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On the Orbit Structure of $\mathbb{R}^n$-Actions on n-Manifolds

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Dedicated to Professor Sotomayor 60^{th} birthday

We begin by proving that a locally free C^2-action of $\mathbb{R}^{n-1}$ on $T^{n-1}\times[0,1]$ tangent to the boundary and without compact orbits in the interior has all non-compact orbits of the same topological type. Then, we consider the set $A^2(\mathbb{R}^n, N)$ of C^2-actions of $\mathbb{R}^n$ on a closed connected orientable real analytic n-manifold N. We define a subset $\mathscr{A}_n \subset A^r(\mathbb{R}^n, N)$ and prove that if $\varphi \in \mathscr{A}_n$ has a $T^{n-1}\times\mathbb{R}$-orbit, then every n-dimensional orbit is also a $T^{n-1}\times\mathbb{R}$-orbit. The subset $\mathscr{A}_n$, is big enough to contain all real analytic actions that have at least one n-dimensional orbit. We also obtain information on the topology of N.

Key Words: Action of $\mathbb{R}^n$, orbit structure.

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1. INTRODUCTION

In this paper we begin by proving the following result that is a generalization of Corollary 2.6 in [5].

Theorem A. *ψ be a locally free C^2-action of $\mathbb{R}^{n-1}$ on $T^{n-1} \times [0,1]$, $n \geq 2$, tangent to the boundary. If there are no compact orbits in the interior, then all non-compact orbits have the same topological type.*

Due to the theorem below, there is no restriction in assuming that the manifold with boundary, in Theorem A, is $T^{n-1} \times [0,1]$.

Theorem B (Chatelet - Rosenberg, [4]). *Let N be a compact orientable n-manifold with non-empty boundary. Suppose that ψ is a C^2 locally free action of $\mathbb{R}^{n-1}$ on N, then N is diffeomorphic to $T^{n-1} \times [0,1]$.*

N will denote a closed connected orientable real analytic n-manifold with $n \geq 2$. Let $\mathcal{H}_n$ be the family of orientable n-manifolds obtained by glueing two copies of $T^{n-2} \times D^2$. $\mathcal{H}_2$ contains only S^2 and $\mathcal{H}_3$ consists of 3-manifolds that admit a Heegaard splitting of genus one. Denote by $A^r(\mathbb{R}^n, N)$ the set of C^r-actions of $\mathbb{R}^n$ on N, $2 \leq r \leq \omega$, with C^r infinitesimal generators. It was proved in [2] that if $\varphi \in A^\omega(\mathbb{R}^n, N)$, then all n-dimensional orbits of φ have the same topological type, i.e., are $T^k \times \mathbb{R}^{n-k}$-orbits for some fixed k, $0 \leq k \leq n$. Moreover, if the type is $T^{n-1} \times \mathbb{R}$, then N is either homeomorphic to T^n or $N \in \mathcal{H}_n$. It is not difficult to construct counterexamples of this results when $r = \infty$. In this paper we define a subset $\mathscr{A}_n \subset A^2(\mathbb{R}^n, N)$, see Definition 2, which contains all actions $\varphi \in A^\omega(\mathbb{R}^n, N)$ that have at least one n-dimensional orbit. Then, we prove:

Theorem C. *If $\varphi \in \mathscr{A}_n$ has one $T^{n-1} \times \mathbb{R}$-orbit, then every n-dimensional orbit is also a $T^{n-1} \times \mathbb{R}$-orbit. Moreover,*
(1) *if $\mathrm{Sing}^c_{n-2}(\varphi) = \emptyset$, then N is a T^{n-1} bundle over S^1;*
(2) *if $\mathrm{Sing}^c_{n-2}(\varphi) \neq \emptyset$, then $\mathrm{Sing}^c_{n-2}(\varphi)$ is the union of two T^{n-2}-orbits and $N \in \mathcal{H}_n$.*

The connection between the two main results is that the first is used in the proof of third. It would be interesting to obtain analogous results for actions in $A^2(\mathbb{R}^n, N)$ that have one $T^k \times \mathbb{R}^{n-k}$-orbit with $0 \leq k < n-1$.

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2. PRELIMINARIES AND PROOF OF RESULTS

M will denote a closed connected and orientable real analytic m-manifold. A C^r-action of a Lie group G on M is a C^r-map $\varphi : G\times M \to M$, $1 \le r \le \omega$, such that $\varphi(e,p) = p$ and $\varphi(gh,p) = \varphi(g,\varphi(h,p))$, for each $g,h \in G$ and $p \in M$, where e is the identity in G. $\mathcal{O}_p = \{\varphi(g,p);\ g \in G\}$ is called the φ-*orbit* of p. $G_p = \{g \in G;\ \varphi(g,p) = p\}$ is called the *isotropy group* of p. For each $p \in M$ the map $g \mapsto \varphi(g,p)$ induces an injective immersion of the homogeneous space G/G_p in M with image $\mathcal{O}_p$. When $G = \mathbb{R}^n$, the possible φ-orbits are injective immersions of $T^k \times \mathbb{R}^\ell$, $0 \le k+\ell \le n$, where $T^k = S^1 \times \cdots \times S^1$, k times.

For each $0 \le i \le n-1$ let $\mathrm{Sing}_i(\varphi) = \{p \in M; \dim \mathcal{O}_p = i\}$ and $\mathrm{Sing}(\varphi) = \cup_{i=0}^{n-1} \mathrm{Sing}_i(\varphi)$. If $p \in \mathrm{Sing}(\varphi)$, $\mathcal{O}_p$ is called a *singular orbit* and when $p \in \mathrm{Sing}_0(\varphi)$, $\mathcal{O}_p$ is also called a *point orbit* and p a *fixed point* by φ. We also write $p \in \mathrm{Sing}_i^c(\varphi)$, $i = 1,\ldots,n-1$, when $\mathcal{O}_p$ is a T^i-orbit. If $\mathrm{Sing}(\varphi) = M$, we call φ a *singular action*.

For each $w \in \mathbb{R}^n \setminus \{0\}$ φ induces a C^r-flow $(\varphi_w^t)_{t\in\mathbb{R}}$ given by $\varphi_w^t(p) = \varphi(tw,p)$ and its corresponding C^{r-1}-vector field X_w defined by $X_w(p) = D_1\varphi(0,p)\cdot w$. If $\{w_1,\ldots,w_n\}$ is a base of $\mathbb{R}^n$ the associated vector fields $X_{w_1},\ldots,X_{w_n}$ determine completely the action φ and are called a set of *infinitesimal generators* of φ. Note that $[X_{w_i},X_{w_j}] = 0$ for any two of them.

Definition 1. Let $\varphi \in A^r(\mathbb{R}^n,N)$ and $p \in N$.

a) φ is of *type* j *at* p, $0 \le j \le n$, if there exists a neighborhood V of p such that the union of the j-dimensional orbits of $\varphi|_V$ form an open and dense subset of V.

b) φ is j-*finite at* p, if there exists a neighborhood V of p that intersects only a finite number of j-dimensional orbits.

Let $V \subset N$. We will denote the boundary of V in N by $\mathrm{Front}(V)$.

Definition 2. We say that $\varphi \in \mathscr{A}_n \subset A^2(\mathbb{R}^n,N)$, where $\dim N = n$, if φ is of type n and n-finite at each $p \in \mathrm{Sing}_i(\varphi)$ with $0 \le i \le n-3$, $\mathrm{Sing}_{n-2}(\varphi) = \mathrm{Sing}_{n-2}^c(\varphi)$ and for each $p \in \mathrm{Sing}_{n-2}^c(\varphi)$ there exists a neighborhood V_p of $\mathcal{O}_p$ in N that satisfies one of the following two properties:

(1) V_p is φ-invariant, homeomorphic to $T^{n-2}\times D^2$, where D^2 is an open disk, $V_p \cap (\cup_{i=1}^{n-2}\mathrm{Sing}_i(\varphi)) = \mathcal{O}_p$ and $\mathrm{Front}(V_p)$ is either a T^{n-1}-orbit or a T^{n-2}-orbit.

(2) V_p contains at most a finite number of i-dimensional orbits with $i = n-1, n$.

Infinitesimal generators adapted to a T^{n-1}-orbit. Assume that $\mathcal{O}_p$ is a T^{n-1}-orbit of $\varphi \in A^r(\mathbb{R}^n,N)$ and let G_p be its isotropy group.

Call G_p^0 the connected component of G_p that contains the origin and let H be a $(n-1)$-dimensional subspace of $\mathbb{R}^n$ such that $\mathbb{R}^n = H \oplus G_p^0$. Note that $G_p \cap H$ is isomorphic to $\mathbb{Z}^{n-1}$. Let $\{w_1, \ldots, w_n\}$ be a base of $\mathbb{R}^n$ such that $\{w_1, \ldots, w_{n-1}\}$ is a set of generators of $G_p \cap H$, $w_n \in G_p^0$ and write $X_i = X_{w_i}\,; i = 1, \ldots, n$. Note that if $q \in \mathcal{O}_p$, then for every $k \in \{1, \ldots, n-1\}$ the orbit of X_k by p is periodic of period one and also $X_n(q) = 0$. We shall say that $X_1, \ldots, X_n$ is a set of *infinitesimal generators adapted to* $\mathcal{O}_p$. The action $\psi_\varphi \in A^r(\mathbb{R}^{n-1}, N)$, $r \geq 2$, with infinitesimal generators $X_1, \ldots, X_{n-1}$ will be called the *action induced* by φ and $\mathcal{O}_p$. The understanding of the holonomy of $\mathcal{O}_p$ as an orbit of ψ_φ will bring light on the orbit structure of φ in the neighborhood of $\mathcal{O}_p$.

Let $\mathcal{O}_p$ be a T^{n-1}-orbit of $\psi \in A^r(\mathbb{R}^{n-1}, N)$, $\{w_1, \ldots, w_{n-1}\}$ be a set of generators of its isotropy group G_p and $X_1 = X_{w_1}, \ldots, X_{n-1} = X_{w_{n-1}}$. For each $k \in \{1, \ldots, n-1\}$, let $\psi_k \in A^r(\mathbb{R}^{n-2}, N)$ be the action defined by $X_1, \ldots, X_{k-1}$, $X_{k+1}, \ldots, X_{n-1}$. Put a Riemannian metric on N and let ξ be the norm one vector field defined in a neighborhood of $\mathcal{O}_p$ that is orthogonal to the orbits of ψ. Let S_k be the circle orbit of X_k through p, $k = 1, \ldots, n-1$, and consider the ring $A = S^1 \times (-1, 1)$ with coordinates (θ, x). Define $f_k : A \to N$ by $f_k(\theta, x) = \xi^x \circ X_k^\theta(p)$ and note that $f_k(S^1 \times \{0\}) = S_k$ and $f_k(0,0) = p$. Fix $k \in \{1, \ldots, n-1\}$. Since S_k, as a submanifold of $\mathcal{O}_p$, is transversal to the orbits of ψ_k, there exists $\varepsilon > 0$ such that f_k restricted to $A_\varepsilon = S^1 \times (-\varepsilon, \varepsilon)$ is an embedding transversal to the orbits of ψ_k. Let $D_k^{n-2}(\delta) = \{t = (t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_{n-1}); t_j \in (-\delta, \delta)\}$ and consider the C^r-map $h_k : A_\varepsilon \times D_k^{n-2}(1) \to N$ defined by $h_k(\theta, x, t) = \psi_k(t, f_k(\theta, x))$. There exists $\delta > 0$ such that h_k restricted to $A_\varepsilon \times D_k^{n-2}(\delta)$ is a diffeomorphism onto its image V_k. Moreover, in these coordinates the infinitesimal generators of φ take the form:

$$\begin{aligned} X_i(\theta, x, t) &= \frac{\partial}{\partial t_i}, \quad i = 1, \ldots, k-1, k+1, \ldots, n-1 \\ X_k(\theta, x, t) &= \sum_{k \neq j = 1}^{n-1} a_{jk}(\theta, x) \frac{\partial}{\partial t_j} + b_k(\theta, x) \frac{\partial}{\partial \theta} + c_k(\theta, x) \frac{\partial}{\partial x}. \end{aligned} \tag{1}$$

A map like h_k will be called a *cylindrical coordinate system adapted to* $\mathcal{O}_p$ *at* S_k. The vector field

$$\widehat{X}_k = b_k(\theta, x) \frac{\partial}{\partial \theta} + c_k(\theta, x) \frac{\partial}{\partial x}$$

defines a local flow on A_ε having $S^1 \times \{0\} \subset A_\varepsilon$ as an orbit. When $\psi = \psi_\varphi$ for some $\varphi \in A^r(\mathbb{R}^n, N)$, then we also have

$$X_n(\theta, x, t) = \sum_{k \neq j = 1}^{n-1} a_{jk}(\theta, x) \frac{\partial}{\partial t_j} + d_k(\theta, x) \frac{\partial}{\partial \theta} + e_k(\theta, x) \frac{\partial}{\partial x}.$$

The vector fields $\widehat{X}_k$ and $\widehat{X}_n = d_k\partial/\partial\theta + e_k\partial/\partial x$ define a local C^r-action $\widehat{\varphi}_k$ of $\mathbb{R}^2$ on A having $S^1 \times \{0\}$ as a singular orbit.

The ring $\Sigma_k = f_k(A_\varepsilon)$ is transversal to the orbits of ψ and so is $J = \cap_{k=1}^{n-1}\Sigma_k$. Note that $p \in J$. The vector fields $\widehat{Y}_k = (f_k)_*\widehat{X}_k$ and $\widehat{Y}_n = (f_k)_*\widehat{X}_n$ are tangent to Σ_k and define a local C^r-action of $\mathbb{R}^2$ on Σ_k. The map $\alpha_k : [0,1] \to \Sigma_k$ given by $\alpha_k(\tau) = \widehat{Y}_k^\tau(p)$ is a parametrization of S_k. Let $\omega_k : (J,p) \to (J,p)$ be the Poincaré map of α_k and

$$\mathrm{Hol} : \pi_1(\mathcal{O}_p\,, p) \cong \mathbb{Z}^k \to \mathrm{Diff}^r(J,p) \tag{2}$$

the holonomy of $\mathcal{O}_p$ as a leaf of the foliation defined by the orbits of ψ. Then, $\omega_k = \mathrm{Hol}([\alpha_k])$. Write J as the union of two intervals $J^+ \cup J^-$ with $J^+ \cap J^- = \{p\}$. Since $\mathcal{O}_p$ is two-sided in N, each ω_i leaves J^+ (J^-) invariant.

Remark 3. Note that $\{X_1, \dots X_{k-1}, \widehat{X}_k, X_{k+1}, \dots, X_{n-1}, \widehat{X}_n\}$ define a local $\mathbb{R}^n$-action $\widehat{\varphi}$ on $A \times D_k^{n-2}(\varepsilon)$ and that $\mathcal{O}_{(\theta,x,t)}(\widehat{\varphi}) = \mathcal{O}_{(\theta,x,t)}(h_k \circ \varphi \circ h_k^{-1})$ for each $(\theta,x,t) \in A \times D_k^{n-2}(\varepsilon)$.

The local C^r-action $\widehat{\varphi}_k$, $k = 1, \dots, n-1$, of $\mathbb{R}^2$ on A and the next lemma will be used in the proof of Proposition 9.

Lemma 4. *Let $\mathcal{O}_p$ be a T^{n-1}-orbit of $\psi \in A^2(\mathbb{R}^{n-1}, N)$ and assume that ψ has no T^{n-1}-orbits, aside $\mathcal{O}_p$, in a neighborhood V of $\mathcal{O}_p$. Then there exists a neighborhood I^+ of p in J^+ such that for each $k \in \{1, \dots, n-1\}$ one of the following statements is verified:*

(1) $\omega_k|_{I^+} = id$; *i.e., every $\widehat{Y}_k$-orbit near S_k is periodic.*

(2) *Either $\omega_k|_{I^+}$ or $(\omega_k|_{I^+})^{-1}$ is a topological contraction, i.e., every $\widehat{Y}_k$-orbit near S_k spirals towards S_k.*

Proof. We give the proof for $k = n-1$; the other cases are similar. Assume that ω_{n-1} does not satisfy (2). Then, there is a sequence $\{q_l \in J^+; l \in \mathbb{N}\}$ such that $\omega_{n-1}(q_l) = q_l$ and $\lim_{l\to\infty} q_l = p$. We claim that p is an isolated fixed point of ω_j for at least one $j \in \{1, \dots, n-2\}$. Otherwise, for each $1 \le j \le n-2$ there exists a sequence $\{q_{jk} \in J^+; k \in \mathbb{N}\}$ such that $\omega_j(q_{jk}) = q_{jk}$ and $\lim_{k\to\infty} q_{jk} = p$. If $q_{jk} \in V$ and $\omega_i(q_{jk}) = q_{jk}$ for each $i \in \{1, \dots, n-1\}$, then the ψ-orbit of q_{jk} is a T^{n-1}-orbit. Therefore, for each $q_{jk} \in V$ there exists $i \neq j$ such that $\omega_i(q_{jk}) \neq q_{jk}$. Let $q_k = \lim_{m\to\infty} \omega_i^m(q_{jk}) \neq p$. It follows from the commutativity of ω_i and ω_j that $q_k \in \mathrm{Fix}(\omega_i) \cap \mathrm{Fix}(\omega_j)$. If $\omega_\ell(q_k) \neq q_k$, then $p \neq \lim_{m\to\infty} \omega_\ell^m(q_k) \in \mathrm{Fix}(\omega_\ell) \cap \mathrm{Fix}(\omega_i) \cap \mathrm{Fix}(\omega_j)$. Repeating this process, if necessary, we obtain a point $q \in \cap_{i=1}^{n-1}\mathrm{Fix}(\omega_i)$ with $q \neq p$. But, this would implie that $\mathcal{O}_q$ is a T^{n-1}-orbit, contradicting one of the hypothesis. Thus, it exists $j \in \{1, \dots, n-2\}$ such that p is an isolated fixed point of ω_j, í.,e., there exists a neighborhood I^+ of p in J^+ such that $I^+ \cap \mathrm{Fix}(\omega_j) = \{p\}$. By N. Kopell Lemma [6], $\omega_{n-1}|_{I^+} = id$. Hence, ω_{n-1} satisfies (1). ▮

In the proof of Lemma 4 it is not essential that ω_k be the Poincaré map of the $\widehat{Y}_k$-orbit S_k. It is only used that the holonomy of $\mathcal{O}_p$ is abelian.

Proof (*of Theorem A*). It is a classical result in foliation theory that the leave structure of a foliation in the neighborhood of a compact leaf is determined by the holonomy of such leaf. The holonomy of the orbit $T^{n-1} \times \{0\}$, given in Lemma 4, guarantees that there exist $d \in (0,1)$ and $s \in \{0, \ldots, n-2\}$ such that every ψ-orbit by points in $T^{n-1} \times [0,d)$ is homeomorphic to $T^s \times \mathbb{R}^{n-s-1}$. We claim that the saturated V of $T^{n-1} \times [0,d)$ by ψ is equal to $T^{n-1} \times [0,1)$ and this would conclude the proof. In fact, if $V \neq T^{n-1} \times [0,1)$, then $\mathrm{Front}(V) \cap T^{n-1} \times \{1\} = \emptyset$. Let $C \neq T^{n-1} \times \{0\}$ be a connected component of $\mathrm{Front}(V)$. C is a compact ψ-invariant subset and contains a minimal subset μ. By a theorem of Sacksteder [7, Theorem 7], μ can not be an exceptional minimal set. Thus μ is a compact orbit. This contradiction proves that $V = T^{n-1} \times [0,1)$. ▮

Proposition 5 below plays an important role in the proof of Theorem C and to prove it we have to prove first Lemma 7.

PROPOSITION 5. *If $\varphi \in A^2(\mathbb{R}^n, N)$ has an orbit $\mathcal{O}$ diffeomorphic to $T^{n-1} \times \mathbb{R}$, then* $\mathrm{Front}(\mathcal{O})$ *is the union of at most two T^k-orbits with $k \in \{n-2, n-1\}$.*

Theorem 6 below will be used in the proof of Lemma 7.

THEOREM 6 ([1]). *Let N be a closed and connected n-manifold, $n \geq 3$, and G a Lie group diffeomorphic to $\mathbb{R}^{n-1}$ acting in class C^2. Assume that the set K of singular points of the action is a non-empty finite subset. Then:*

(i) *K contains only one point,*

(ii) *N is homeomorphic to S^n.*

LEMMA 7. *Let $\psi \in A^2(\mathbb{R}^{m-1}, M^m)$, $m \geq 3$, and $p \in \mathrm{Fix}(\psi)$ be an isolated singularity. Then, there exists a neighborhood V of p in M^m which does not contain any T^{m-1}-orbit.*

Proof. Suppose the assertion of the lemma is false. Let V be a neighborhood of p homeomorphic to open n-disk such that $V \cap \mathrm{Sing}(\psi) = \{p\}$ and assume that there is a T^{m-1}-orbit $\mathcal{O} \subset V$. $V \setminus \mathcal{O} = C_1 \cup C_2$, where C_1 and C_2 are disjoint connected components. Assume that $\mathrm{Front}(C_1) = \mathcal{O}$. There are two possibilities, either $p \in C_1$ or $p \in C_2$. If $p \in C_2$, then ψ is locally free on $N = C_1 \cup \mathcal{O}$ and, by Theorem B, N should be homeomorphic to $T^{m-1} \times [0,1]$. This contradicts the fact that $\mathrm{Front}(C_1) = \mathcal{O}$. If

$p \in C_1$, then ψ is an action on N having p as the only singular point. Let P be the double of N and $\xi \in A^2(\mathbb{R}^{m-1}, P)$ such that ξ restricted to N is equal to ψ. Then ξ has exactly two singular points and this contradicts the Theorem 6. ▌

Proof (*Proof of Proposition* 5). Front($\mathcal{O}$) has at most two connected components and each one of them contains at least a compact orbit. Let C be a connected component and $\mathcal{O}_0 \subset C$ a T^k-orbit with isotropy group G_0. We will show that $C = \mathcal{O}_0$. Assume, for a moment, that C contains another orbit $\mathcal{O}_1$ and take $p \in \mathcal{O}_0$ and $q \in \mathcal{O}_1$. Let $u_1, \dots, u_{n-1}$ be a set of generators of G, the isotropy group of $\mathcal{O}$. Then, all X_{u_i}-orbits through $\mathcal{O}$ are periodic of period one. Even more, there are sequences $\{p_j \in \mathcal{O}; j \in \mathbb{N}\}$ and $\{t_{ij} \in [0,1]; i = 1, 2 \dots, n-1 \text{ and } j \in \mathbb{N}\}$ such that

$$\lim_{j\to\infty} p_j = p \quad \text{and} \quad \lim_{j\to\infty} \varphi\Big(\sum_{i=1}^{n-1} t_{ij} u_i, p_j\Big) = q.$$

For each $i = 1, \dots, n-1$, we can assume, extracting a subsequence if necessary, that $t_{ij} \to t_i \in [0,1]$. Then $\varphi(\sum_{i=1}^{n-1} t_i u_i, p) = q$, which contradicts the fact that $\sum_{i=1}^{n-1} t_i u_i \in G_0$.

Now, we will show that $k \in \{n-2, n-1\}$. Assume, for a moment, that $k < n-2$, then $m = n-k > 2$. Let $p \in \mathcal{O}_0$ and $\{X_1, \dots, X_n\}$ be a set of infinitesimal generators of φ such that the orbit through p of X_i, $i = 1, \dots, k$, is periodic and $X_i(p) = 0$, $i = k+1, \dots, n$. Let $h : V_p \to D^n_\varepsilon$ be a chart of N such that if $(\theta, x) \in D^n_\varepsilon = D^k_\varepsilon \times D^{n-k}_\varepsilon$, then the vector fields X_i in this chart can be writen

$$\begin{aligned} X_i(\theta,x) &= \frac{\partial}{\partial\theta_i}, \quad i = 1,\dots,k \\ X_{k+i}(\theta,x) &= \sum_{j=1}^{k} a_{ji}(x)\frac{\partial}{\partial\theta_j} + \sum_{j=k+1}^{n} a_{ji}(x)\frac{\partial}{\partial x_j}, \quad i = 1,\dots,m \end{aligned} \tag{3}$$

A chart like the one above is called *adapted to* $\mathcal{O}_0$ *at* p. The vector fields

$$\widehat{X}_i = \sum_{j=k+1}^{n} a_{ji}(x)\frac{\partial}{\partial x_j}, \quad i = 1,\dots,m,$$

define a local action φ_T of $\mathbb{R}^m$ on D^m_ε having $0 \in D^m_\varepsilon$ as a fixed point. The image of $\mathcal{O} \cap V_p$ by h intersects D^m in a $T^{m-1} \times \mathbb{R}$-orbit $\widehat{\mathcal{O}}$ of φ_T such that $0 \in D^m$ is a connected component of Front($\widehat{\mathcal{O}}$) and, therefore, an isolated singular point of φ_T. The generators $u_1, \dots, u_{m-1}$ of the isotropy group of $\widehat{\mathcal{O}}$ define a local action of $\mathbb{R}^{m-1}$ on D^m and this action has T^{m-1}-orbits arbitrarily close to 0. This contradicts Lemma 7 and proves that $k \in \{n-2, n-1\}$. ▮

The closure of $V \subset N$, in N, is denoted by $\mathrm{cl}(V)$.

Corollary 8. *Let $\mathcal{O}$ be an T^{n-1}-orbit of $\varphi \in A^1(\mathbb{R}^n, N)$ and V be a neighborhood of $\mathcal{O}$ such that $V \setminus \mathcal{O}$ has exactly two connected components V_1 and V_2. Assume that there exist points $p, q \in V_1$ such that $\dim \mathcal{O}_p = n$, $\mathcal{O}_p \neq \mathcal{O}_q$ and $\mathrm{cl}(\mathcal{O}_p) \supset \mathcal{O} \subset \mathrm{cl}(\mathcal{O}_q)$. Then, $\mathcal{O}_p$ is not a $T^{n-1} \times \mathbb{R}$-orbit.*

Proposition 9. *Assume that $\varphi \in A^2(\mathbb{R}^n, N)$ with $n \geq 2$. If $\mathcal{O}$ is a $T^k \times \mathbb{R}^{n-k}$-orbit with $k \leq n-2$, then $\mathrm{cl}(\mathcal{O})$ can not contain a T^{n-1}-orbit.*

Let us prove this proposition. If $X \in \mathfrak{X}^r(M^2)$, let $\mathcal{C}(X)$ be the set of diffeomorphisms $f \in \mathrm{Diff}^r(M^2)$ that preserve orbits of X. Assume that the orbit γ_p of X by p is periodic of period τ. Let Σ be a cross section to X at p and $P_X : (\Sigma, p) \to (\Sigma, p)$ be Poincaré map.

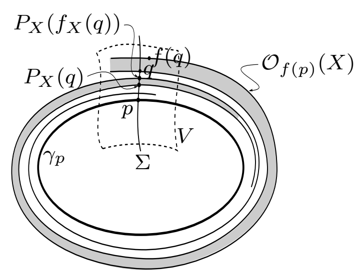

FIG. 1.

Lemma 10. *There exists a cross section $\Sigma_p \subset \Sigma$ and a neighborhood $\mathscr{V} \subset \mathrm{Diff}^r(M^2)$ of the identity map in the C^0 topology such that every $f \in \mathcal{C}(X) \cap \mathscr{V}$ induces a local diffeomorphism $f_X : (\Sigma_p, p) \to (\Sigma_p, p)$ of class C^r, with $f_X \circ P_X = P_X \circ f_X$.*

Proof. Let $\tau > 0$ be the period of γ_p. We can assume that $\Sigma = h^{-1}(\{0\} \times (-1,1))$, where $h : V \to (-1,1)^2$ is a flow box for X at p. Let $\pi : V \to \Sigma$ be the projection along of the orbits of X and recall that $P_X = \pi \circ X^\tau$. Put $U = h^{-1}((-1/2, 1/2)^2)$ and $\Sigma_\varepsilon = h^{-1}(\{0\} \times (-\varepsilon, \varepsilon))$. There exists a neighborhood $\mathscr{V}$ of $id \in \mathrm{Diff}^r(M^2)$, in the C^0 topology, and $\varepsilon > 0$ such that $f(U) \subset V$ and $f(\Sigma_\varepsilon) \subset U$ for each $f \in \mathscr{V}$ and also $X^\tau(\Sigma_\varepsilon) \subset U$. Choose $\Sigma_p = \Sigma_\varepsilon$. For each $f \in \mathscr{V}$ there is defined a map $f_X : \Sigma_p \to \Sigma$ by $f_X(q) = \pi(f(q))$. We are going to show that $f_X \circ P_X = P_X \circ f_X$. Let $[x,y] \subset \Sigma$ be the arc with extremes x and y and define $x \leq y$ if $[p,x] \subseteq [p,y]$. Assume that $P_X(q) \leq q$ for every $q \in \Sigma_p$. For a fixed $f \in \mathscr{V}$ there are two possibilities $f_X(q) \geq q$ or $f_X(q) \leq q$. Let us consider the case $P_X(q) \leq q$ and $f_X(q) \geq q$. If $P_X(q) = q$, i.e., the orbit of q by X is periodic, then the orbit by $f_X(q)$ is also closed, therefore $f_X \circ P_X(q) = f_X(q) = P_X \circ f_X(q)$. If $P_X(q) < q$, then $f_X(P_X(q))$ belongs to the orbit of X by $f_X(q)$ and $P_X(q) \leq f_X(P_X(q)) < f_X(q)$. Thus, $f_X \circ P_X(q) = P_X \circ f_X(q)$. The other cases are analogous. ∎

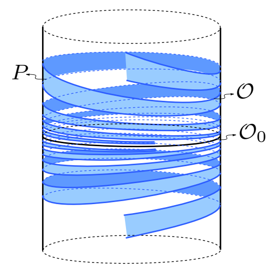

FIG. 2.

Proof (*of Proposition 9*). We begin proving this proposition for $n = 2$. Let $\varphi \in A^2(\mathbb{R}^2, N)$ and assume that there exist an $\mathbb{R}^2$-orbit P such that its closure, $\mathrm{cl}(P)$, contains an S^1-orbit $\mathcal{O}_0$. Let U be neighborhood of $\mathcal{O}_0$ homeomorphic to a cylinder such that every φ-orbit through a point in U has dimension greater or equal to one. It exists due to the fact that N is orientable, see Figure 2. If $U \setminus \mathcal{O}_0 = U_1 \cup U_2$, then either $P \subset U_1$ or $P \subset U_2$. Assume that $P \subset U_1$. Let G_0 be the isotropy group of $\mathcal{O}_0$ and $X_i = X_{w_i}$, $i = 1, 2$, where $\{w_1, w_2\}$ is a set of generators of G_0 such that $w_1 \notin G_0^0$ and $w_2 \in G_0^0$. Note that $X_2|_{\mathcal{O}_0} \equiv 0$ and $\mathcal{O}_0$ is a periodic orbit of X_1. Let Σ be a transversal section to X_1 at $p \in \mathcal{O}_0$ and $P_{X_1} : (\Sigma, p) \to (\Sigma, p)$ be the Poincaré. There exists a neighborhood Σ_0 of p in $\Sigma \cap \mathrm{cl}(U_1)$ such that P_{X_1} has no fixed points in Σ_0. Otherwise, there would be circle orbits in U_1 arbitrarily close to $\mathcal{O}_0$ not allowing the plane P to approach $\mathcal{O}_0$. Thus, either $P_{X_1}|_{\Sigma_0}$ or $(P_{X_1}|_{\Sigma_0})^{-1}$ is a topological contraction. Then, there exists at least one $\mathbb{R}$-orbit $\mathcal{O}$ contained in $\mathrm{Front}(P)$ such that $\mathrm{cl}(\mathcal{O}) \supset \mathcal{O}_0$. This implies that $G_{\mathcal{O}}$, the isotropy group of $\mathcal{O}$, coincides with G_0^0. Let $\mathscr{V}$ be as in the Lemma 10 and $\delta > 0$ such that $X_2^t \in \mathscr{V}$ for all $t \in (-\delta, \delta)$. Fix a $t \neq 0$ in the interval $(-\delta, \delta)$ and put $f = X_2^t$. It follows from the commutativity of X_1 with X_2 that

$f \in \mathcal{C}(X_1)$. Note that $f_{X_1}(q) = q$ if $q \in \mathcal{O} \cap \Sigma_0$ and that $f_{X_1}(q) \neq q$ if $q \in P$. By Lemma 10, $f_{X_1} \circ P_{X_1} = P_{X_1} \circ f_{X_1}$ and by N. Kopell Lemma $f_{X_1} = id$. This contradiction completes the proof in the case $n = 2$.

Assume now that $n \geq 3$ and that there exist a $T^k \times \mathbb{R}^{n-k}$-orbit P, with $k < n-1$, such that $\mathrm{cl}(P)$ contains a T^{n-1}-orbit $\mathcal{O}_0$. If G_0 and G_P are the isotropy groups of $\mathcal{O}_0$ and P, respectively, then $G_P \subset G_0$. Let $X_1, \ldots, X_n$ the infinitesimal generators adapted to $\mathcal{O}_0$ such that the linear $(n-2)$-subspace H of $\mathbb{R}^n$ generated by $\{w_1, \ldots, w_{n-2}\}$ contains G_P. There exists a neighborhood V of $\mathcal{O}_0$ such that the action $\psi_\varphi \in A^2(\mathbb{R}^{n-1}, N)$ induced by φ and $\mathcal{O}_0$, i.,e., generated by $X_1, \ldots, X_{n-1}$, has not T^{n-1}-orbits inside V. Otherwise, by Corollary 8, either P is an $T^{n-1} \times \mathbb{R}$-orbit or $\mathcal{O}_0 \not\subset \mathrm{cl}(P)$. If $\mathcal{O}$ is a $T^s \times \mathbb{R}^{n-s-1}$-orbit of φ such that $\mathcal{O} \subset \mathrm{cl}(P)$, then $\mathcal{O}_0 \subset \mathrm{cl}(\mathcal{O})$ and $s < n-1$. Consequently, if $G_{\mathcal{O}}$ is the isotropy group of $\mathcal{O}$, then $G_{\mathcal{O}} \subset G_0$ and $G^0_{\mathcal{O}} = G^0_0$. Since $k, s < n-1$, we can assume that $H \cap G_{\mathcal{O}}$ is isomorphic to $\mathbb{Z}^s$ and $w_{n-1}, w_n \notin H$. Let $p \in \mathcal{O}_0$ and consider $\omega_1, \ldots, \omega_{n-1}$ as in the proof of Lemma 4. By Lemma 4 ω_{n-1} (or ω^{-1}_{n-1}) is a topological contraction. Therefore, if $q \in \Sigma_{n-1} \cap P$, then $\mathcal{O}_{h^{-1}_{n-1}(q)}(\widehat{\varphi}_{n-1})$ is a $\mathbb{R}^2$-orbit in A that contains the $\widehat{\varphi}_{n-1}$-orbit $S^1 \times \{0\}$ in its closure. By the first part of the proof this is a contradiction. ∎

Remark 11. Let $\varphi \in \mathscr{A}_n$, $p \in \mathrm{Sing}^c_{n-2}(\varphi)$ and V_p a neighborhood of $\mathcal{O}_p$.
(a) Assume that V_p satisfies (1) and $\mathrm{Front}(V_p)$ is a T^{n-1}-orbit. Since $r \geq 2$, it follows by Proposition 9, that there is no $T^s \times \mathbb{R}^{n-s}$-orbit with $s \neq n-1$ inside V_p. Thus, we can say that one of the following possibilities is satisfied:

(a1) $V_p \setminus \mathcal{O}_p$ is a $T^{n-1} \times \mathbb{R}$-orbit;
(a2) V_p contains infinitly many $T^{n-1} \times \mathbb{R}$-orbits;
(a3) $V_p \setminus \mathcal{O}_p$ contains only $(n-1)$-dimensional orbits.

(b)Assume now that V_p satisfies (2), then:

(b1) if there is one $T^s \times \mathbb{R}^{n-s-1}$-orbit, $s \neq n-1$, such that its closure contains $\mathcal{O}_p$, then every n-orbit in $V_p \backslash \mathcal{O}_p$ is not homeomorphic to $T^{n-1} \times \mathbb{R}$;

(b2) if there are no $T^s \times \mathbb{R}^{n-s-1}$-orbits, $s \neq n-1$, which contain $\mathcal{O}_p$ in its closure, then there is only one n-orbit and it is homeomorphic to $T^{n-1} \times \mathbb{R}$.

PROPOSITION 12. *Let $\mathcal{O}_0$ be a T^{n-1}-orbit of $\varphi \in A^2(\mathbb{R}^n, N)$. Then, there exists a φ-invariant neighborhood V_0 of $\mathcal{O}_0$, such that every connected component U of $V_0 \setminus \mathcal{O}_0$ satisfies one of the following properties:*

(1) *U is an $T^{n-1}\times\mathbb{R}$-orbit;*

(2) *U contains infinitely many T^{n-1}-orbits approaching $\mathcal{O}_0$ and every n-dimensional orbit inside U is a $T^{n-1}\times\mathbb{R}$-orbit;*

(3) *There exist $s\in\{0,1,\ldots,n-2\}$ such that U is the union of $T^s\times\mathbb{R}^{n-s-1}$-orbits.*

Proof. By the continuity of the infinitesimal generators of φ there is a neighborhood V of $\mathcal{O}_0$ such that every orbit by points in V_0 has dimension at least $n-1$. Since $r\geq 2$, it follows from Proposition 9 that there are no $T^s\times\mathbb{R}^{n-s}$-orbits, $s\neq n-1$ approaching $\mathcal{O}_0$. Therefore, if U is one connected component of $V\setminus\mathcal{O}_0$, then there are two possibilities:
(i) *There are infinitely many n-orbits in U approaching F.* In this case we will show that (2) is verified. The other possibility is that there exists a sequence $\{\mathcal{O}_i\}_{i\in\mathbb{N}}$ of n-dimensional orbits inside U, which are not homeomorphic to $T^{n-1}\times\mathbb{R}$ and that approach $\mathcal{O}_0$. Since $\mathrm{cl}(\mathcal{O}_i)\cap\mathrm{Sing}_j(\varphi)=\emptyset$ for each $j\in\{0,\ldots,n-2\}$, then the compact set $\mathrm{cl}(\mathcal{O}_i)\setminus\mathcal{O}_i$ is the union of $(n-1)$-dimensional orbits, of which at least one is compact, this contradicts the Proposition 9.
(ii) *There are only a finite number of n-orbits in U.* If it happens to exist a $T^{n-1}\times\mathbb{R}$-orbit $\mathcal{O}$ such that $\mathrm{Front}(\mathcal{O})\supset\mathcal{O}_0$, then we can assume that $U=\mathcal{O}$, therefore (1) is verified. If there is no such $T^{n-1}\times\mathbb{R}$-orbit, then we can assume that the orbit of each $p\in U$ is $(n-1)$-dimensional. We also can assume that φ has no T^{n-1}-orbits inside U, otherwise (2) is verified. Reducing the size of V, if necessary, it follows from Lemma 4 that there exist $s\in\{0,\ldots,n-2\}$ such that $\mathcal{O}_p$ is a $T^s\times\mathbb{R}^{n-s-1}$-orbit for each $p\in U$. Thus, (3) is verified and this completes the proof. ∎

Figure 3 illustrates Theorem C for some $\varphi\in\mathscr{A}_2$.

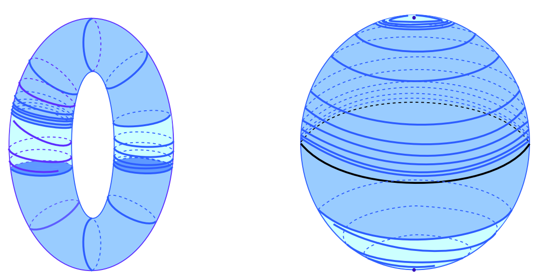

FIG. 3.

Proof (*of Theorem C*). Let $\mathcal{O}$ be a $T^{n-1}\times\mathbb{R}$-orbit and $\mathscr{U}$ the family of all φ-invariant neighborhoods $U\supset\mathcal{O}$ homeomorphic to $T^{n-1}\times\mathbb{R}$ that do not contain a $T^s \times \mathbb{R}^{n-s}$-orbit with $s < n-1$. The inclusion relation defines a parcial order in $\mathscr{U}$ and by Zorn's Lemma there exists a maximal element U_M in $\mathscr{U}$. We are going to show that $N \setminus U_M$ is either a T^{n-1}-orbit or the union of two T^{n-2}-orbits. Assume that $N \setminus U_M$ has non-empty interior, then $\mathrm{cl}(U_M) \setminus U_M$ has two connected components. By Definition 2 at each $p \in \mathrm{Sing}_i(\varphi)$, $i = 1, \dots, n-3$, φ is of type n and n-finite. This fact and Lemma 5 implie that $(\mathrm{cl}(U_M) \setminus U_M) \cap \mathrm{Sing}_i(\varphi) = \emptyset$, $i = 1, \dots, n-3$. Moreover, there exists one connected component F of $\mathrm{cl}(U_M) \setminus U_M$ that is not a T^{n-2}-orbit. We know that F is φ-invariant and will show that $F \cap \mathrm{Sing}^c_{n-2}(\varphi) = \emptyset$. In fact, if there exists $p \in F$ such that $\mathcal{O}_p$ is a T^{n-2}-orbit and V_p is a neighborhood of $\mathcal{O}_p$ that satisfies Definition 2 (1), then $U_M \cup V_p$ would be a member of $\mathscr{U}$ containing U_M properly. If V_p satisfies condition (2), then, since $F \neq \mathcal{O}_p$, there are $T^s \times \mathbb{R}^{n-s-1}$-orbits, $s \neq n-1$, arriving at $\mathcal{O}_p$ and by Remark 11 we would have $T^l \times \mathbb{R}^{n-l}$-orbits, $l \neq n-1$, inside U_M. Therefore F is an T^{n-1}-orbit. If (1) or (2) of Proposition 12 is verified, then there exists an open φ-invariant set V homeomorphic to $T^{n-1} \times \mathbb{R}$, which does not contain $T^s \times \mathbb{R}^{n-s}$-orbits with $s \neq n-1$ and such that $\mathrm{Front}(V) \subset F$. If (3) of Proposition 12 is verified, then by Theorem B there exists an open φ-invariant set V homeomorphic to $T^{n-1} \times \mathbb{R}$ and such that $\mathrm{Front}(V) \subset F$. The open set $U_M \cup V \in \mathscr{U}$ and contains properly U_M, but this contradicts the fact that U_M is maximal. Thus, $\mathrm{Front}(U_M) = N \setminus U_M$.

Assume that $\mathrm{Sing}^c_{n-2}(\varphi) = \emptyset$, then $\mathrm{Front}(U_M)$ is homeomorphic to T^{n-1}. Therefore, N is T^{n-1} bundle over S^1. If $\mathrm{Sing}^c_{n-2}(\varphi) \neq \emptyset$, then $\mathrm{Front}(U_M)$ is the union of two T^{n-2}-orbits consequently, $N \in \mathcal{H}_n$. ∎

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