QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **4**, 181–203 (2004)
ARTICLE NO. 60

Dynamical and Topological Aspects of Lyapunov Graphs

Maria Alice Bertolim*

Instituto de Matemática, Estatística e Computação Científica, Unicamp
Campinas, SP, Brazil
E-mail: bertolim@ime.unicamp.br

and

Margarida Pinheiro Mello†

Instituto de Matemática, Estatística e Computação Científica, Unicamp
Campinas, SP, Brazil
E-mail: margarid@ime.unicamp.br

and

Ketty Abaroa de Rezende‡

Instituto de Matemática, Estatística e Computação Científica, Unicamp
Campinas, SP, Brazil
E-mail: ketty@ime.unicamp.br

Submitted: June 12, 2003 *Accepted*: July 5, 2004

Dedicated to Professor Sotomayor 60^{th} birthday

In this survey we present the interplay between topological dynamical systems theory with network flow theory in order to obtain a continuation result for abstract Lyapunov graphs $L(h_0, \ldots, h_n, \kappa)$ in dimension n with cycle number κ. We also show that an abstract Lyapunov graph satisfies the Poincaré-Hopf inequalities if and only if it satisfies the Morse inequalities and the first Betti number γ_1 is greater than or equal to κ. We define the Morse polytope determined by the Morse inequalities and describe some of its geometrical properties.

Key Words: Lyapunov graph, Morse inequalities, Poincaré-Hopf inequalities, Conley index, network-flow theory, integral polytope.

* Supported by FAPESP under grant 02/08400-3.
† Supported by CNPq-PRONEX Optimization.
‡ Partially supported by FAPESP under grant 00/05385-8, 02/102462 and CNPq under grant 300072.

1. INTRODUCTION

It is well known that the interaction among different areas very often produces interesting results. In this survey we use network flow theory techniques to obtain topological dynamical system results for Lyapunov graphs. This article is written in a discursive style and it is a compendium of [1], [2] and [3], where the results collected below are proved in all dimensions. As often is the case in this area, odd, $0 \mod 4$ and $2 \mod 4$ cases need to be analyzed separately. The big picture is the same but each case entails the consideration of myriad smaller details. In order to spare the reader the tiresome repetition of similar arguments we chose to illustrate the results in the simpler odd dimensional case.

Lyapunov graphs carry dynamical information of gradient-like flows as well as topological information of its phase space which is taken to be a closed n-manifold. See Section 2. The main results concern dynamical and topological properties of Lyapunov graphs.

The first result below mimics the continuation results for gradient flows, [11], with a more constructive approach and uses completely different techniques in the proof. Roughly, an abstract Lyapunov graph may represent a very complicated gradient-like flow and Theorem 1 continues it to an abstract Lyapunov graph of Morse type which in turn represents a Morse flow.

Theorem 1 (Continuation). *Consider an abstract Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$. It admits continuations to abstract Lyapunov graphs of Morse type with cycle rank greater or equal to κ if and only if it satisfies the Poincaré-Hopf inequalities* (1)–(2) *at each vertex, where $\kappa \leq \min\{h_1 - (h_0 - 1), h_{n-1} - (h_n - 1)\}$. Moreover, the number of possible continuations is obtained.*

The second result partially addresses a difficult question which has been answered in some cases, which is that of the realization of an abstract Lyapunov graph as a flow on some manifold. Morse's classical result asserts that if an abstract dynamical data list does not satisfy the Morse inequalities for a closed manifold M then there exists no flow on M with this data. It is important to observe that an abstract Lyapunov graph carries dynamical data but carries no information of the manifold it is realizable on. In other words, the Morse inequalities cannot be used to verify the realizability or not of this dynamical data. We present the Poincaré-Hopf inequalities for closed manifolds which are verifiable for the dynamical data of an abstract Lyapunov graph. As a consequence of the theorem below, if the graph does not satisfy the Poincaré-Hopf inequalities, it will not satisfy the Morse inequalities (3) for any choice of Betti number vector and hence we screen out Lyapunov graphs which cannot be realized as a continuous flow on any manifold.

THEOREM 2 (Equivalence). *Given an abstract Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$, it satisfies the Poincaré-Hopf inequalities for closed manifolds if and only if it satisfies the Morse inequalities (3) and the inequality $\gamma_1 \geq \kappa$ for some Betti number vector $(\gamma_0, \ldots, \gamma_n)$.*

This article is divided in the following sections. Section 2 is background material. Section 3 will present the Poincaré-Hopf inequalities with connectivity parameter κ and the Morse inequalities. Section 4 presents the explosion algorithm with cycles which originates a linear h_κ^{cd}-system. The nonnegative solutions of this system correspond to the different continuations of the graph. In Section 5, we sketch a proof of the continuation theorem 1 for abstract Lyapunov graphs in the presence of cycles using network flow theory. In Section 6 we show the main techniques involved in the proof of the equivalence theorem. In Section 7 we describe properties of the Morse polytope considering the connectivity parameter κ.

2. LYAPUNOV GRAPHS, SEMI-GRAPHS AND ISOLATING BLOCKS

It is easy to see that there is a natural connection between a closed manifold and a Lyapunov graph. To establish a correspondence with an isolating block we need to define a Lyapunov semi-graph. We extend the notion of a directed graph to allow for a distinguished vertex, which we will denote by ∞.

Given a finite set V we define a *directed semi-graph* $G' = (V', E')$ as a pair of sets $V' = V \cup \{\infty\}$, $E' \subset V' \times V'$. As usual, we call the elements of V' vertices and since we regard the elements of E' as ordered pairs, these are called *directed edges*. Furthermore the edges of the form (∞, v) and (v, ∞) are called *semi-edges* (or dangling edges as in [8]). Note that whenever G' does not contain semi-edges G' is a graph in the usual sense. The graphical representation of the graph will have the semi-edges cut short.

A Lyapunov semi-graph L_N, consisting of one vertex labelled with the dimensions of the Conley homology indices of an isolated invariant set Λ and entering and exiting labelled semi-edges can be associated to an isolating block N for Λ with entering set for the flow N^+ and exiting set for the flow N^-. The number of incoming (outgoing) edges e^+ (e^-) correspond to the number of connected components of N^+ (N^-). The labels on the edges correspond to the Betti numbers of the closed codimension one submanifolds N^+ and N^-.

A singularity, respectively a vertex, labelled with $h_\ell = 1$ is ℓ-d if it has the algebraic effect of increasing the ℓ-th Betti number of N^+ or respectively, the corresponding β_ℓ label on the incoming edge. A singularity, respectively a vertex labelled with $h_\ell = 1$ is $(\ell-1)$-c if it has the algebraic effect of decreasing the $(\ell-1)$-th Betti number of N^+ or respectively, the corresponding $\beta_{\ell-1}$ label on the incoming edge. In the case $n = 2i = 0$ mod 4, a singularity, respectively a vertex labelled with $h_i = 1$ is β-i, if all Betti numbers are kept constant. See the corresponding semi-graphs in Figure 1. See [6].

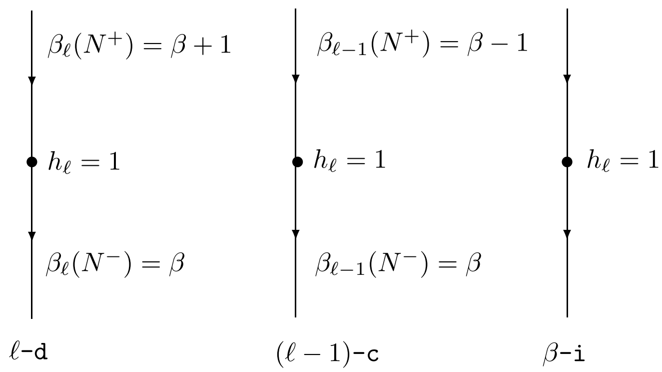


FIG. 1. The three possible algebraic effects.

Lyapunov graphs were initially introduced by Franks in [9]. A Lyapunov graph is defined by the following equivalence relation on M: $x \sim_f y$ if and only if x and y belong to the same connected component of a level set of f, where $f : M \to \mathbb{R}$ is a Lyapunov function. We call $M/\sim_f$ a *Lyapunov graph*. Each vertex v_k represents the component R_k of the chain recurrent set R and hence can be labelled with dynamical invariants. Each edge represents a level set times an interval and hence can be labelled with topological invariants of the level set.

An *abstract Lyapunov graph in dimension n* is a finite, connected, oriented graph, that has no oriented cycles. Also, each vertex is labelled with a chain recurrent flow on a compact n-dimensional space and each edge is labelled with topological invariants of a closed $(n-1)$-dimensional manifold. In this article we choose to label the vertex v_k of an abstract Lyapunov graph with the dimensions of the Conley homology indices, $\dim CH_j(R_k) = h_j(v_k)$, with $j = 0, \dots n$.

Each vertex is labelled with a list of nonnegative integers $(h_0(v_k), \dots, h_n(v_k), \kappa(v_k))$, where $\kappa(v_k)$ is the *cycle number of the vertex* v_k, which is a nonnegative integer weight on v_k. An alternative notation is to label the vertex with $h_j(v_k) = n_j$ whenever $n_j \neq 0$. And $\kappa(v_j) = k_j$ whenever $k_j \neq$

0. This latter notation is convenient whenever $(h_0(v_k),\ldots,h_n(v_k),\kappa(v_k))$ has many zero entries.

We label the edges with the Betti numbers of a closed $(n-1)$-dimensional manifold, a Betti number vector. A *Betti number vector* is a list of non-negative integers $(\gamma_0,\gamma_1,\ldots,\gamma_{n-1},\gamma_n)$, where $\gamma_{n-k}=\gamma_k$, $\gamma_0=\gamma_n=1$ and $\gamma_{n/2}$ is even if n is even.

Given an abstract Lyapunov graph L with vertex set V and cycle rank[1] κ_L, we will denote it by $L(h_0,\ldots,h_n,\kappa)$ where $h_j=\sum_{v_k\in V} h_j(v_k)$ and $\kappa_V=\sum_{v_k\in V}\kappa(v_k)$ and $\kappa=\kappa_L+\kappa_V$. We will refer to κ as the *cycle number of the graph.* This definition is easily extended to Lyapunov semi-graphs.

An *abstract Lyapunov graph of Morse type* is an abstract Lyapunov graph that satisfies the following:

1. every vertex is labelled with $h_j=1$ for some $j=0,\ldots,n$ and the cycle number of each vertex equal to zero.

2. the number of incoming edges, e^+, and the number of outgoing edges, e^-, of a vertex must satisfy:

(i) if $h_j=1$ for $j\neq 0,1,n-1,n$ then $e^+=1$ and $e^-=1$;

(ii) if $h_1=1$ then $e^+=1$ and $0<e^-\leq 2$; if $h_{n-1}=1$ then $e^-=1$ and $0<e^+\leq 2$;

(iii) if $h_0=1$ then $e^-=0$ and $e^+=1$; if $h_n=1$ then $e^+=0$ and $e^-=1$.

3. every vertex labelled with $h_\ell=1$ must be of type ℓ-d or $(\ell-1)$-c. Furthermore if $n=2i=0 \mod 4$ and $h_i=1$ then v may be labelled with β-i.

It is easy to see that the cycle number of an abstract Lyapunov graph of Morse type is equal to its cycle rank.

In order to define continuation of abstract Lyapunov graphs we introduce the notion of *vertex explosion.* Let v be a vertex on an abstract Lyapunov graph labelled with $(h_0(v),h_1(v)\ldots,h_n(v),\kappa_v)$. A vertex v can be exploded if v can be removed and replaced by an abstract Lyapunov graph I of Morse type with cycle rank greater or equal to κ_v. The graph I must respect the orientations and labels of the incoming and outgoing edges of v. In other words, the new graph obtained must be oriented and with cycle number greater or equal to κ_v. The incoming (outgoing) edges of v, must be incoming (outgoing) edges on vertices of I and all labels on the edges

[1] The *cycle rank* of a graph is the maximum number of edges that can be removed without disconnecting the graph.

must respect the restrictions of the Morse type vertices. Moreover,

$$h_\lambda(v) = \sum_{j=1}^{k} h_\lambda(v_j), \quad \text{for } \lambda = 1, \dots, n-1, \text{ where } v_j \in I.$$

An abstract Lyapunov graph $L(h_0, \dots, h_n, \kappa)$ admits a *continuation* to an abstract Lyapunov graph of Morse type L_M if each vertex can be exploded such that L_M has cycle rank greater or equal to κ.

Given an abstract Lyapunov graph $L(h_0, \dots, h_n, \kappa)$ with cycle number equal to κ, one can define a *graph implosion of* L as an abstract Lyapunov graph L_C with:

1. one saddle type vertex ν labelled with $(h_1(\nu), \dots, h_{n-1}(\nu), \kappa)$ where

$$\sum_{v \in V} h_\lambda(v) = h_\lambda(\nu),$$

where V is the set of vertices of L;

2. the vertex ν will have $\sum_{v \in V} h_n(v) = e^+$, incoming edges and will have $\sum_{v \in V} h_0(v) = e^-$ outgoing edges;

3. the incoming edges of ν are outgoing edges of e^+ vertices labelled with $h_n = 1$ and the outgoing edges of ν are incoming edges of e^- vertices labelled with $h_0 = 1$;

4. the labels of all the edges satisfy $B_j^+ = B_j^- = 0$, for all $j \neq 0, n-1$, and $B_j^+ = B_j^- = 1$, for $j = 0, n-1$.

3. POINCARÉ-HOPF INEQUALITIES WITH CONNECTIVITY PARAMETER κ FOR ISOLATING BLOCKS

In [1] we consider the Poincaré-Hopf inequalities (1)–(2), in the case $\kappa = 0$ for an isolated invariant set Λ with isolating block N, with entering set for the flow N^+ and exiting set for the flow N^-. These inequalities were obtained from the analysis of long exact sequences of the index pairs (N, N^-) and (N, N^+), for Λ and for the isolated invariant set of the reverse flow, Λ', where $\operatorname{rank} H_i(N, N^-) = h_i$, $\operatorname{rank} H_i(N, N^+) = h_{n-i}$, (see [10]), $\operatorname{rank} H_0(N^-) = e^-$, $\operatorname{rank} H_0(N^+) = e^+$, $\operatorname{rank} H_0(N) = 1$ and $\operatorname{rank}(H_i(N^\pm)) = B_i^\pm$.

In [3] we consider these inequalities in the presence of a parameter κ for an isolating block N. Thus, the Poincaré-Hopf inequalities for isolating blocks with this parameter will be the collection of the following constraints (1)–(2).

$$
\left\{
\begin{array}{l}
\left\{
\begin{array}{l}
h_j \geq -(B^+_{j-1} - B^-_{j-1}) + (B^+_{j-2} - B^-_{j-2}) + - \dots \\
\qquad \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \\
\qquad -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \dots \\
\qquad \pm (h_{n-1} - h_1) \pm [(h_n - h_0) + (e^+ - e^-)] \\
\\
h_{n-j} \geq - \left[-(B^+_{j-1} - B^-_{j-1}) + (B^+_{j-2} - B^-_{j-2}) + - \dots \right. \\
\qquad \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \\
\qquad -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \dots \\
\qquad \left. \pm (h_{n-1} - h_1) \pm [(h_n - h_0) + (e^+ - e^-)] \right]
\end{array}
\right. \\
\qquad \vdots \\
\left\{
\begin{array}{l}
h_2 \geq -(B^+_1 - B^-_1) - (h_{n-1} - h_1) + (h_n - h_0) + (e^+ - e^-) \\
h_{n-2} \geq - \left[-(B^+_1 - B^-_1) - (h_{n-1} - h_1) + (h_n - h_0) + (e^+ - e^-) \right]
\end{array}
\right. \\
\\
\left\{
\begin{array}{l}
h_1 \geq h_0 - 1 + e^- + \kappa \\
h_{n-1} \geq h_n - 1 + e^+ + \kappa
\end{array}
\right.
\end{array}
\right.
\tag{1}
$$

Furthermore, the Poincaré-Hopf equality must be considered in the odd-dimensional case $n = 2i + 1$:

$$
\mathcal{B}^+ - \mathcal{B}^- = e^- - e^+ + \sum_{j=0}^{2i+1} (-1)^j h_j \tag{2}
$$

where

$$
\mathcal{B}^+ = \frac{(-1)^i}{2} B^+_i \pm B^+_{i-1} \pm \dots - B^+_1
$$

$$
\mathcal{B}^- = \frac{(-1)^i}{2} B^-_i \pm B^-_{i-1} \pm \dots - B^-_1
$$

In [2] we consider a particular case of the Poincaré-Hopf inequalities for isolating blocks (1)–(2), which are the Poincaré-Hopf inequalities for closed manifolds.

Given a Lyapunov graph $L(h_0, \dots, h_n, \kappa)$ with cycle number equal to κ, the implosion L_C of $L(h_0, \dots, h_n, \kappa)$ is a graph with only one saddle vertex ν labelled with $(0, h_1, \dots, h_{n-1}, 0, \kappa)$ and has the following properties: $e^+ = h_n$, $e^- = h_0$, $B^-_j = B^+_j = 0$. By substituting this information of the vertex ν of L_C in the Poincaré-Hopf inequalities for isolating blocks (1)–(2), the Poincaré-Hopf inequalities for closed manifolds are obtained.

We say that a Lyapunov graph (respectively, a Lyapunov semi-graph) $L(h_0, \dots, h_n, \kappa)$ satisfies the Poincaré-Hopf inequalities if the data $(h_0, \dots,$

$h_n, \kappa)$ satisfies the Poincaré-Hopf inequalities for closed manifolds (respectively, (1)–(2)).

We now present the generalized Morse inequalities which appear in the equivalence theorem. The proof can be found in [4] where γ_i is the i-th Betti number of M and h_i is the dimension of the i-th Conley homology index as defined previously.

$$
\begin{array}{rcll}
\gamma_n - \gamma_{n-1} + - \ldots \pm \gamma_1 \pm \gamma_0 & = & h_n - h_{n-1} + - \ldots \pm h_1 \pm h_0 & \text{(n)} \\
\gamma_{n-1} - \gamma_{n-2} + - \ldots \pm \gamma_1 \pm \gamma_0 & \leq & h_{n-1} - h_{n-2} + - \ldots \pm h_1 \pm h_0 & \text{(n} - 1) \\
\vdots & & \vdots & \\
\gamma_j - \gamma_{j-1} + - \ldots \pm \gamma_1 \pm \gamma_0 & \leq & h_j - h_{j-1} + - \ldots \pm h_1 \pm h_0 & \text{(j)} \\
\gamma_{j-1} - \gamma_{j-2} + - \ldots \pm \gamma_1 \pm \gamma_0 & \leq & h_{j-1} - h_{j-2} + - \ldots \pm h_1 \pm h_0 & \text{(j} - 1) \\
\vdots & & \vdots & \\
\gamma_2 - \gamma_1 + \gamma_0 & \leq & h_2 - h_1 + h_0 & (2) \\
\gamma_1 - \gamma_0 & \leq & h_1 - h_0 & (1) \\
\gamma_0 & \leq & h_0 & (0)
\end{array}
\tag{3}
$$

4. EXPLOSION ALGORITHM IN THE PRESENCE OF CYCLES

In this section we present an explosion algorithm of a vertex on an abstract Lyapunov semi-graph in order to continue it to an abstract Lyapunov semi-graph of Morse type.

A vertex on an abstract Lyapunov graph labelled with $(h_0, h_1, \ldots, h_n, \kappa)$ is a *repeller vertex* if it has indegree zero and $h_n > 0$ (respectively, is an *attractor vertex* if it has outdegree zero and $h_0 > 0$). Otherwise, a vertex with positive indegree and outdegree is a *generalized saddle vertex*. A particular case of this occurs when the vertex is labelled with $(h_0 = 0, h_1, \ldots, h_n = 0, \kappa)$ and will be called a *saddle vertex*.

Our explosion algorithm consists of initially performing a partial explosion on a vertex which reduces it to a saddle type vertex. This explosion is possible if the last two inequalities in (1) are satisfied. Since this part of the algorithm is less relevant in the subsequent development of the material in this article we refer the reader to [1] and [3].

Without loss of generality we assume our vertex to be of saddle type labelled with $(0, h_1, \ldots, h_{n-1}, 0, \kappa)$.

4.1. Explosion of a saddle type vertex in the presence of cycles

Let v be a saddle type vertex labelled with $(0, h_1, h_2, \ldots, h_{n-1}, 0, \kappa)$ and incoming edges labelled with $((\beta_0^+, \ldots, \beta_{n-1}^+)_i)_{i=1}^{e^+}$ and outgoing edges labelled with $((\beta_0^-, \ldots, \beta_{n-1}^-)_i)_{i=1}^{e^-}$, where i denotes the edge. Let $B_j^+ = \sum_{i=1}^{e^+}(\beta_j^+)_i$ and $B_j^- = \sum_{i=1}^{e^-}(\beta_j^-)_i$. See Figure 2. Observe that $B_0^- = e^-$ e $B_0^+ = e^+$.

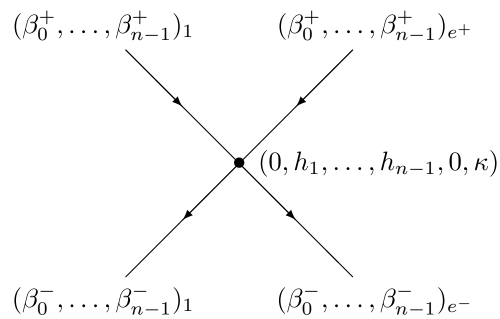


FIG. 2. Vertex to be exploded.

4.1.1. Adjusting the incident edges

In this step we wish to define G^+ and G^-.

Choose $e^- - 1$ vertices labelled with $h_1 = 1$ of type 0-c. This is possible by the last inequality in (1). By choosing this number of vertices labelled with 1-singularities, G^- is formed with e^- outgoing edges and one incoming edge. Singularities of type 0-c do not alter the β_i with $0 < i < n-1$. This type of singularity decreases β_0 and by duality β_{n-1}. Hence, the incoming edge of G^- has $B_0^- = B_{n-1}^- = 1$ and $B_j^- = \sum_{i=1}^{e^-}(\beta_j^-)_i$ with $j \in \{1, \ldots, n-2\}$. See Figure 4. Similarly, the graph G^+ is formed by choosing $e^+ - 1$ vertices labelled with $h_{n-1} = 1$ of type $n-1$-d.

4.1.2. The insertion of cycles

An *elementary cycle* is a pair of (h_1^c, h_{n-1}^d) with one edge labelled with $(1, 0, \ldots, 0, 1)$ and the other edge labelled with $(1, \beta_1, \ldots, \beta_{n-2}, 1)$. See Figure 3.

Without loss of generality, attach to G^-, κ elementary cycles where $(1, \beta_1, \ldots, \beta_{n-2}, 1) = (1, B_1^-, \ldots, B_{n-2}^-, 1)$. Of course this attachment can also be done to G^+. It is clear that once κ cycles are inserted the number of vertices labelled with $h_1 = 1$ of type 0-c is greater or equal to κ. Similarly, the number of vertices labelled with $h_{n-1} = 1$ of type $n-1$-d is greater or equal to κ.

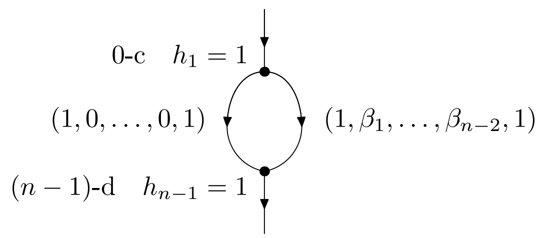


FIG. 3. (h_1^c, h_{n-1}^d) pair

Hence all together we have inserted $h_1^c = \kappa + e^- - 1$ vertices labelled with $h_1 = 1$ of type 0-c. Similarly, we have inserted $h_{n-1}^d = \kappa + e^+ - 1$ vertices labelled with $h_{n-1} = 1$ of type $(n-1)$-d. This is possible due to the last two inequalities in (1), which asserts that for this saddle vertex v $(h_0 = h_n = 0)$:

$$\begin{cases} h_1 \geq -1 + e^- + \kappa \\ h_{n-1} \geq -1 + e^+ + \kappa \end{cases} \tag{4}$$

However, more cycles can appear in the explosion.

The last cycle inserted has incoming edge labelled with $(1, B_1^-, B_2^-, \dots, B_{mid}^-, \dots, B_{n-2}^-, 1)$.

4.1.3. The linear explosion without middle dimensions (Step 1, …, Step $\boldsymbol{\ell}$*)*

This is done by an induction argument. For more details see [1].

Assume that the adjustments of B_j for $j < \ell$ and by duality B_{n-j-1} for $j < \ell$ have been made in increasing order for j. Hence, several linear graphs have been added to G^+ forming at this point a graph $G^+ \cup \displaystyle\bigcup_{i=1}^{\ell-1} L_i^+$ whose outgoing edge is labelled with

$$(1, B_1, \dots, B_{\ell-1}, B_\ell^+, \dots, B_{mid}^+, \dots, B_{n-\ell-1}^+, B_{n-\ell}, \dots, B_{n-2}, 1).$$

Similarly, several linear graphs have been added to G^- forming at this point a linear graph $G^- \cup \displaystyle\bigcup_{i=1}^{\ell-1} L_i^-$ whose incoming edge is labelled with

$$(1, B_1, \dots, B_{\ell-1}, B_\ell^-, \dots, B_{mid}^-, \dots, B_{n-\ell-1}^-, B_{n-\ell}, \dots, B_{n-2}, 1).$$

In order to adjust B_ℓ and by duality $B_{n-\ell-1}$ add to the graph above L_ℓ^- formed with h_ℓ^d vertices $h_\ell = 1$ of type ℓ-d, and $h_{\ell+1}^c$ vertices $h_{\ell+1} = 1$

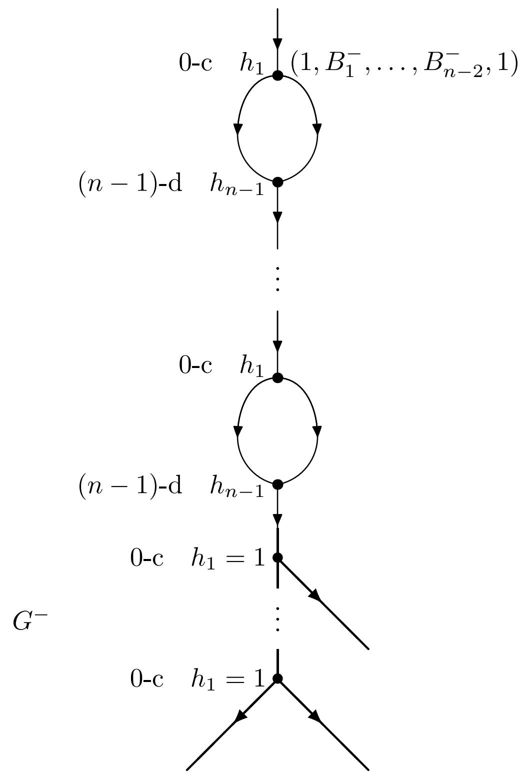


FIG. 4. Outgoing edges exploded and cycles inserted.

of type ℓ-c forming $G^- \cup \bigcup_{i=1}^{\ell} L_i^-$. Similarly, the insertion of $h_{n-\ell}^c$ vertices $h_{n-\ell} = 1$ of type $(n-\ell-1)$-c and the insertion of $h_{n-\ell-1}^d$ vertices $h_{n-\ell-1} = 1$ of type $(n-\ell-1)$-d will form $G^+ \cup \bigcup_{i=1}^{\ell} L_i^+$.

Since the insertion of any other type of vertex will not alter the ℓ-th and the $(n-\ell-1)$-th Betti number it is necessary that

$$B_\ell = B_\ell^- + h_\ell^d - h_{\ell+1}^c = B_\ell^+ - h_{n-\ell-1}^d + h_{n-\ell}^c. \tag{5}$$

The labels of B_ℓ and $B_{n-\ell-1}$ for $0 < \ell < mid$ have all been adjusted. It remains to adjust the middle dimensional labels. This is done in the next step.

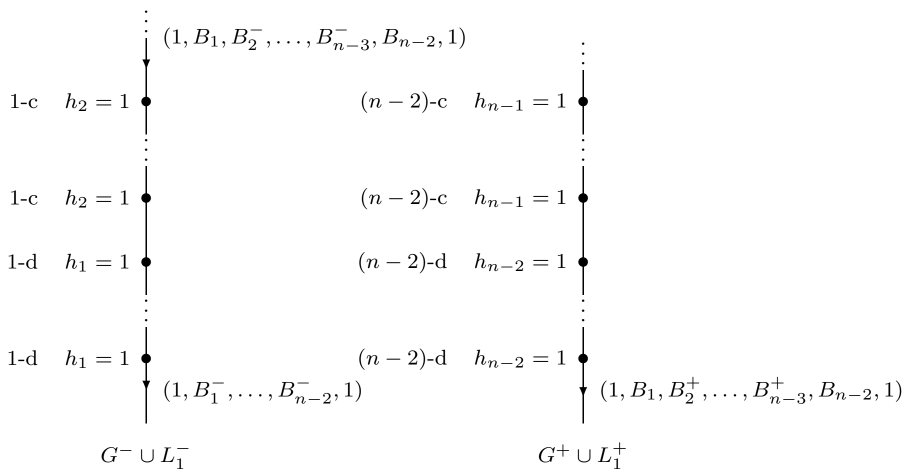


FIG. 5. Linear explosion: L^- and L^+.

4.1.4. Middle dimensional explosion

Since $n-1$ is even, there is only one middle dimensional label $B_{\frac{n-1}{2}}$ and in this case the insertion of the vertices alters the labels as in,

$$B_i = B_i^- + 2h_i^d = B_i^+ + 2h_{i+1}^c. \tag{6}$$

4.1.5. $\boldsymbol{h_\kappa^{cd}}$-Systems

All these adjustments are recorded in the following h^{cd}_{κ}-system which describes the explosion of a saddle vertex labelled with $(h_0 = 0, h_1, \ldots, h_{n-1}, h_n = 0, \kappa)$. Hence, this linear system of equations must be solved for $(h^c_1, h^d_1, \ldots,\ \beta_i,\ \ldots,\ h^c_{2i}, h^d_{2i-1}, \kappa)$ in order for the saddle explosion algorithm to work.

$$n = 2i+1 \left\{ \begin{array}{l} e^- - 1 - h^c_1 + \kappa = 0 \\ \{h_j = h^c_j + h^d_j,\ \ j = 1, \ldots, 2i \\ e^+ - 1 - h^d_{2i} + \kappa = 0 \\ \left\{ \begin{array}{l} -(B^+_1 - B^-_1) + h^d_1 - h^c_2 - h^c_{2i} + h^d_{2i-1} = 0 \\ -(B^+_2 - B^-_2) + h^d_2 - h^c_3 - h^c_{2i-1} + h^d_{2i-2} = 0 \\ \vdots \\ \frac{-(B^+_i - B^-_i)}{2} + h^d_i - h^c_{i+1} = 0 \end{array} \right. \end{array} \right. \tag{7}$$

5. CONTINUATION RESULTS IN THE PRESENCE OF CYCLES

The explosion algorithm of last section gives rise to a linear system that provides an intermediate stepping stone for the proof of the Continuation theorem 1. Proposition 3 establishes the equivalence between the existence of a continuation of the abstract Lyapunov graph and the existence of integral nonnegative solution(s) to the h^{cd}_{κ}-system (7). The next result, Proposition 5 states that the h^{cd}_{κ}-system has integral nonnegative solution(s) if and only if the same is true for the Poincaré-Hopf inequalities.

PROPOSITION 3. *A saddle vertex v labelled with $(0, h_1, \ldots, h_{n-1}, 0, \kappa)$ can be exploded to a Lyapunov semi-graph of Morse type with cycle rank greater or equal to κ and $\kappa \leq \min\{h_1 - (e^- - 1), h_{n-1} - (e^+ - 1)\}$, if and only if the appropriate h^{cd}_{κ} system* (7) *has a nonnegative integer solution* $(h^c_1, h^d_1, \ldots, h^c_{n-1}, h^d_{n-1})$.

Proof. This follows directly from the fact that the steps in the saddle explosion algorithm are described by the h^{cd}_{κ} systems. ∎

In order to use results already established in [1], we reproduce in (8) below the generic form of one of the linear systems considered therein:

$$n = 2i+1 \begin{cases} h_1^c = b_0 \\ \{h_j^c + h_j^d = b_j, \quad j = 1, \dots, 2i \\ h_{2i}^d = b_{2i+1} \\ \begin{cases} h_1^d - h_2^c - h_{2i}^c + h_{2i-1}^d = \delta_1 \\ h_2^d - h_3^c - h_{2i-1}^c + h_{2i-2}^d = \delta_2 \\ \vdots \\ h_i^d - h_{i+1}^c = \delta_i \end{cases} \end{cases} \tag{8}$$

The special case (7) may be obtained with the following substitutions:

$$n = 2i+1 \begin{cases} b_0 &= -1 + e^- + \kappa \\ b_j &= h_j, \quad \text{for } j = 1, \dots, 2i \\ b_{2i+1} &= -1 + e^+ + \kappa \\ \delta_j &= B_j^+ - B_j^-, \quad \text{for } j = 1, \dots, i-1 \\ \delta_i &= (B_i^+ - B_i^-)/2 \end{cases} \tag{9}$$

It was shown in [1] that system (8) may be recast as a network-flow problem by a suitable change of sign of half of the equations. The h^{cd} variables are interpreted as flows on the arcs of the network and each equation may be read as "flow in − flow out = node constant". The network corresponding to the case $n = 2i+1$, i odd, is depicted in Figure 6, with the node constant shown inside the node. In the planar embedding adopted in this picture, the zig-zag shape of the digraph component of the network resembles the lateral structure of a clotheshorse. Arcs corresponding to flow variables $(h_1^c, h_1^d, h_2^c, h_2^d, \dots, h_{2i}^c, h_{2i}^d)$, in this order, form an Eulerian nonoriented path covering the whole digraph. The network corresponding to the case $n = 2i+1$, i even, has the same structure, but the positions of the node constants and arcs in the last piece of the network (the losangle shaped graph) are inverted, see [3].

Both networks contain a chain of $i-1$ cycles of length four and the arc sequence associated with $(h_1^c, h_1^d, h_{2i}^c, h_{2i}^d)$ forms a nonoriented path that is adjacent to the first cycle. The arcs in the j-th cycle are associated with variables h_{j+1}^d, h_{2i-j}^c, h_{2i-j}^d and h_{j+1}^c, and the orientation of the first two arcs is opposite to the orientation of the last two, with respect to an arbitrary orientation of the cycle.

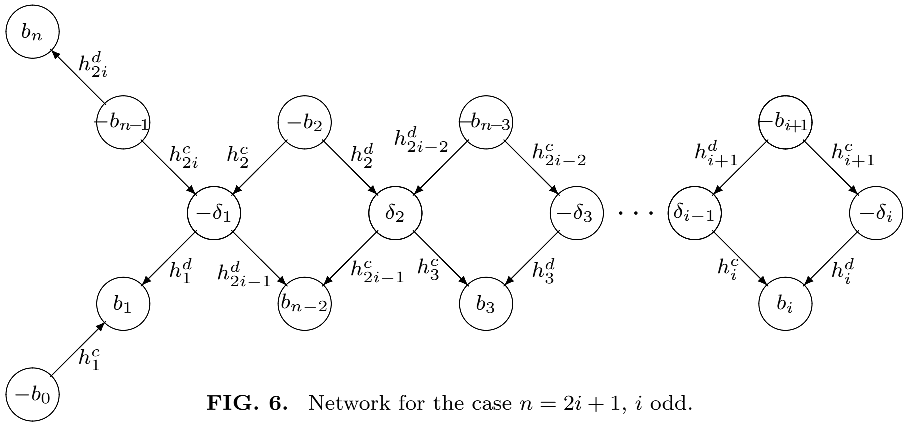


FIG. 6. Network for the case $n = 2i + 1$, i odd.

The general solution of system (8) is the sum of a particular solution and a solution of the homogeneous version of the system, that is, a solution that satisfies the condition "flow in = flow out" at every node. The former one is called a *flow* and, the latter one, a *circulation*. Thus the set of circulations is a linear subspace, the null space of the coefficient matrix of the linear system. Rockafellar [12] showed that a vector of a linear space is the sum of elementary vectors of this space. An *elementary vector* of a subspace is a vector of this subspace with minimal support. It is well known that the elementary circulations are those whose support corresponds to a simple cycle in the network. Thus the network considered has $i - 1$ distinct and unique (up to multiples) simple circulations, which are easy enough to construct. Finally, if we remove one arc from each cycle, the remaining arcs form a tree. The columns of the coefficient matrix of the h^{cd}_κ-system associated with the remaining arcs (unknowns) are linearly independent and thus this subsystem has a unique solution, if it has a solution (this submatrix has more rows than columns). These remaining arcs also constitute a spanning tree of the network. The corresponding solution is easy to compute: start at the leaves of the tree (nodes with only one incident arc) and work your way in. In algebraic parlance, this is equivalent to permutating the rows and columns of the associated submatrix to make it lower triangular and then solve the corresponding system by back substitution. One particular such solution, called *complementary solution*, described in Lemma 4, is of special interest in the development that follows.

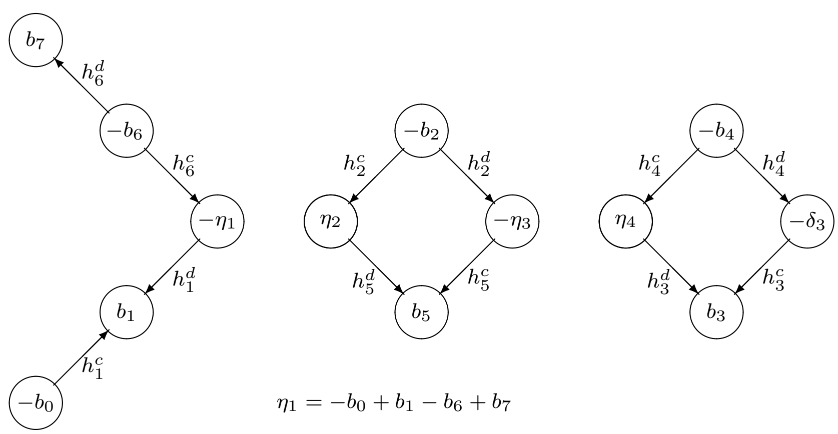
LEMMA 4. *If the h^{cd}_κ-system* (8) *has a nonnegative integral solution, then there exists a unique nonnegative integral solution that satisfies the* *complementarity condition*

$$h_j^c h_{n-j}^d = 0, \quad \textit{for } j = 2, \ldots, i. \tag{10}$$

Proof. Given an arbitrary nonnegative integral flow h^{cd} satisfying (8), one can reduce to zero at least one of the flows in (10) by sending through the $(j-1)$-th cycle a circulation of value $\min\{h_j^c, h_{2i+1-j}^d\}$ in the direction opposite to that of the arc associated with h_j^c. Since both h_j^c and h_{2i+1-j}^d are integral, the resulting solution will be nonnegative and integral. ▮

PROPOSITION 5. *The h_κ^{cd}-system (7) has nonnegative integral solutions if and only if the Poincaré-Hopf inequalities (1)–(2), for isolating blocks are satisfied. Moreover, the set of all solutions to the h_κ^{cd}-system may be obtained as sums of the complementary solution and multiples of the elementary circulations of the network.*

Proof. The trick to show the equivalence between the network-flow problems and the Poincaré-Hop inequalities in [1] was to split the network-flow problem into a set of i independent smaller network-flow problems. Figure 7 shows the decomposition of the problem corresponding to the case $n = 7$.



$\eta_1 = -b_0 + b_1 - b_6 + b_7$

$\eta_2 = -\delta_1 + \eta_1$

$\eta_3 = \eta_2 - b_2 + b_5$

$\eta_4 = \delta_2 + \eta_3$

FIG. 7. Decomposition of network, for $n = 7$.

While the subproblem corresponding to the leftmost network in Figure 7 has a unique solution (the underlying graph is a tree), the remaining $i-1$ subproblems are special cases of the 4-node-cycle network, exemplified by the two networks on the right of Figure 7. The existence of solutions to such a subproblem may be recast as the existence of solutions to a set of inequalities involving the node constants. The collection of all such inequalities, obtained from the decompostion of the original network-flow problem into i smaller network-flow problems, turn out to be precisely the Poincaré-Hopf inequalities produced by the long exact sequences, presented in Section 3. Details can be found in [1] for $\kappa = 0$ and in [3] for $\kappa \geq 0$. ∎

Propositions 3 and 5 imply in Theorem 1, since each vertex explosion corresponds to a nonnegative integral solution of the associated h^{cd}_{κ}-system.

In other words, we have shown that if every vertex of a Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$ satisfies the Poincaré-Hopf inequalities for isolating blocks (1)–(2), the graph admits a continuation to a Morse type Lyapunov graph with cycle rank greater or equal to κ where $\kappa \leq \min\{h_1 - (h_0 - 1),\ h_{n-1} - (h_n - 1)\}$. See [3] for more details.

The number of continuations, i.e., the number of nonnegative integral flows, has been calculated in [1] for $\kappa = 0$. From Proposition 5, this number is the number of admissible multiples of elementary circulations of the network. Since the values of h^c_1, h^d_1, h^c_{2i} and h^d_{2i} are uniquely determined, this is the number of nonnegative integral flows of the smaller network obtained after the elimination of these four variables from the system of equations (8). It is straightforward to verify that this elimination will result in the same subnetwork, regardless of the value of κ. Thus the total number of continuations is just the number of possible values of κ $(1+\min\{h_1-(h_0-1), h_{n-1}-(h_n-1)\})$ times the number of continuations for $\kappa = 0$.

6. MORSE INEQUALITY RESULTS

In this section we consider the generalized Morse inequalities (3) and the Poincaré-Hopf inequalities for closed manifolds. In this context we want to consider an h^{cd}_{κ}-system which we obtain by setting $B^-_j = B^+_j$, $e^- = h_0$ and $e^+ = h_n$ in (7). We refer to it as the reduced h^{cd}_{κ}-system.

The mappings $\Gamma(\cdot)$ and $H^{cd}(\cdot)$, whose definitions follow Lemma 6, provide the key to a constructive proof, again with the mediation of the h^{cd}_{κ}-system, of the Equivalence theorem 2, theme of the present section. The complementary solution of the reduced h^{cd}_{κ}-system will play a special role in one direction of the proof. Likewise, a particular class of Betti number vectors is specially suited for the opposite direction of the proof. Finally, these mappings help in the characterization and construction of the Morse polytope undertaken in Section 7.

LEMMA 6. *If γ is Betti number vectors that satisfy the Morse inequalities* (3) *and $\gamma_1 \geq \kappa$, then γ^* where $\gamma_j^* = \gamma_j$ for $0 \leq j \leq 2i+1$, $i \neq j \neq i+1$, and γ_i^*, γ_{i+1}^* satisfy*

$$\mathbb{Z} \ni \gamma_i^* = \gamma_{i+1}^* \in \begin{cases} [\kappa, h_i + (-1)^{i+1} \sum_{j=0}^{i-1} (-1)^{j+1}(h_j - \gamma_j^*)], & \text{if } i = 1 \\ [0, h_i + (-1)^{i+1} \sum_{j=0}^{i-1} (-1)^{j+1}(h_j - \gamma_j^*)], & \text{if } i > 1 \end{cases} \tag{11}$$

also is a Betti number vector that solves (3) *and satisfies $\gamma_1^* \geq \kappa$.*
In particular, if γ_i^* and γ_{i+1}^* coincide with the right endpoint of the interval in (11) *then γ^* saturates the middle inequality in* (3), *i.e.,*

$$\sum_{j=0}^{i} (-1)^{j+1} \gamma_j^* = \sum_{j=0}^{i} (-1)^{j+1} h_j. \tag{12}$$

Proof. All linear inequalities in (3) except the middle one contain the difference $\pm(\gamma_i - \gamma_{i+1})$ (which is, by the way, zero, since $\gamma_i = \gamma_{i+1}$) or do not contain neither γ_i nor γ_{i+1}. Thus if we let $\gamma_j^* = \gamma_j$ for $i \neq j \neq i+1$, the new vector γ^* will be a nonnegative solution of (3) and satisfy $\gamma_1 \geq \kappa$ as long as its middle components γ_i and γ_{i+1} satisfy following inequalities are satisfied:

$$\begin{aligned} \gamma_i^* &= \gamma_{i+1}^* && (13) \\ \gamma_i^* - \gamma_{i-1}^* + \cdots + (-1)^i \gamma_0^* &\leq h_i - h_{i-1} + \cdots + (-1)^i h_0 && (14) \\ \gamma_i^* &\geq \begin{cases} \kappa, & \text{if } i = 1 \\ 0, & \text{if } i > 1. \end{cases} && (15) \end{aligned}$$

These conditions are summarized in (11).

Finally, the closed interval in (11) is nonempty (it contains the current value of γ_i) and has integral-valued endpoints, since γ is by assumption integral. Setting γ_i^* and γ_{i+1}^* equal to the right endpoint of the interval we assure that γ^* is a Betti number vector that satisfies (3) and (12). ∎

The mappings $\Gamma : \mathbb{R}^{4i} \to \mathbb{R}^{2i+2}$ and $H^{cd} : \mathbb{R}^{2i+2} \to \mathbb{R}^{4i}$ are defined as follows:

$$\Gamma_0(h^{cd}) = \Gamma_{2i+1}(h^{cd}) = 1$$

$$\Gamma_j(h^{cd}) = \begin{cases} h^d_1 - h^c_2 + \kappa, & \text{if } j = 1 \\ h^d_j - h^c_{j+1}, & \text{if } 2 \le j < i \\ h^d_i, & \text{if } j = i \\ h^c_{i+1}, & \text{if } j = i+1 \\ -h^d_{j-1} + h^c_j, & \text{if } i+2 \le j \le 2i-1 \\ -h^d_{2i-1} + h^c_{2i} + \kappa, & \text{if } j = 2i \end{cases} \tag{16}$$

$$H^d_{2i}(\gamma) = -\sum_{j=0}^{2i} (-1)^{j+1}(h_j - \gamma_j) + \kappa, \tag{17}$$

$$H^d_{2i+1-\ell}(\gamma) = (-1)^\ell \sum_{j=0}^{2i+1-\ell} (-1)^{j+1}(h_j - \gamma_j), \quad \text{for } 2 \le \ell \le i \tag{18}$$

$$H^c_{2i+2-\ell}(\gamma) = (-1)^\ell \sum_{j=0}^{2i+1-\ell} (-1)^{j+1}(h_j - \gamma_j), \quad \text{for } i+2 \le \ell \le 2i \tag{19}$$

$$H^c_1(\gamma) = h_0 - \gamma_0 + \kappa, \tag{20}$$

$$H^d_\ell(\gamma) = \gamma_\ell + H^c_{\ell+1}(\gamma), \quad \text{for} 1 \le \ell \le i-1 \tag{21}$$

$$H^c_\ell(\gamma) = \gamma_\ell + H^d_{\ell-1}(\gamma), \quad \text{for } i+2 \le \ell \le 2i \tag{22}$$

$$H^d_i(\gamma) = \gamma_i \tag{23}$$

$$H^c_{i+1}(\gamma) = \gamma_{i+1}. \tag{24}$$

THEOREM 7. *Suppose fixed the pre-assigned index data* $(h_0, \ldots, h_n)$. *The reduced* h^{cd}_κ*-system has nonnegative integral solutions if and only if there exists nonnegative integral Betti number vectors* γ *that satisfies the Morse inequalities* (3) *and the inequality* $\gamma_1 \ge \kappa$.

Proof. If the reduced h^{cd}_κ-system has nonnegative integral solutions, then, by Lemma 4, it admits a complementary (nonnegative integral) solution h^{*cd}. Then it can be shown, see [3], that $\Gamma(h^{*cd})$ is a Betti number vector that satisfies the Morse inequalities (3) and $\gamma_1 \ge \kappa$ (For more background on this last inequality see [5], [7]).

Now suppose there exist Betti number vector(s) satisfying (3) and $\gamma_1 \ge \kappa$. Let γ^* be such a vector that further satisfies (12). Then $H^{cd}(\gamma^*)$ is a nonnegative integral solution of the reduced h^{cd}_κ-system (8). The proof in [2], for the case $\kappa = 0$, may be easily adapted. An alternative proof may be found in [3]. ▮

Hence we have shown in this section that the reduced h^{cd}_{κ}-system has nonnegative integral solutions if and only if there exists nonnegative integral Betti number vectors γ that satisfies the Morse inequalities (3) and the inequality $\gamma_1 \geq \kappa$. On the other hand in the previous section we have shown that the h^{cd}_{κ}-system (7) has nonnegative integral solutions if and only if the Poincaré-Hopf inequalities (1)–(2), for isolating blocks are satisfied. Of course this result holds in the particular case of the reduced h^{cd}_{κ}-system and the Poincaré-Hopf inequalities for closed manifolds. Hence the equivalence theorem follows.

7. MORSE POLYTOPE

Consider a fixed pre-assigned index data set $(h_0, h_1, \ldots, h_n)$ and let κ be an integer in the interval $[0, \min\{h_1 - (h_0 - 1), h_{n-1} - (h_n - 1)\}]$. The possibility of cycles implies in the inclusion of the extra inequality $\gamma_1 \geq \kappa$ to the set of Morse inequalities. This larger set of inequalities, plus the boundary constraints $\gamma_0 = \gamma_n = 1$, the duality constraints $\gamma_j = \gamma_{n-j}$, for $j = 0, \ldots, n$, and the nonnegative constraints $\gamma \geq 0$, define a polyhedron $\mathcal{P}_\kappa(h_0, \ldots, h_n) \subset \mathbb{R}^{n+1}$, which for simplicity we refer to as $\mathcal{P}_\kappa$, the subject of this section. The first interesting fact is that $\mathcal{P}_\kappa$ is in fact a polytope, that is, a bounded polyhedron, since $0 \leq \gamma \leq h_j$ (upper bound is implied by inequalities (j) and $(j-1)$ of (3)), for $1 \leq j \leq n-1$. All properties of $\mathcal{P}_0$ obtained in [2], where the case $\kappa = 0$ was analyzed, may be extended to the more general case $\kappa \geq 0$, as shown in [3]. We summarize the main results below, the proof and other results may be found in [3].

THEOREM 8. *The polytope $\mathcal{P}_\kappa$ satisfies the following properties:*

1. The set $\mathcal{P}_\kappa$ is an integral polytope, given by the convex hull of the Betti number vectors that satisfy the Morse inequalities (3) *and $\gamma_1 \geq \kappa$.*
2. $\mathcal{P}_\kappa$ has the maximum element $\Gamma(h^{*cd})$, where h^{*cd} is the complementary solution of the reduced h^{cd}_{κ}-system.
3. If $\kappa \geq \kappa'$ then $\mathcal{P}_\kappa \subseteq \mathcal{P}_{\kappa'}$.

The set of nonnegative solutions to the reduced h^{cd}_{κ}-system also constitutes a polytope. It is remarkable that both polytopes have integral vertices, and that there is a relationship (though not 1-to-1) between the integral elements in each polytope. The next example strives to illustrate some of the geometrical properties of $\mathcal{P}_\kappa$.

Example Let $n = 2i + 1 = 7$ and $(h_0, \ldots, h_7) = (1, 5, 11, 10, 5, 3, 4, 3)$. The complementary solution of the reduced h_0^{cd}-system is $h^{*cd} = (0, 5, 3, 8, 5, 5, 5, 0, 3, 0, 2, 2)$, and the elementary circulations are $\mathrm{circ}^1 = (0, 0, 1, -1, 0, 0, 0, 0, -1, 1, 0, 0)$ and $\mathrm{circ}^2 = (0, 0, 0, 0, 1, -1, -1, 1, 0, 0, 0, 0)$, corresponding to cycles 1 and 2 depicted in Figure 8. The maximum element of $\mathcal{P}_0$ is $\gamma^* = \Gamma(h^{*cd}) = (1, 2, 3, 5, 5, 3, 2, 1)$. Although $\mathcal{P}_\kappa \subset \mathbb{R}^8$, we may use the duality and boundary contraints to eliminate γ_0, γ_4, ..., γ_7 and obtain the reduced polytope $\mathcal{P}_\kappa^r \subset \mathbb{R}^3$, whose elements are in a 1-to-1 relationship with the elements of $\mathcal{P}_\kappa$. This allows the depiction of the polytope, furnished in Figure 9, and the visualization of the mentioned geometric properties. The vector of remaining variables is denoted by $\gamma^r = (\gamma_1, \gamma_2, \gamma_3)$.

It can be shown that $\mathcal{P}_\kappa^r$ is given by the inequalities

$$
\begin{array}{lllllll}
\gamma_1 & & & & & \leq & 2 \\
\gamma_1 & - & \gamma_2 & & & \geq & -1 \\
\gamma_1 & - & \gamma_2 & + & \gamma_3 & \leq & 4 \\
\gamma_1 & & & & & \geq & \kappa \\
 & & \gamma_1, & \gamma_2, & \gamma_3 & \geq & 0
\end{array}
$$

$\hat{\gamma}^r - \tilde{\gamma}^r$	$\hat{h}^{cd} - \tilde{h}^{cd}$
$(-1, -1, 0)$	circ^1
$(0, -1, -1)$	circ^2

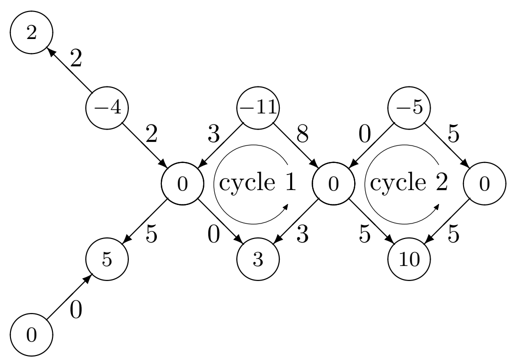
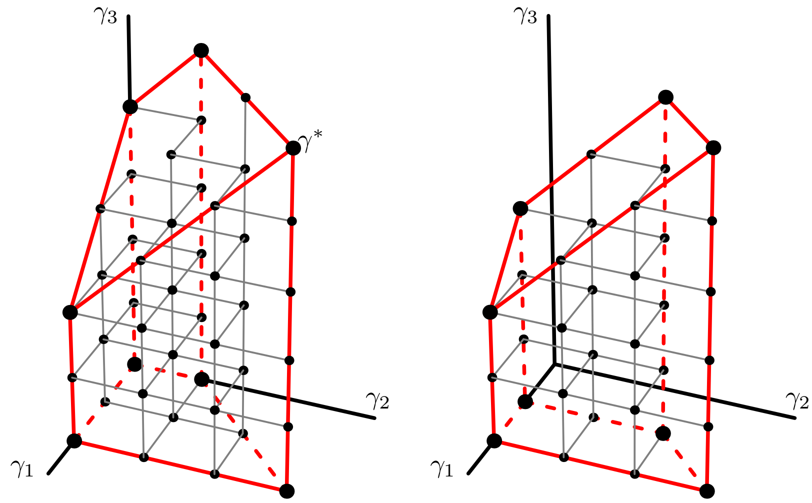


FIG. 8. Relationship between γ's and h^{cd}'s and solution $\tilde{h}^{cd}$ of example.

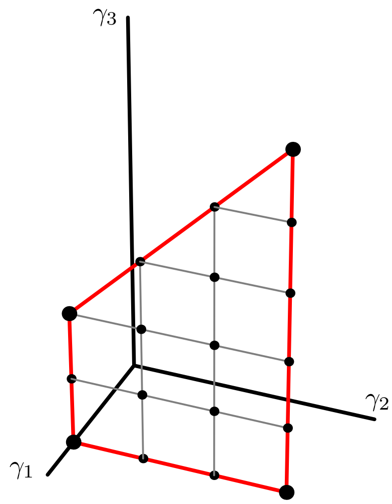
In this case κ may assume the values 0, 1 and 2. Figure 9 depicts the three corresponding polytopes, delineating their edges and emphasizing their integral elements. Notice that all vertices are integral. The relationship between two γ^r's in the top facet and their corresponding h^{cd}'s may be obtained from the basic ones given in the table in Figure 8 above. Thus if $\hat{\gamma}^r = (1, 2, 5) = \gamma^{*r} + (-1, -1, 0)$ then $\hat{\gamma} = \Gamma(\hat{h}^{cd})$, where $\hat{h}^{cd} = h^{*cd} + \mathrm{circ}^1 = (0, 5, 4, 7, 5, 5, 5, 0, 2, 1, 2, 2)$. Similarly, if $\hat{\gamma}^r =$

$(1,1,4) = \gamma^{*r} + (-1,-2,-1) = \gamma^{*r} + (-1,-1,0) + (0,-1,-1)$, then $\hat{h}^{cd} = h^{*cd} + \mathrm{circ}^1 + \mathrm{circ}^2 = (0,5,4,7,6,4,4,1,2,1,2,2)$. Finally, notice that $\mathcal{P}_0^r \supset \mathcal{P}_1^r \supset \mathcal{P}_2^r$.



(a) $\mathcal{P}_0^r$ and its integral elements.

(b) $\mathcal{P}_1^r$ and its integral elements.



(c) $\mathcal{P}_2^r$ and its integral elements.

FIG. 9. Polytopes $\mathcal{P}_\kappa^r$.

REFERENCES

1. M. A. Bertolim, M. P. Mello and K. A. de Rezende, *Lyapunov graph continuation*, Ergodic Theory and Dynamical Systems **23** (2003), 1–58.
2. M. A. Bertolim, M. P. Mello and K. A. de Rezende, *Poincaré-Hopf Inequalities*, to appear in Trans. Amer. Math. Soc.
3. M. A. Bertolim, M. P. Mello and K. A. de Rezende, *Poincaré-Hopf and Morse inequalities for Lyapunov graphs*, to appear in Ergodic Theory and Dynamical Systems.
4. C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conference Series in Mathematics, **38**. American Mathematical Society, Providence, R.I., 1978.
5. O. Cornea, *The genus and the fundamental group of high-dimensional manifolds*, Stud. Cerc. Mat. **41** (1989), 169–178.
6. R. N. Cruz and K. A. de Rezende, *Gradient-like flows on high-dimensional manifolds*, Ergodic Theory and Dynamical Systems **19** (1999), 339–362.
7. R. N. Cruz and K. A. de Rezende, *Cycle rank of Lyapunov graphs and the genera of manifolds*, Proceedings of the American Mathematical Society **126** (1998), 3715–3720.
8. K. A. de Rezende, *Smale flows on the three-sphere*, Trans. Amer. Math. Soc. **303** (1987), 283–310.
9. J. Franks, *Nonsingular Smale flows on S^3*, Topology **24** (1985), 265–282.
10. C. McCord, *Poincaré-Lefschetz duality for the homology Conley index*, Trans. Amer. Math. Soc. **329** (1992), 233–252.
11. J. F. Reineck, *Continuation to Gradient Flows*, Duke Mathematical Journal **64** (1991), 261–269.
12. R. T. Rockafellar, The elementary vectors of a subspace of $\mathbb{R}^n$, in R. C. Bose and T. A. Dowling, editors, *Combinatorial Mathematics and its Applications*, 104–127. University of North Carolina Press, 1969.