0 & 0 \\ 0 & 0 & 0 & . & \lambda & 0 \\ 0 & 0 & 0 & 0 & 1 & \lambda \end{pmatrix}. \tag{41}$$

Let D be the matrix whose principal subdiagonal is $(1, 1, \cdots 1)$ and any other entry is zero, that is

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & . & 0 & 0 & 0 \\ 0 & 0 & . & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{42}$$

LEMMA 10. *The linear differential systems $\dot{x} = (\lambda I + D)\, x$ and $\dot{x} = Dx$ have the same invariant hypersurfaces h=0 and the same exponential factors F=exp(f/g), when h, f and g are homogeneous polynomials with f and g of the same degree. Here, I is the identity matrix with the same size as D.*

Proof: We will prove that the invariant hypersurfaces and the exponential factors of the above form for the system $\dot{x} = (\lambda I + D)\, x$, are invariant hypersurfaces and exponential factors for the system $\dot{x} = Dx$. The converse direction is proved in a similar way.

Let $f = 0$ be an invariant hypersurface of $\dot{x} = (\lambda I + D)\, x$ with cofactor k where f is a homogeneous polynomial of degree d. Hence

$$< (\lambda I + D)x, \nabla f > = \; kf.$$

Therefore

$$\lambda < x, \nabla f > + < Dx, \nabla f > = \; kf.$$

From the Euler's Theorem for homogeneous functions, we get $< x, \nabla f > \; = \; df$. We denote by $k_1 = k - \lambda d$, hence f is a solution of the equation

$$< Dx, \nabla f > = \; k_1 f.$$

That is, $f = 0$ is an invariant hypersurface of the system

$$\dot{x} = Dx.$$

Analogously, let $F = \exp(f/g)$ be an exponential factor of $\dot{x} = (\lambda I + D)\, x = Ax$ with cofactor k. Then

$$< Ax, \nabla F >= F \left\langle Ax, \nabla \frac{f}{g} \right\rangle = kF.$$

From

$$\left\langle Ax, \nabla \frac{f}{g} \right\rangle = \left\langle Ax, \frac{g\nabla f - f\nabla g}{g^2} \right\rangle,$$

we have that

$$< (\lambda I + D)\, x, g\nabla f - f\nabla g >= kg^2.$$

Since f and g are homogeneous polynomials with the same degree, the last equation is equivalent to

$$< Dx, g\nabla f - f\nabla g >= kg^2.$$

Indeed, $< (\lambda I + D)\, x, g\nabla f - f\nabla g >=< \lambda x, g\nabla f - f\nabla g > + < (\lambda I + D)\, x, g\nabla f - f\nabla g >$. The Euler's Theorem for homogeneous functions implies $< \lambda x, g\nabla f - f\nabla g >= 0$.

This prove that $F = \exp(f/g)$ is an exponential factor of the linear system $\dot{x} = Dx$. ∎

From Lemma 10, the search of first integrals of the system

$$\dot{x} = (\lambda I + D)\, x, \tag{43}$$

using the Darboux Theory of integrability based on the invariant hypersurfaces and exponential factors can be reduced to find first integrals of the system

$$\dot{x} = Dx. \tag{44}$$

Remark 11. We observe in the proof of Lemma 10 that the cofactor of the exponential factor does not depend on λ and it is the same for both systems (43) and (44). However, if $f = 0$, is an invariant hypersurface of degree d of system (44) with cofactor k_1, then $f = 0$ is an invariant hypersurface of system (43), with cofactor $k = k_1 + \lambda d$. In particular, if f is a homogeneous polynomial first integral of degree d of (44), then $f = 0$ is just an invariant hypersurface of (43) with cofactor $k = \lambda d$.

LEMMA 12. *Let f be a homogeneous polynomial. Then $f = 0$ is an invariant hypersurface of system (44) if and only f is a polynomial first integral of the same system.*

Proof: Let $f(x_1, x)$ be a homogeneous polynomial of degree r. Here, we denote $x = (x_2, \cdots, x_n)$. Hence, $f(x_1, x) = x_1^r + x_1^{r-1}h_1(x) + x_1^{r-2}h_2(x) + \cdots + x_1 h_{r-1}(x) + h_r(x)$, where h_i is a homogeneous polynomial of degree i. Assume that f satisfies the equation

$$< D(x_1, x), \nabla f(x_1, x) >= kf(x_1, x). \tag{45}$$

The gradient of f can be written as

$$\nabla f(x_1, x) = B(x_1, x) + G(x_1, x),$$

where $B(x_1, x) = h_{r-1}(x)\nabla x_1 + \nabla h_r(x)$, and $G = \nabla f - B$. So x_1 is a factor of G. Then, $< D(x_1, x), \nabla f(x_1, x) >=< D(x_1, x), B(x_1, x) > + < D(x_1, x), G(x_1, x) >$.

Note that x_1 is a factor of $< D(x_1, x), G(x_1, x) >$ and since $B(x_{1,}x) = (h_{r-1}(x), \frac{\partial h_r}{\partial x_2}, \cdots, \frac{\partial h_r}{\partial x_n})$, we obtain that

$$\begin{aligned} < D(x_{1,}x), B(x_1, x) > &= \left\langle (0, x_1, \cdots, x_{n-1}), \left(h_{r-1}(x), \frac{\partial h_r}{\partial x_2}, \cdots, \frac{\partial h_r}{\partial x_n} \right) \right\rangle \\ &= x_1 \frac{\partial h_r}{\partial x_2} + x_2 \frac{\partial h_r}{\partial x_3} + \cdots + x_{n-1} \frac{\partial h_r}{\partial x_n}. \end{aligned}$$

From the equality

$$< D(x_1, x), B(x_1, x) > + < D(x_1, x), G(x_1, x) >= kf(x_1, x),$$

we get

$$x_2 \frac{\partial h_r}{\partial x_3} + \cdots + x_{n-1} \frac{\partial h_r}{\partial x_n} = kh_r(x).$$

This is equivalent to

$$< D^1 x, \nabla h_r(x) >= kh_r(x), \tag{46}$$

where D^1 is the matrix D in dimension $n-1$. The last equation with $n-1$ variables is similar to equation (45). Therefore, it determines a recurrence relation. So, the lemma holds if we prove that (45) implies $k = 0$, when the dimension of system (44) is 2.

Let $f(x, y) = a_r x^r + a_{r-1}x^{r-1}y + a_{r-2}x^{r-2}y^2 + \cdots + a_0 y^r$ be a homogeneous polynomial in two variables of degree r. We assume that the equation $< D(x, y), \nabla f(x, y) > = kf(x, y)$ is satisfied with D of dimension 2. We identify the homogeneous polynomial f with the vector $(a_r, a_{r-1}, a_{r-2}, \cdots, a_0), of\ \mathbb{R}^{r+1}$. Then the homogeneous polynomial

$x\ \partial f/\partial y$ can be identified with $(a_{r-1}, 2a_{r-2}, \cdots, ra_0, 0)$. Hence, equation (46) can be written in the form

$$(ka_r, ka_{r-1}, ka_{r-2}, \cdots, ka_0) = (a_{r-1}, 2a_{r-2}, \cdots, ra_0, 0).$$

This last equation corresponds to a homogeneous linear equation in the variables a_0, $a_1, \ldots, a_r$ having determinant $(-k)^{r+1}$. So it has a non–zero solution if and only if $k = 0$. ▮

PROPOSITION 13. *System (44) with $x = (x_1, \ldots x_n)$ has the following independent first integrals.*

(a)$H_1 = x_1$.

(b)$H_l = \sum\limits_{j=1}^{l} (-1)^{j+1} x_j x_{l+1-j}$ *with* $3 \leq l \leq n$, l *odd.*

(c)$H_k = x_1^{k-2} x_k - \dfrac{1}{(k-1)!} x_2^{k-1} + \sum\limits_{j=2}^{k-1} \dfrac{(-1)^j}{(k-j)!} x_1^{j-2} x_2^{k-j} x_j$ *with* $4 \leq k \leq n$, k *even. Moreover,*

(d)$f(x) = \exp(x_2/x_1)$ *is an exponential factor with cofactor* 1.

Proof: The proof of (a) is trivial. For each odd l such that $3 \leq l \leq n$, we have that $\nabla H_l = 2(x_l, -x_{l-1}, \cdots, x_1, 0, \cdots, 0)$. Therefore, $\langle Dx, \nabla H_l \rangle = 2(0, x_1, \cdots x_{l-1}, x_l,$
$\cdots x_{n-1})(x_l, -x_{l-1}, \cdots, x_1, 0, \cdots, 0) = 0$. So, the function H_l is a first integral and (b) is proved.

We consider H_k as in the statement (c) of the lemma. After some computations, we have

$$\begin{aligned} \frac{\partial H_k}{\partial x_2} &= \sum_{j=2}^{k-1} \frac{(-1)^j}{(k-j-1)!} x_1^{j-2} x_2^{k-j-1} x_j, \\ \frac{\partial H_k}{\partial x_r} &= \frac{(-1)^r}{(k-r)!} x_1^{r-2} x_2^{k-r}, \quad 3 \leq r \leq k. \end{aligned}$$

Then

$$\begin{aligned} \sum_{j=1}^{k-1} x_j \frac{\partial H_k}{\partial x_{j+1}} &= x_1 \frac{\partial H_k}{\partial x_2} + \sum_{j=3}^{k} x_{j-1} \frac{\partial H_k}{\partial x_j} \\ &= \sum_{j=3}^{k} \frac{(-1)^{j-1}}{(k-j)!} x_1^{j-2} x_2^{k-j} x_{j-1} + \sum_{j=3}^{k} \frac{(-1)^j}{(k-j)!} x_1^{j-2} x_2^{k-j} x_{j-1} = 0. \end{aligned}$$

So, the function H_k is a first integral, and (c) is proved.

Statement (d) follows easily by direct computations. Finally, we will prove that the first integrals given in (a), (b) and (c) are linearly independent. Any set of vectors $\{v_j, j = 1, \cdots, r\}$, where the jth component $v_{j,j}$ of v_j is not zero and any other component in the right hand side of $v_{j,j}$ is zero, is a linearly independent set. Hence, the first integrals H_1, H_l and H_k, are independent at every point with $x_1 \neq 0$, since the gradient ∇H_l is of the form $(\alpha_1, \cdots, \alpha_{l-1}, x_1, 0, \cdots, 0)$ and ∇H_k is of the form $\left(\beta_1, \cdots, \beta_{k-1}, x_1^{k-2}, 0, \cdots, 0\right)$. ∎

The next corollary follows easily from Proposition 2 and Remark 1.

COROLLARY 14. *If system (3) is given by the matrix $A = \lambda I + D$, then*

(a) The function $F(x) = \exp(x_2/x_1)$ is an exponential factor with cofactor 1.

(b) The hypersurfaces $H_1(x) = x_1 = 0$, $H_l = \sum\limits_{j=1}^{l} (-1)^{j+1} x_j x_{l+1-j} = 0$ for odd l such that $3 \leq l \leq n$, and $H_l = x_1^{l-2} x_k - \dfrac{1}{(l-1)!} x_2^{l-1} + \sum\limits_{j=2}^{l-1} \dfrac{(-1)^j}{(l-j)!} x_1^{j-2} x_2^{l-j} x_j = 0$ for even l such that $4 \leq l \leq n$, are invariant with cofactor λ, 2λ and $(l-1)\lambda$, respectively.

Now, we shall prove Theorems 1 and 2. For convenience, we use the notation introduced in Section 2.

Proof of Theorem 1: Note that A restricted to $\left(\bigoplus\limits_{\tau=1}^{\alpha} V_\tau\right)$ has a complete spectrum. Let$\{v_{\tau,t}\}_{t=1}^{r_\tau}$ be an ordered basis of V_τ, where $v_{\tau,t}$ is an eigenvector of A. So, we get d_r eigenvectors of A, which generate the invariant hypersurfaces $f_{\tau,t}(x) =< v_{\tau,t}, x >= 0$, $\tau = 1, \cdots, \alpha$, $t = 1, \cdots, r_\tau$, with cofactor λ_τ. Hence, statements (a) and (b) of Theorem 1 follows from Proposition 9.

Let $\{\omega_{j,t}\}_{t=1}^{s_j}$ be an ordered basis of $U_{j.}$ Suppose that the matrix S_j has the form (41) for each $j = 1, \cdots, \beta$. So, from Corollary 14 we obtain $s_j - 1$ invariant hypersurfaces $g_{j,t}(x) = H_t\left(U\pi^j x\right) = 0$, $t = 1, 3, \cdots, s_j$ with cofactor $\sigma_t \mu_j$. Here, the function H_t is given as in Corollary 14, according to the parity of the index t. Using Lemma 5 each couple of functions $(f_{1,1}, g_{j,t})$ generates the first integral of the form (9) described in statement (c) of Theorem 1.

We can check easily that $F_j(x) = \exp(U\pi_2^j x / U\pi_1^j x)$ is an exponential factor with cofactor μ_j (see Corollary 14(a)). In this way, using Lemma 5 we obtain β couples of functions $(f_{1,1}, F_j)$ which determine the first integrals of the form (10) in Theorem 1(d). The independence of this set of $n-1$ first integrals follows easily from the decomposition (6) of $\mathbb{R}^n$. ∎

Proof of Theorem 2: If $\alpha = 0$, then $\mathbb{R}^n = \bigoplus_{j=1}^{\beta} U_j$. From Lemma 10 and Corollary 14, it follows that each subspace U_j has $s_j - 1$ invariant hypersurfaces given by $g_{j,t}(x) = H_t\left(U\pi^j x\right) = 0$, $t = 1, 3, \cdots, s_j$, and the exponential factors $F_j(x) = \exp\left(U\pi_2^j x / U\pi_1^j x\right)$, $j = 1, \cdots, \beta$. Using Lemma 5 the couple of functions $(g_{1,1}, g_{1,t})$, for $3 \le t \le s_1$, generate the first integrals of the form (11) in Theorem 2(a). The first integrals of the form (12) are constructed with the functions $(g_{1,1}, g_{j,t})$, where $2 \le j \le \beta$, $t \ne 2$ and $t \le s_j$. Finally, the couples $(g_{1,1}, F_j)$, $j = 1, \cdots, \beta$, are used to obtain the integrals of the form (13).

The independence of this set of $n-1$ first integrals follows easily from the decomposition (6) of $\mathbb{R}^n$. ∎

If the matrix A has no real eigenvalues, then it is necessary to analyze how to construct first integrals of linear differential systems whose matrix is a real Jordan block of the form (14) or (15).

Let Q be the following square $2m \times 2m$ matrix

$$Q = \begin{pmatrix} \Lambda & 0 & \cdots & 0 \\ \epsilon_1 I & \ddots & 0 & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \epsilon_{m-1} I & \Lambda \end{pmatrix}. \tag{47}$$

Here $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $\epsilon_j \in \{0, 1\}$ for each $j = 1, \cdots, m-1$.

Now, we associate to the linear system

$$\dot{x} = Qx, \tag{48}$$

defined in the $2m-$dimensional real space, $\{x = (x_1, \cdots, x_{2m}) \mid \ x_j \in \mathbb{R}\}$

the complex linear system

$$\begin{pmatrix} \dot{z}_1 \\ \vdots \\ \vdots \\ \dot{z}_m \\ \dot{\overline{z}}_1 \\ \vdots \\ \vdots \\ \dot{\overline{z}}_m \end{pmatrix} = \begin{pmatrix} \lambda & 0 & & & \cdots & & & 0 \\ \epsilon_1 & & 0 & & & & & \\ 0 & \ddots & & \ddots & & & & \\ & 0 & \epsilon_{m-1} & \lambda & \ddots & & & \vdots \\ & & \ddots & 0 & \overline{\lambda} & \ddots & & \\ \vdots & & & \ddots & \epsilon_1 & & 0 & \\ & & & & \ddots & \ddots & & 0 \\ 0 & & \cdots & & & 0 & \epsilon_{m-1} & \overline{\lambda} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ \vdots \\ z_m \\ \overline{z}_1 \\ \vdots \\ \vdots \\ \overline{z}_m \end{pmatrix},$$

where $\lambda = a + bi$ and $z_j = x_{2j-1} + x_{2j}i$, for all $j = 1, \cdots, m$. This system is denoted by

$$\dot{Z} = B(\lambda) Z, \tag{49}$$

where $Z = (z_1, \cdots, z_m, \overline{z}_1, \cdots, \overline{z}_m) = (z, \overline{z})$, $B(\lambda) = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$, $B_1 = \lambda I + D_1$, $B_2 = \overline{\lambda} I + D_1$ and

$$D_1 = \begin{pmatrix} 0 & \cdots & & 0 \\ \epsilon_1 & \ddots & & \vdots \\ 0 & \ddots & & 0 \\ 0 & 0 & \epsilon_{m-1} & 0 \end{pmatrix}. \tag{50}$$

We define the partial derivatives with respect to z_j and with respect to $\overline{z}_j$ by

$$\frac{\partial H}{\partial z_j} = \frac{1}{2}\left(\frac{\partial H}{\partial x_{2j-1}} - \frac{\partial H}{\partial x_{2j}} i\right), \quad \frac{\partial H}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial H}{\partial x_{2j-1}} + \frac{\partial H}{\partial x_{2j}} i\right),$$

respectively. Now, we have the following result.

LEMMA 15. *If $F(Z) = 0$ is an invariant hypersurface or F is an exponential factor of system (49), with cofactor k, then $G(x_1, \cdots, x_{2m}) = F(Z)$, determines an invariant hypersurface or an exponential factor of system (48) with cofactor k,respectively. Moreover, if F is a first integral of (49) then $Re(F)$ and $Im(F)$ are first integrals of (48).*

Proof: We denote the derivative $\dfrac{\partial F}{\partial u}$ by F_u. Given a function $F = F(Z)$, we get the equality

$$\begin{aligned}&\langle B(\lambda) Z, \nabla_Z F(Z)\rangle = \lambda z_1 F_{z_1} + (\epsilon_1 z_1 + \lambda z_2) F_{z_2} + \cdots + (\epsilon_{m-1} z_{m-1} + \\ &\lambda z_m) F_{z_m} + \overline{\lambda}\overline{z}_1 F_{\overline{z}_1} + \left(\epsilon_1 \overline{z}_1 + \overline{\lambda}\overline{z}_2\right) F_{\overline{z}_2} + \cdots + \left(\epsilon_{m-1}\overline{z}_{m-1} + \overline{\lambda}\overline{z}_m\right) F_{\overline{z}_m} \\ &= \left[\lambda z_1 F_{z_1} + \overline{\lambda}\overline{z}_1 F_{\overline{z}_1}\right] + \epsilon_1\left[z_1 F_{z_2} + \overline{z}_1 F_{\overline{z}_2}\right] + \left[\lambda z_2 F_{z_2} + \overline{\lambda}\overline{z}_2 F_{\overline{z}_2}\right] + \cdots + \\ &\epsilon_{m-1}\left[z_{m-1} F_{z_m} + \overline{z}_{m-1} F_{\overline{z}_m}\right] + \left[\lambda z_m F_{z_m} + \overline{\lambda}\overline{z}_m F_{\overline{z}_m}\right].\end{aligned}$$

On the other hand, it is easy to see that $(\alpha + \beta i) f_u + (\alpha - \beta i) f_{\overline{u}} = \alpha f_{u_1} + \beta f_{u_2}$, where $f = f(u) = f(u_1 + u_2 i)$. Hence

$$\begin{aligned}&\langle BZ, \nabla_Z F(Z)\rangle = [(ax_1 - bx_2) F_{x_1} + (bx_1 + ax_2) F_{x_2}] + \epsilon_1[x_1 F_{x_3} + \\ &x_2 F_{x_4}] + [(ax_3 - bx_4) F_{x_3} + (bx_3 + ax_4) F_{x_4}] + \cdots + \epsilon_{m-1}[x_{2m-3} F_{x_{2m-1}} + \\ &x_{2m-2} F_{x_{2m}}] + \left[(ax_{2m-1} - bx_{2m}) F_{x_{2m-1}} + (bx_{2m-1} + ax_{2m}) F_{x_{2m}}\right] \\ &= (ax_1 - bx_2) F_{x_1} + (bx_1 + ax_2) F_{x_2} + (\epsilon_1 x_1 + ax_3 - bx_4) F_{x_3} + \\ &(\epsilon_1 x_2 + bx_3 + ax_4) F_{x_4} + \cdots + (\epsilon_{m-1} x_{2m-3} + ax_{2m-1} - bx_{2m}) F_{x_{2m-1}} + \\ &(\epsilon_{m-1} x_{2m-2} + bx_{2m-1} + ax_{2m}) F_{x_{2m}} = \langle Qx, \nabla_x G(x)\rangle.\end{aligned}$$

The lemma follows easily from this equality. ∎

In the proofs of Theorems 3 and 4 we use again the notation introduced in Section 2. The $2m \times 2m$ diagonal matrix whose m first diagonal entries take the value α and the remaining diagonal entries take the value β is denoted by $L_m(\alpha, \beta)$.

Proof of Theorem 3: Let ν_k be the eigenvalue of $A\,|_{W_k}$, $k = 1, \cdots, \gamma$. In correspondence with system (3) restricted to W_k, we consider the system $\dot{Z} = L_{\omega_k/2}(\nu_k, \overline{\nu}_k) Z$, where $Z = (z, \overline{z})$, $z \in W_k^c$. This system has ω_k invariant hypersurfaces given by

(i) $h_j(Z) = z_j = 0$, with cofactor ν_k, and $g_j(Z) = \overline{z}_j = 0$ with cofactor $\overline{\nu}_j$, $j = 1, \cdots, \omega_k/2$.

Analogously, let ξ_u be the eigenvalue of $A\,|_{M_u}$, $u = 1, \cdots, \delta$. System (3) restricted to M_u corresponds to the system $\dot{Z} = B(\xi_u) Z$, where $Z = (z, \overline{z})$, $z \in M_u^c$. As in Corollary 14 this system has $m_u - 2$ invariant hypersurfaces and 2 exponential factors, which are constructed in the following way:

(ii) $m_u/2 - 1$ invariant hypersurfaces of the form $G(z, \bar{z}) = H_l(z) = 0$, $l = 1, 3, \cdots, m_u$, whose cofactors are $\sigma_l \xi_u$.

(iii) $m_u/2 - 1$ invariant hypersurfaces of the form $G(z, \bar{z}) = H_l(\bar{z}) = 0$, $l = 1, 3, \cdots, m_u$, whose cofactors are $\sigma_l \bar{\xi}_u$.

(iv) Two exponential factors given by $F_1(z, \bar{z}) = \exp(z_2/z_1)$ and $F_2(z, \bar{z}) = \exp(\bar{z}_2/\bar{z}_1)$, with cofactor 1.

We assume now that $\gamma > 0$. From (i) and Lemma 15, it follows that system (3) has $\omega_k/2$ invariant hypersurfaces of the form $f_{j,k} = W^c\pi_j^k(C(W\pi^k x)) = 0$, with cofactor ν_k, and $\omega_k/2$ invariant hypersurfaces $\overline{f}_{j,k} = \overline{W^c\pi_j^k\left(C\left(W\pi^k x\right)\right)} = 0$, with cofactor $\bar{\nu}_k$. The first integrals of type (17) are given by $\operatorname{Re}\left(\dfrac{f_{1,1}}{f_{j,1}}\right)$, for $j = 2, \cdots, \omega_1/2$; and the ones of type (18) are given by $\operatorname{Re}\left(\dfrac{f_{1,1}^{\bar{\nu}_1}}{\bar{f}_{j,1}^{\nu_1}}\right)$, for $j = 1, \cdots, \omega_1/2$. The first integrals of type (19) and (20) are obtained from $\dfrac{f_{1,1}^{\nu_k}}{f_{j,1}^{\nu_1}}$ and $\dfrac{f_{1,1}^{\bar{\nu}_k}}{\bar{f}_{j,1}^{\nu_1}}$, with $k = 2, \cdots, \gamma$, $j = 1, \cdots, \omega_k/2$. Again from Lemma 15 and (iii), it follows that system (3) has $m_u/2 - 1$ invariant hypersurfaces of the form $g_{l,u} = H_l\left(C\left(M\pi^u x\right)\right) = 0$, whose cofactors are $\sigma_l\xi_u$; and $m_u/2 - 1$ invariant hypersurfaces of the form $\bar{g}_{l,u} = H_l\left(\overline{C\left(M\pi^u x\right)}\right) = 0$, whose cofactors are $\sigma_l\xi_u$. So, the first integrals of type (21) are defined by $\operatorname{Re}\left(\dfrac{f_{1,1}^{\sigma_l\xi_u}}{g_{l,u}^{\nu_1}}\right)$, and the ones of type (22) are defined by $\operatorname{Re}\left(\dfrac{f_{1,1}^{\sigma_l\bar{\xi}_u}}{\bar{g}_{l,u}^{\nu_1}}\right)$. Here, $l = 1, 3, \cdots, m_u/2$ and $u = 1, \cdots, \delta$.

Now, from (iv) we get two exponential factors

$$F_u = \exp\left[M^c\pi_2^u\left(C\left(M\pi^u x\right)\right)/M^c\pi_1^u\left(C\left(M\pi^u x\right)\right)\right],$$

$$\bar{F}_u = \exp\left[\overline{M^c\pi_2^u\left(C\left(M\pi^u x\right)\right)}/\overline{M^c\pi_1^u\left(C\left(M\pi^u x\right)\right)}\right],$$

with cofactor 1, for each $u = 1, \cdots, \delta$. The first integrals of types (23) and (24) can be obtained from $\dfrac{f_{1,1}}{F_u^{\nu_1}}$ and $\dfrac{f_{1,1}}{\bar{F}_u^{\nu_1}}$, respectively by taking the real part of them.

We consider $\tau = 1, \cdots, \alpha$, and $j = 1, \cdots, r_\tau$. Given an eigenvector a_j with eigenvalue λ_j, from Lemma 8 it follows that $p_{j,\tau}(x) = \langle a_j, x\rangle = 0$ is an invariant hypersurface with cofactor λ_j. A direct computation shows that $h = \left(W\pi_1^1 x\right)^2 + \left(W\pi_2^1 x\right)^2 = 0$ is an invariant hypersurface with cofactor $2\operatorname{Re}(\nu_1)$. So, the integrals of type (25) are defined by $\dfrac{h^{\lambda_j}}{p_{j,\tau}^{2\operatorname{Re}(\nu_1)}}$. From Corollary 14, $H_k\left(U\pi^j x\right) = 0$ is an invariant hypersurface with cofactor $\sigma_k\mu_j$, and $\exp\left(U\pi_2^j x/U\pi_1^j x\right)$ is an exponential factor with cofactor 1, for each $j = 1, \cdots, \beta$ and $k = 1, \cdots, s_j$. Therefore, the functions

$\dfrac{h^{\sigma_k \mu_j}}{\left(H_k\left(U\pi^j x\right)\right)^{2\mathrm{Re}(\nu_1)}}$ and $\dfrac{h}{\left(\exp\left(U\pi_2^j x/U\pi_1^j x\right)\right)^{2\mathrm{Re}(\nu_1)}}$ determine the first integrals of types (26) and (27), respectively. ▮

Proof of Theorem 4: The first integrals of types (17)−(24) do not exist anymore, when $\gamma = 0$. The first integrals of types (28) and (29) are constructed with the quotients $\dfrac{g_{1,1}}{g_{l,1}}$ and $\dfrac{g_{1,1}}{\bar{g}_{j,1}}$, respectively, with $3 \leq l \leq m_1/2$ and $j = 1, \cdots, m_1/2$. Similarly, to construct the first integrals of types (30) and (31), we use the function $g_{1,1}$ and the functions $g_{l,u}$ and $\bar{g}_{l,u}$, respectively, with $u = 2, \cdots, \delta$, and $l = 1, \cdots, m_u/2$. The first integrals of types (32) and (33) are defined by $\dfrac{g_{1,1}}{F_u}$ and $\dfrac{g_{1,1}}{\bar{F}_u}$, respectively, with $u = 1, \cdots, \delta$. Since $g = \left(M\pi_1^1 x\right)^2 + \left(M\pi_2^1 x\right)^2 = 0$ is an invariant hypersurface with cofactor $2\mathrm{Re}\left(\xi_1\right)$, the first integrals of types (34), (35) and (36) are constructed in the same way as in the proof of Theorem 3. ▮

We conclude this work with a final remark about the first integrals of a complex linear system

$$\dot{z} = Az, \tag{51}$$

where A is a complex $n \times n$ matrix and $z \in \mathbb{C}^n$. In this case, the canonical Jordan Form of A consists of blocks of the form (41) and diagonal blocks (of course, some of these kind of blocks could be missed). Therefore, the first integrals of (51) are obtained from Theorem 1 and Theorem 2, substituting x by z.

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