QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **4**, 413–424 (2004)
ARTICLE NO. 69

Bifurcation of Unimodal Maps*

Welington de Melo

Instituto Nacional de Matemática Pura e Aplicada, Estrada Dona Castorina 110, 22460-320, Rio de Janeiro – Brazil.
E-mail: demelo@impa.br

Submitted: August 7, 2003 *Accepted*: February 2, 2004
To Jorge Sottomayor on his 60th birthday.

Here we review some recent results that give a rather complete description of the dynamics of almost all mappings in real analytic families of unimodal maps

1. THE RESULTS

We will review here some recent results on the rich dynamics and bifurcation structure in one dimensional families of real analytic unimodal maps.

The domain of our maps will be normalized to be the interval $I = [-1, 1]$, also called the phase space. A unimodal map is a smooth map $f: I \to I$ with a unique non-degenerate critical point. Normalizing, we can assume that the critical point is 0 and $f(1) = f(-1) = -1$. The non- degeneracy hypothesis on the critical point is that the second derivative $D^2 f(0)$ is negative. Let $\mathcal{U}^\omega$ denotes the set of real analytic maps so normalized. The parameter space P is a compact interval. An analytic family of unimodal map is a mapping $t \in P \mapsto f_t \in \mathcal{U}^\omega$ such that the $(t, x) \mapsto f_t(x)$ is real analytic in both variables. A prototype example is the quadratic family: $q_t(x) = -tx^2 + t - 1$ where $P = [\frac{1}{2}, 2]$.

We say that f is $S - unimodal$ if its Schwarzian derivative,

$$Sf = \frac{D^3 f}{Df} - \frac{3}{2}\left(\frac{D^2 f}{Df}\right)^2$$

* This work has been partially supported by FAPERJ grants E-26/151.896/2000, E-26/151.189/2002.

is negative for all $x \neq 0$. If such map has a periodic attractor then the basin of this periodic point contains the critical point and also has full Lebesgue measure in I. Hence, in this case, for almost all points x in the interval, the limit

$$\lambda(f) = \lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)|$$

exists and is negative. An important result by Keller, [9], states that the above limit exists almost everywhere for any map with negative Schwarzian derivative. Furthermore, $\lambda_f > 0$ if and only if there exists a probability measure μ which is f-invariant and absolutely continuous with respect to Lebesgue and

$$\lambda_f = \int \log |Df| d\mu$$

is the *Lyapunov exponent* of the measure μ. Also, if δ_x denotes the Dirac measure at x we have that

$$\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$$

converges weakly to μ for Lebesgue almost all $x \in I$; the support of the measure μ is a cycle of intervals, i.e., a finite number of disjoint closed intervals that are in the same orbit of f . Therefore, if $\lambda_f \neq 0$ there is a unique invariant probability measure that describes the statistical behavior of Lebesgue almost all orbits, the measure being a Dirac measure if $\lambda_f < 0$ or absolutely continuous if $\lambda_f > 0$. In the first case we say that the dynamics is *regular* and in the second one it is *stochastic*. More generally, we say that an invariant measure μ of a dynamical system f is a *physical measure* if the time average $\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$ converges weakly to μ for a set of positive Lebesgue measure called the ergodic basin of μ. Hence, if $\lambda_f \neq 0$, f has a unique physical measure which is either atomic or absolutely continuous. The borderline case, $\lambda_f = 0$, different situations may occur: i) there is a unique atomic physical measure supported on a parabolic fixed point; ii) there is a unique physical measure supported in a Cantor set which is the closure of the critical orbit (infinitely renormalizable maps) or some more pathological cases like iii) there is a physical measure supported on a repelling periodic orbit or iv) there is no physical measure, [7].

An analytic family of unimodal maps f_t is *non-trivial* if there is a parameter value s such that the critical value of f_s lands in a repelling periodic point and this combinatorics is not constant. Notice that this is a very mild condition: a family is trivial if either the combinatorics is constant in the family or the period of each periodic point of each map in the family is a power of two.

As a consequence of the main result in [1], in any non-trivial family of real analytic unimodal maps with negative Schwarzian derivative and quadratic critical point, the set of parameter values corresponding to maps in the borderline case has Lebesgue measure zero. On the other hand by [10], the set of parameter values corresponding to maps with regular dynamics ($\lambda(f) < 0$), is open and dense. By [8], the set of parameter values corresponding to maps with stochastic dynamics ($\lambda(f) > 0$) has positive measure. As we will explain below, this result holds for all non-trivial family of real analytic maps and also for typical families of smooth maps. In fact even more is true as was proved by Avila-Moreira [2]: we can extend the set of excluded parameter values and give a very precise description of the remaining stochastic maps and still prove that the set of all excluded parameters has not only Lebesgue measure zero but also Hausdorff dimension smaller than one. In the next section we will give more details about this development.

2. SOME TOOLS

2.1. Renormalization and Principal Nest

We start with a short discussion on the combinatorics of unimodal interval maps. Let J be an interval around the critical point and $D_J \subset J$ be the set of points $x \in J$ such that there exists an integer $r(x) \geq 1$ with $f^{r(x)}(x) \in J$ and $f^j(x) \neq J$ for $j < r(x)$. The mapping $R_J\colon D_J \to J$, $R_J(x) = f^{r(x)}(x)$ is the *first return map* to J. To get a nice structure for the first return map it is convenient to consider only the so called *nice* intervals: the forward orbit of the boundary of J does not meet the interior of J, [24]. In this case, if a component of the domain of the first return map does not contain the critical point it is mapped diffeomorphically onto J and if it contains the critical point it is folded into J and its boundary is mapped in one of the boundary points of J; this component is called the *central interval* of J and it is also a nice interval. If the domain of the first return map coincides with J we say that J is a *restrictive* interval and that f is *renormalizable*. Clearly, in this case, J is a periodic interval and two iterates of the interior of J either coincide or are pairwise disjoint. The first return maps is then conjugate by a Moebius transformation (or affine transformation if the original map is even) to a unimodal map. The simplest example of a nice interval is the interval $T_0 = [\alpha', \alpha]$ where $\alpha > 0$ is the orientation reversing fixed point of f and $f(\alpha') = \alpha$. It generates a sequence of nice intervals as follows: T_1 is the central interval associated to the first return map R_0 to T_0 and, inductively, T_{i+1} is the central interval associated to the first return map R_i to T_i. This sequence, which Lyubich in [13] calls *the principal nest*, is infinite, unless the critical point never returns to T_i for some i. We say that R_i is a *central return* if $R_i(0)$

belongs to T_{i+1}. If the number of non-central returns is finite, i.e, all the returns to T_i are central returns for $i \geq i_0$ then there exists an n such that $R_i(0) = f^n(0)$ for all $i \geq i_0$ and the intersection of the $T_i's$ is a restrictive interval of period n. Therefore, if the critical point is recurrent and non-periodic, then either the sequence of non-central returns is infinite or the mapping is renormalizable. For each i we can also consider the mapping L_i, called the *landing map*, that to each point not in T_i but whose positive orbit intersects T_i associates the first iterate that belongs to T_i. Clearly, the first return map R_i to T_i is the composition of f with the landing map L_i.

2.2. Cross-Ratio and Distortion Estimates

To describe the distortion properties and the geometry of the domain of the return maps the main tool, introduced in [28], is the control of the distortion of the cross-ratio of two nested intervals under diffeomorphic iterates. If M is an interval compactly contained in another interval T, the cross-ratio of the pair (T, M) is

$$C(T, M) = \frac{|T||M|}{|L||R|}$$

where $|T|$ denotes the length of the interval T and R, L are the components of $T \setminus M$. If $h: T \to \mathbb{R}$ is a diffeomorphism onto its image, the distortion of the cross-ratio is denoted by

$$C(h; T, M) = \frac{C(h(T), h(M))}{C(T, M)}.$$

The crucial fact is that if f is a C^2 unimodal map and $l > 0$ then there exists a positive constant C_l such that if $\sum_{i=0}^{n-1} |f^i(T)| \leq l$ and $f^n|T$ is a diffeomorphism, then $C(f^n; T, M) > C_l$. The constant C_l is equal to one for all l if f has negative Schwarzian derivative. In general it depends only on f and converges to 1 as $l \to 0$.

The relevance of this estimate is related to the *real Koebe Lemma*: there exists a positive constant $D = D(\tau, C)$ such that for any C^1 diffeomorphism $h: T \to \mathbb{R}$ such that $C(h; T', M') > C$ for every $M' \subset T' \subset T$ and also $C(h(T), h(M)) < \tau$ [1] then the distortion of f at the middle interval M, $D(f, M) = \sup_{x,y \in M}\{\frac{|Df(x)|}{|Df(y)|}\}$, is bounded by D. The constant D tends to 1 as $(\tau, C) \to (0, 1)$. This estimate from below of the distortion of the cross-ratio under iteration of a C^2 unimodal map is the main ingredient of the proof in [28] of the non-existence of wandering intervals [2] which

[1] This means that M is well inside T.

[2] The non-existence of wandering intervals was proved earlier for maps of negative Schwarzian derivative in [5]

implies that each component of the closure of the backward critical orbit is eventually mapped into a periodic interval whose return map is monotone. In particular if the critical point is recurrent but non-periodic and the map is non-renormalizable, then the lengths of the intervals in the principal nest converges to zero. Also if such map is infinitely renormalizable, the lengths of the restrictive intervals converge to zero. Another important consequence of this estimates obtained in [22] is that the number of such periodic intervals disjoint from the critical orbit is finite as is the period of non-hyperbolic periodic points. In particular a real analytic unimodal map has at most a finite number of non-hyperbolic and of attracting periodic points.

2.3. Hyperbolicity

Another important notion in dynamics is that of *hyperbolicity* which in one dimension is the following: a closed, forward f-invariant set Λ is *hyperbolic* if there exit constants $C > 0$ and $\lambda > 1$ such that

$$|Df^n(x)| \geq C\lambda^n, \forall x \in \Lambda, n \in \mathbb{N}.$$

An important criteria for hyperbolicity was established in [19] for C^2 maps (and earlier in [20] for maps with negative Schwarzian derivative): if Λ is closed, forward f-invariant, does not contain critical point and all periodic points in Λ are hyperbolic and repelling, then Λ is a hyperbolic set. It is not difficult to see that the set of parameter values in a non-trivial real analytic family corresponding to maps that have at least one non-hyperbolic periodic point is countable. Hence for almost all parameter values, any invariant set that does not contain the critical point is hyperbolic. In particular, for a unimodal map f with all periodic points hyperbolic, the so called *Kupka-Smale* maps, the set of points whose orbit does not intersect a neighborhood J of the critical point and are not in the basin of some attracting periodic point is hyperbolic.

A first important consequence of hyperbolicity is that if an interval T lies in a small neighborhood of a hyperbolic set as well as all its iterates up to $f^n(T)$ then $f^n(T)$ is exponentially bigger than T, the sum of the lengths of its iterates up to $f^n(T)$ is bounded by a constant (that depends only on C, λ), times the length of the last iterate and the distortion of f^n in T is bounded by a constant that depends only on the hyperbolicity constants. This implies, in particular, using Lebesgue density theorem, that every hyperbolic set has zero Lebesgue measure.

A second consequence of hyperbolicity is its persistence under C^2 perturbations of the mapping: if Λ is a hyperbolic set of f then there exists a neighborhood $\mathcal{N}$ of f and for each $g \in \mathcal{N}$ a hyperbolic set Λ_g of g and a continuous family of homeomorphisms $h_g\colon \Lambda \to \Lambda_g$ that conjugates f and g, $h_g \circ f = g \circ h_g$, see [29].

2.4. Non-recurrent critical orbits

If all periodic points of a unimodal map f are hyperbolic and the critical point is not recurrent then either the critical point is in the basin of a periodic attractor and the mapping is structurally stable under C^2 perturbations or the critical value $f(0)$ belongs to a hyperbolic set. The second situation is not persistent under perturbation, in fact, it was proved in [31] that the set of parameter values in a non-trivial real analytic family of mappings for which this situation occur has zero Lebesgue measure, see also [2].

2.5. Recurrent Critical Orbit

So we are led to consider maps with recurrent critical points and having all periodic points hyperbolic and also perturbations of such maps.

For C^3 unimodal maps with all periodic points hyperbolic one can combine the above two distortion estimates to give a lower bound estimate for the cross-ration distortion $C(f^n;T,J)$ that depends only on the length of the last iterate as long as it does not intersect the immediate basin of periodic attractor, [11]. Refining this idea it was proved in [6] that if f is real analytic, non-renormalizable with all periodic points hyperbolic then if i is big enough, there exists a neighborhood $\mathcal{N}$ of f in the C^3 topology and a real analytic diffeomorphism $h\colon I \to I$ such that the $i's$ interval T_i of the principal nest of f has a continuation $T_i(g)$ such that the first return map of $h \circ g \circ h^{-1}$ to $h(T_i(g))$ has negative Schwarzian derivative. Combining this with the finiteness of the periodic attractors one can extend all the results proved for maps with negative Schwarzian derivatives to slightly modified results for maps without this restriction. In particular we can state the following result for the decay of geometry.

Theorem 1. *Let f be a C^3 non-renormalizable map with all periodic points hyperbolic and recurrent critical point. Let T_i be the principal nest of f and $k(n)$ be the sequence of non-central returns. Then there exist constants $C > 0$ and $\lambda < 1$ such that*

$$\frac{|T_{k(n)}|}{|T_{k(n)-1}|} \leq C\lambda^n$$

The above theorem was proved by Lyubich in [13] for maps with negative Schwarzian derivatives using heavy results from holomorphic dynamics and was extended by Kozlowski to smooth maps using the arguments we discussed before. Recently Weixiao Shen gave a new and simple proof in [32] using only the real methods we have discussed.

Let now f be a map with recurrent critical points and at most a finite number of restrictive intervals. Let T_i be the principal next of the first return map to the smallest restrictive interval. Let J_1 denotes the domain of the first landing map of f to $J_0 = T_i$ that contains the critical value $f(0)$ and $J_2, J_3, \dots$ denote the other components. Clearly f maps each J_k diffeomorphically onto either J_0 or to another component $J_{l(k)}$ and there exists an integer $m(k)$ such that the restriction of the landing map to J_k coincides with $f^{m(j)}$ which maps an interval $\tilde{J}_k \supset J_k$ diffeomorphically onto T_{i-1}. Assuming furthermore that all periodic points of f are hyperbolic we get from the cross-ratio estimates we have discussed before that the non-linearity (which is the logarithmic of the distortion) of $f^{m(k)}$ in J_k is bounded by a constant times the scaling factor $\frac{|T_i|}{|T_{i-1}|}$ and hence it is very small at deep intervals of non-central return in the principal nest. Furthermore, the complement of the domain of the first landing map is the union of a hyperbolic set with some connected components of the basin of attracting periodic points. This structure persists in a neighborhood $\mathcal{N}$ of f: for each $g \in \mathcal{N}$ there is a homeomorphism h_g depending continuously on g such that $g^{m(k)}(h_g(J_k)) = h_g(J_0)$, $h_g(J_0)$ is the i's element of the principal nest of the return map of g to the corresponding restrictive interval.

A map $f \in \mathcal{U}^3$ is *quasiquadratic* if any nearby map $g \in \mathcal{U}^3$ is topologically conjugate to some quadratic map. We denote by $\mathcal{QQ} \subset \mathcal{U}^3$ the space of quasiquadratic maps. By the theory of Milnor-Thurston [21] and Guckenheimer [5], a map $f \in \mathcal{U}^3$ with negative Schwarzian derivative is quasiquadratic, so the quadratic family is contained in $\mathcal{QQ}$.

THEOREM 2 (Martens & Nowicki [25]). *Let f be a non-renormalizable quasiquadratic map and $\lambda_n = |T_n|/|T_{n-1}|$ be its scaling factors. If*

$$\sum \sqrt{\lambda_n} < \infty$$

then f is stochastic.

This result together with Theorem 1 implies the following combinatorial criterion:

THEOREM 3. *Let f be a mapping with recurrent critical point which is not infinitely renormalizable. Suppose that the first return map of f to its smallest restrictive interval is a quasiquadratic map and that all but finitely many levels in the principal nest of this quasiquadratic map are non-central. Then f is stochastic.*

So we are led to consider three subsets of real analytic maps: the set $\mathcal{IR}$ of infinitely renormalizable maps; the set $\mathcal{IC}$ of Kupka-Smale mappings that renormalizes to a quasiquadratic map with infinitely many intervals of central return and of non-central return in the principal nest and the set $\mathcal{NQ}$ of Kupka-Smale mappings whose renormalization to the smallest restrictive interval has a periodic attractor but the critical orbit is recurrent and non-periodic. The issue now is to prove that in a nontrivial family of real analytic maps the set of parameter values corresponding to maps in one of these three subsets has zero Lebesgue measure. For this we will need some very strong complex analytic tools.

In the special but fundamental case of the quadratic family the third set is empty and Lyubich proved in [16] that the first two subsets have Lebesgue measure zero by a strong geometric reason: each interval in the parameter space contains a sub-interval in the complement of these sets with comparable size. The main strategy in [1] is to prove that the parameter space of a non-trivial real analytic family of quasi-quadratic maps contains a closed and countable subset such that for each component of the complement there exists a quasi-symmetric [3] homeomorphism ϕ onto a sub-interval of $[\frac{1}{2}, 2]$ such that f_t is topologically conjugate to $q_{\phi(t)}$. Since the above geometric conditions are preserved by quasi-symmetric homeomorphisms the result remains true for the family f_t. This decomposition of the parameter space is accomplished by proving that some Banach manifolds of quasiquadratic maps decomposes in a union of connected codimension one submanifolds, each in the same conjugacy class, and that this decomposition has a lamination structure with quasi-symmetric holonomy almost everywhere. The result follows by proving that a non-trivial analytic family in such Banach manifolds is transversal to the leaves of this lamination except probably in a countable closed subset of the parameter space.

Let Ω_a be the set of points in the complex plane whose distance to the interval I is at most a. The set $\mathcal{H}_a$ of complex analytic maps on Ω_a having a continuous extension to the closure of Ω_a is a complex Banach space if endowed with the supp norm. The subset of real maps F, i.e, $\overline{F(z)} = F(\overline{z})$, is a real Banach space. The subset $\mathcal{U}_a$ of such mappings whose restriction to the interval I is unimodal is an open subset of an affine subspace together with the codimension one submanifold of *Ulam-Neumann* (or *Chebyshev*) maps f, i.e, $f(0) = 1$. Any real analytic family of unimodal maps is an analytic curve in $\mathcal{U}_a$ for some a. The *hybrid class* of a mapping $f \in \mathcal{U}_a$ is the set $\mathcal{H}(f)$ of mappings that are topologically conjugate to f and, if f is regular, has the same multiplier as f in the attracting periodic point. To each mapping f we associate the complex vector space $T_f(\mathcal{H}(f))$ of all complex holomorphic vector fields v on Ω_a that extend continuously to the boundary, that vanish, together with their first derivative, at 0 and such that there exists a quasiconformal vector field α [4] such that the equation

---

[3] A homeomorphism h is k-quasi-symmetric if for any consecutive intervals L, R of the same size, $\frac{1}{k} < \frac{|h(L)|}{|h(R)|} < k$

[4] A continuous vector field $v \equiv v(z)/dz$ on an open set $U \subset \overline{\mathbb{C}}$ is called K-quasiconformal if it has locally integrable distributional partial derivatives ∂v and $\bar{\partial} v$, and $\|\bar{\partial} v\|_\infty \leq K$.

$$v(z) = \alpha(f(z)) - f'(z)\alpha(z). \qquad (1)$$

is satisfied along the critical orbit. This is clearly a complex vector space. Using the implicit function theorem it is easy to prove that if some iterate of the critical point of f is a repelling periodic point then the hybrid class is a real analytic submanifold whose tangent space is the real slice of the above vector space. To prove that the hybrid classes define a lamination with quasisymmetric transversal holonomy in a neighborhood of an infinitely renormalizable map one uses the complex bounds of [33], [12], [17], the implicit function theorem and the lamination structure of germs of quadratic like maps established in [16].

Now suppose that the mapping $f \in \mathcal{IC}$ extends to a holomorphic mapping on Ω_a, the set of points in the complex plane whose distance to the interval I is at most a. Let T_i be a very deep interval in the principal nest of the return map to the smallest restrictive interval having a very big scaling factor and so that all the components of the domain of the landing map to T_i are much smaller than a. Starting with a round disk U_0 with diameter J_0, and using some elementary facts of hyperbolic geometry, we construct a family $U_k, k \geq 0$ of disk neighborhoods of J_k such that $f^{m(k)}$ maps U_k holomorphically onto U_0 and extends to a univalent mapping from $\tilde{U}_k \supset U_k$ onto a conformal disk $\tilde{U}_0 \supset U_0$ with the modulus of the annulus $\tilde{U}_0 \setminus \overline{U_0}$ very big. It follows that the closure of two disks U_k are disjoint and the image of U_0 is either disjoint of U_k or contains the closure of U_k and the complement is an annulus of big modulus. The restriction of f to the union of the conformal disks U_k is called in [1], *puzzle map* and the $U_k's$ are called *puzzle pieces*. Now this puzzle structure persists under perturbations of f in the space of complex analytic maps on Ω_a having a continuous extension to the closure of Ω_a which is a complex Banach space when endowed with the supp norm. This means that there exists a neighborhood $\mathcal{N}$ of f and for each $g \in \mathcal{N}$ a homeomorphism $H_g\colon \mathcal{N} \to \mathcal{N}$ such that $g^{m(k)}$ maps $H_g(U_k)$ onto $H_g(U_0)$ univalently and H_g conjugates f with g in the boundary of the puzzle pieces. Furthermore, for every $x \in \mathcal{N}$, the mapping $g \mapsto H_g(x)$ is holomorphic. This is what is called *a holomorphic motion* of the complex plane [5] Let $X_\lambda = h_\lambda X_*$. The construction of this holomorphic motions is done by observing firstly that since the set of boundary points of the puzzle pieces in the real line is a hyperbolic set for f it does move holomorphically in a neighborhood of f. We then construct by hand a holomorphic motion of a finite number of puzzle pieces. The remaining puzzle pieces are very

[5] Given a domain $\mathcal{V}$ in a complex Banach space E with a base point $*$ and a set $X_* \subset \mathbb{C}$, a *holomorphic motion* of X_* over $\mathcal{V}$ is a family of injections $h_\lambda : X_* \to \mathbb{C}$, $\lambda \in \mathcal{V}$, such that $h_* = \mathrm{Id}$ and $h_\lambda(z)$ is holomorphic in λ for any $z \in X_*$.

near this moving hyperbolic set and its motion is controlled by hyperbolicity. Then we use the fundamental theorem on extension of holomorphic motion of [34], [3]: a holomorphic motion of any set over a ball on a complex Banach space extends to a holomorphic motion of the whole plane over a smaller ball. Furthermore the corresponding homeomorphisms are quasi-conformal and the quasi-conformality constant is uniformly bounded if we shrink the neighborhood. Using the control on the distortion, an infinitesimal pull-back theorem in [1] and the non-existence of invariant line field in the filled Julia set of the puzzle map, one obtain the following key estimate.

$$\|\alpha\|_{\mathrm{qc}} \leq L\|v\|_a, \quad v \in T_g \tag{2}$$

where $\|\alpha\|_{\mathrm{qc}}$ is the supremum of the L^∞ norm of $\overline{\partial}\beta$ for all normalized quasiconformal vector fields that coincide with α at the critical orbit.

The final step of the proof of the lamination structure is the existence of a transversal vector, i.e, a holomorphic vector field that does not satisfy the equation 1. This is an infinitesimal C^1 closing-lemma type argument together with holomorphic approximation.

To extend the result beyond the set of quasiquadratic maps it is necessary to prove that the parameter values in a non trivial analytic family corresponding to maps in the set $\mathcal{NQ}$ has zero Lebesgue measure. In this case, since the mapping has a periodic attractor that does not attract the critical point, we cannot relate it to the quadratic family. The strategy in [2] is to construct a decomposition of the parameter space in neighborhoods of the original maps corresponding to some dynamical decomposition of the phase space that persists. To find bounds for the geometry of this decomposition the main tool is again the theory of holomorphic motions in the same spirit as in [15], where this strategy was used in the quadratic family.

REFERENCES

1. A. Avila, M. Lyubich and W. de Melo, *Regular or stochastic dynamics in real analytic families of unimodal maps*, Inventiones Mathematicae n.3, **154** (2003), 451–550.

2. A. Avila and C. G. Moreira, *Phase-Parameter relation and sharp statistical properties in general families of unimodal maps.* http://www.math.sunysb.edu/ artur/

3. L. Bers and H.L. Royden, *Holomorphic families of injections*, Acta Math. **157** (1986), 259–286.

4. A. Douady and J.H. Hubbard, *On the dynamics of polynomial-like maps*, Ann. Sc. Éc. Norm. Sup. **18** (1985), 287–343.

5. J. Guckenheimer, *Sensitive dependence to initial conditions for one-dimensional maps*, Comm. Math. Physics. **70** (1979), 133–160.

6. J. Graczyk, D. Sands and G. Swiatek, *Schwarzian derivative in unimodal dynamics*, C.R. Acad/ Sci. Paris **332** (2001), no. 4, 329–332.

7. F. Hofbauer and G. Keller, *Quadratic maps without asymptotic measure*, Comm. Math. Physics **127** (1990), 319–337.

8. M. Jacobson,*Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Comm. Math. Phys. **81** (1981), 39–88.

9. G. Keller, *Exponents, attractors, and Hopf decompositions for interval maps*, Erg. Th. & Dyn. Syst. **10** (1990), 717–744.

10. O.S. Kozlovsky, *Axiom A maps are dense in the space of unimodal maps in the C^k topology*, Ann. of Math. (2) **157**, no. 1, (2003) 1–43.

11. O.S. Kozlovsky, *Getting Rid of the negative Schwarzian derivative condition*, Ann. Math. **152** (2000), 743–762.

12. G. Levin and S. van Strien, *Local connectivity of Julia sets of real polynomials*, Ann. of Math. **147** (1998), 471–541.

13. M. Lyubich, *Combinatorics, geometry and attractors of quasi-quadratic maps*, Ann. Math **140** (1994), 347–404.

14. M. Lyubich, *Dynamics of quadratic polynomials, I-II*, Acta Math. **178** (1997), 185–297.

15. M. Lyubich, *Dynamics of quadratic polynomials, III. Parapuzzle and SBR measure*, Asterisque **261** (2000), 173–200.

16. M. Lyubich, *Feigenbaum-Coullet-Tresser Universality and Milnor's Hairiness Conjecture*, Ann. Math. **149** (1999), 319–420.

17. M. Lyubich and M. Yampolsky, *Dynamics of quadratic polynomials: Complex bounds for real maps*, Ann. Inst. Fourier **47** (1997), 1219–1255.

18. G. Levin and S. van Strien, *Local connectivity of Julia sets of real polynomials*, Annals of Math. **147** (1998), 471–541.

19. R. Mañé, *Hyperbolicity, sinks and measures for one-dimensional dynamics*, Comm. Math. Phys. **100** (1985), 495–524.

20. M. Misiurewicz, *Absolutely continuous measures for certain maps of an interval*, Publ. Math. I.H.E.S **53** (1981), 17–51.

21. J. Milnor and W. Thurston, *On iterated maps of an interval I, II*, Springer Lecture Notes in Math. vol. 1342, (1988), 465–563.

22. M. Martens, W. de Melo and S. van Strien, *Julia-Fatou-Sullivan theory for real one dimensional dynamics*, Acta Math. **168** (1992), 273–318.

23. R. Mañé, P. Sad and D. Sullivan, *On the dynamics of rational maps*, Ann. scient. Ec. Norm. Sup. **16** (1983), 193–217.

24. M. Martens, *Distortion results and invariant Cantor sets for unimodal maps*, Erg. Th. & Dyn. Syst. **14** (1994), 331–349.

25. M. Martens and T. Nowicki, *Invariant measures for Lebesgue typical quadratic maps*, Asterisque **261** (2000) 239–252.

26. C. McMullen, *Complex dynamics and renormalization*, Annals of Math. Studies **135**, Princeton University Press, 1994.

27. C. McMullen, *Renormalization and 3-manifolds which fiber over the circle*, Annals of Math. Studies **142**, Princeton University Press, 1996.

28. W. de Melo and S. van Strien, *A structure theorem in one- dimensional dynamics*, Ann. Math. **129** (1989), 519–546.

29. W. de Melo and S. van Strien, *One-dimensional dynamics*, Springer, 1993.

30. D. Singer, *Stable orbits and bifurcations of maps of the interval*, SIAM J. App.Math. **35** (1978), 260–267.

31. D. Sands, *Misiurewwixz maps are rare*, Comm. Math. Phys. **197**, no.1, , (1998) 109–129.

32. W. Shen, *Decay of Geometry for Unimodal maps: an elementary proof.* http://www.maths.warwick.ac.uk/staff/wxshen.html

33. D. Sullivan, *Bounds, quadratic differentials, and renormalization conjectures*, AMS Centennial Publications **2**: Mathematics into Twenty-first Century (1992).

34. D. Sullivan and W. Thurston, *Extending holomorphic motions*, Acta Math. **157** (1986), 243–257.

35. Z. Slodkowsky, *Holomorphic motions and polynomial hulls*, Proc. Amer. Math. Soc. **111** (1991), 347–355.