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# On Properties of the Vertical Rotation Interval for Twist Mappings II*

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Dedicated to Professor Sotomayor $60^{th}$ birthday

In this paper we consider twist mappings of the torus, $\overline{T} : \mathrm{T}^2 \to \mathrm{T}^2$, and their vertical rotation intervals $\rho_V(T) = [\rho_V^-, \rho_V^+]$, which are closed intervals such that for any $\omega \in ]\rho_V^-, \rho_V^+[$ there exists a compact $\overline{T}$-invariant set $\overline{Q}_\omega$ with $\rho_V(\overline{x}) = \omega$ for any $\overline{x} \in \overline{Q}_\omega$, where $\rho_V(\overline{x})$ is the vertical rotation number of $\overline{x}$. In case $\omega$ is a rational number, $\overline{Q}_\omega$ is a periodic orbit (this study began in [1] and [2]). Here we analyze how $\rho_V^-$ and $\rho_V^+$ behave as we perturb $\overline{T}$ when they assume rational values. In particular we prove that, for an interesting class of mappings, these functions are locally constant at rational values.

*Key Words:* twist mappings, vertical rotation set, topological methods, saddle-nodes, analyticity, bifurcations.

## 1. INTRODUCTION AND MAIN RESULTS

Twist mappings are $C^1$ diffeomorphisms of the cylinder (or annulus, torus) onto itself that have the following property: The angular component of the image of a point increases as the radial component of the point increases (more precise definitions will be given below). Such mappings were first studied in connection with the three body problem by Poincaré and it was found that first return mappings for many problems in Hamil-

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tonian dynamics actually have the twist property. Although extensively studied, there are still many open questions about their dynamics. A great progress has been achieved in the nearly integrable case, by means of KAM theory (see [21]) and many important results have been proved in the general case, concerning the existence of periodic and quasi-periodic orbits (Aubry-Mather sets), see [19], [7], [16].

In this paper we continue studying twist mappings of the torus and their vertical rotation intervals (see [1], [2], [5], [3] and [4]). In fact, here we answer (at least for certain mappings) a question raised in [4]. Just to situate the problem, we remember that in [2] we proved that given an exact "area" preserving twist mapping $T : S^1\times\mathbb{R}\rightarrow S^1\times\mathbb{R}$ which induces a diffeomorphism $\overline{T} : \mathrm{T}^2 \rightarrow \mathrm{T}^2$ homotopic to the Dehn twist $(\phi, I) \rightarrow (\phi+I \ mod\ 1, I\ mod\ 1)$, there exists a closed interval $\rho_V(T) = [\rho_V^-, \rho_V^+]$ with the following property:

For $\omega \in int(\rho_V(T))$, there are 2 different situations.

1) $\omega = \frac{p}{q}$ is a rational number. In this case there is a $q$-periodic point $\overline{x}$ for $\overline{T}$ (in fact there are at least 2 such points) that lifts to a point $x \in S^1\times\mathbb{R}$ such that $T^q(x) = x + (0, p)$.

2) $\omega$ is an irrational number. Then there is a compact, $\overline{T}$-invariant set $\overline{Q} \subset \mathrm{T}^2$ that lifts to a set $Q \subset S^1\times\mathbb{R}$ such that for any $x \in Q$ we have,

$$\lim_{n\rightarrow\infty} \frac{p_2 \circ T^n(x) - p_2(x)}{n} = \omega,$$

where $p_2 : S^1\times\mathbb{R}\rightarrow\mathbb{R}$ is the projection on the vertical coordinate.

We also proved that $int(\rho_V(T)) \neq \emptyset$ if and only if $T$ does not have rotational invariant curves. Note that in this case $0 \in \rho_V(T)$.

As we pointed out in [4], even without the "area" preservation and exactness hypotheses, we still have a closed interval (the vertical rotation interval) $\rho_V(T) = [\rho_V^-, \rho_V^+]$ associated to $T$ with the same properties 1) and 2) above. But in this case it may happen that $0 \notin \rho_V(T)$ and we can not give a simple criteria to guarantee the non-degeneracy of $\rho_V(T)$ to a point.

The main goal of this paper is to study the following problem:

**Problem 1)** Suppose that $\rho_V^+$ (or $\rho_V^-$) is a rational number. Is it possible that by certain arbitrarily $C^1$-small perturbations one can increase its value and by another type of perturbation, one can decrease its value? In other words is the function $\rho_V^+(T)$ locally constant at rational values?

This question has a very natural motivation from the study of circle homeomorphisms. In particular when the rotation number of a circle homeomorphism is rational, then it is locally constant in the $C^0$ topology, provided that the mapping is not conjugate to a rigid rotation. Similar results for circle endomorphisms have been proved by Boyland in [8].

The solution we give to problem 1) does not have the same generality as the results from [4]. More precisely we prove that $\rho_V^+(T)$ is locally constant at rational values if we restrict ourselves to mappings of the following form:

$$T : \begin{cases} \phi' = f(\phi) + I' \ mod\ 1 \\ I' = I + k\log(1/f'(\phi)) \end{cases} , \tag{1}$$

where $f : S^1 \to S^1$ is any analytic circle diffeomorphism and $k > 0$ is a real number. We denote the above class of mappings by $D_{\log}$.

As a consequence of our methods, we also get that $\rho_V^+(T)$ is locally constant at rational values for a generic set of twist mappings on the torus in the $C^1$ topology. But unfortunately, until the present moment we were not able to prove this result in full generality. One may say that $D_{\log}$ is an artificial class of mappings used just to make things work, as for instance it does not contain the well-known standard mapping,

$$S_M : \begin{cases} \phi' = \phi + I' \ mod\ 1 \\ I' = I + \frac{k}{2\pi}\sin(2\pi\phi) \end{cases} .$$

In the following we will try to show that this is not the case. In fact, from a physical point of view, some mappings belonging to $D_{\log}$ are much more "realistic" than the standard mapping and their dynamics is richer, having a mixture of conservative and dissipative dynamics.

When studying four-dimensional real analytic Hamiltonian systems, a family of mappings belonging to $D_{\log}$ appears in the following way. Suppose the system has an equilibrium of saddle-center type (namely, it has pure real, $\pm\nu \neq 0$, and pure imaginary, $\pm\omega i \neq 0$, eigenvalues) and additionally there exists an orbit $\Gamma$ homoclinic to this equilibrium. The union of $\Gamma$ and the equilibrium is called saddle-center loop. Under a mild hypothesis on the topology of the energy level set of the saddle-center loop (see [14]) it is possible to define a Poincaré mapping to a transverse section to the saddle-center loop. This construction, due to Lerman [18] and Mielke et al. [20], is analogous to the more familiar one of a Poincaré mapping to a periodic orbit. The leading order term of the Poincaré mapping to the saddle-center loop, that we call saddle-center loop mapping, restricted to the energy level set of the saddle-center loop is given by the following area preserving mapping

$$(z_1, z_2) \to \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

where: $z = (z_1, z_2) \in \mathbb{R}^2$ has sufficiently small norm, $\theta = -\gamma \log \|z\|^2/2$, $\gamma = \omega/\nu > 0$, and $\alpha \geq 1$ is a parameter obtained from the flow of the system linearized at the orbit $\Gamma$ (it is analogous to a Floquet coefficient of a periodic orbit). The origin, $z=(0,0)$ represents the intersection of $\Gamma$ with the transverse section and it is, by definition, a fixed point of the saddle-center loop mapping.

In [15] and [6] it has been proved that many properties of the saddle-center loop mapping extend to the real Poincaré mapping and thus to the Hamiltonian flow. For instance, in [15] it is shown that for certain parameter values $(\alpha,\gamma)$ the stability (instability) of the origin under iterations of the saddle-center loop mapping implies a sort of stability (instability) of the saddle-center loop under the action of the Hamiltonian flow. In [6] it is shown that if the origin is unstable under iterations of the saddle-center loop mapping (what necessarily happens, for instance, if $\gamma(\alpha-\alpha^{-1})>1$, see [15]) then the set of orbits that escape from it may contain some special points that in some sense "attract" most of the escaping orbits. The reason for using the strange word "attract" in this conservative setting will become clearer below.

The logarithmic singularity of the saddle center loop mapping can be removed with the following choice of polar coordinates that blow up the origin

$$\begin{cases} z_1=\sqrt{2}\mathrm{e}^{\widetilde{I}/(2\gamma)}\cos(\widetilde{I}-\widetilde{\phi}) \\ z_2=\sqrt{2}\mathrm{e}^{\widetilde{I}/(2\gamma)}\sin(\widetilde{I}-\widetilde{\phi}). \end{cases}$$

In these coordinates the saddle-center loop mapping writes as:

$$\widetilde{F}:\begin{cases} \widetilde{\phi}'=\mu(\widetilde{\phi})+\widetilde{I}' \\ \widetilde{I}'=\gamma\log J(\widetilde{\phi})+\widetilde{I} \end{cases}, \text{ where} \tag{2}$$

$$\begin{aligned} J(\widetilde{\phi}) &= \alpha^2\cos^2(\widetilde{\phi})+\alpha^{-2}\sin^2(\widetilde{\phi}) \\ \mu(\widetilde{\phi}) &= \arctan\left(\frac{\tan(\widetilde{\phi})}{\alpha^2}\right), \text{with } \mu(0)=0 \end{aligned}$$

Using these coordinates we can easily identify the punctured plane with a cylinder $(\phi,I)\in S^1\times\mathbb{R}$ where $S^1=\mathbb{R}/\pi\mathbb{Z}$, or $\phi=\phi \; mod \; \pi$. We denote as $F$ the mapping induced by $\widetilde{F}$ (see (2)) on this cylinder. Clearly this mapping belongs to $D_{\log}$, because $\mu'(\widetilde{\phi})=1/J(\widetilde{\phi})$. The area preserving property of the saddle-center loop mapping implies that $F$ preserves the following measure on the cylinder:

$$\mu(A)=\int_A e^{I/\gamma}d\phi\wedge dI \tag{3}$$

All mappings belonging to $D_{\log}$ are invariant under vertical translations by 1 (in case of $F$ written as in (2) the translations are of multiples of $\pi$, so $F(\phi, I+\pi) = F(\phi, I)+\pi$). Thus, $F$ induces a mapping on the 2-torus, that we denote as $\overline{F}$, just by taking $I = I \; mod \; \pi$. Notice that the measure $\mu$ (3) is not invariant under $I$-translations by $\pi$ which implies that it does not define a measure on the quotient 2-torus $\phi \; mod \; \pi, I \; mod \; \pi$. So, the original measure preserving property of $F$ is lost in the quotient construction of $\overline{F}$. Clearly the same happens for any mapping belonging to $D_{\log}$.

If we consider parameters $(\alpha, \gamma)$ such that $F$ does not have rotational invariant circles, then $\overline{F}$ has certain properties of a conservative and a dissipative system. For instance all periodic points of $\overline{F}$ that are also periodic points of $F$ have a conservative character, that is, $\overline{F}$ can have elliptic points surrounded by invariant curves and also hyperbolic saddles with eigenvalues $\lambda$, $\lambda^{-1}$. Periodic points of $\overline{F}$ which represent vertical $I$-translations for $F$ have necessarily a dissipative or expansive character. In [1] and [5] the reader can find examples of complicated invariant sets of $\overline{F}$ with dissipative dynamics like, for instance, sinks and Hénon attractors. The consequences of the existence of attractors for $\overline{F}$ on the dynamics of the original Hamiltonian flow is not completely understood, yet. In [6] we prove several results in this direction. The main difficulty we find is that we still do not have a good description for the dynamics of $F$ or $\overline{F}$ for most values of the parameters $(\alpha, \gamma)$. In order to better explain this point let us get back to the analogy between the dynamics near a saddle-center loop and the dynamics near a periodic orbit. For the saddle-center loop the role played by the linear part of the Poincaré mapping to a periodic orbit is replaced by $F$. The dynamics of a linear mapping is very simple for any value of its eigenvalues while the dynamics of $F$, and its dependence on $(\alpha, \gamma)$, is very complicated. For instance when the dynamics of the linear mapping is unstable the set of orbits that escape from the periodic orbit has a very simple description given by the unstable manifold theorem. In the analogous case where the origin of the saddle-center loop mapping is unstable the unstable set of the saddle-center loop is very complicated and still not fully understood. An interesting partial result proved in [6] says that if the origin of the saddle-center loop mapping is unstable and $\overline{F}$ has a periodic topological sink, then there is a set of points of positive measure that escapes under the action of the original Hamiltonian flow from any sufficiently small neighborhood of the saddle-center loop following (or clustering around) a finite number of escaping orbits that correspond to the periodic sink of $\overline{F}$. Finally, in [5] we proved that $\overline{F}$ has attractive periodic orbits of all periods and all positive vertical rotation numbers for open sets of values of $(\alpha, \gamma)$.

We believe that the above application is a good enough reason for studying the class of mappings described in (1).

This paper is organized as follows. In the rest of this section we present some definitions and the statement of our main result. In the next section we present the statements of some previous results we use. In section 3 we present the proofs of our results.

*Notation and definitions* :

0) Let $(\phi, I)$ denote the coordinates for the cylinder $S^1 \times \mathbb{R} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$, where $\phi$ is defined modulo 1. Let $(\widetilde{\phi}, \widetilde{I})$ denote the coordinates for the universal cover of the cylinder, $\mathbb{R}^2$. For all mappings $T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ we define $(\phi', I') = T(\phi, I)$ and $(\widetilde{\phi}', \widetilde{I}') = \widetilde{T}(\widetilde{\phi}, \widetilde{I})$, where $\widetilde{T} : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of $T$.

1) $D^1_r(\mathbb{R}^2) = \{\widetilde{T} : \mathbb{R}^2 \to \mathbb{R}^2 \ / \ \widetilde{T}$ is a $C^1$ diffeomorphism of the plane, $\widetilde{I}'(\widetilde{\phi}, \widetilde{I}) \overset{\widetilde{I} \to \pm\infty}{\to} \pm\infty$, $\partial_{\widetilde{I}} \widetilde{\phi}' > 0$ (twist to the right), $\widetilde{\phi}'(\widetilde{\phi}, \widetilde{I}) \overset{\widetilde{I} \to \pm\infty}{\to} \pm\infty$ and $\widetilde{T}$ is the lift of a $C^1$ diffeomorphism $T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}\}$.

2) $Diff^1_r(S^1 \times \mathbb{R}) = \{ T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R} \ / \ T$ is induced by an element of $D^1_r(\mathbb{R}^2)\}$.

3) Let $p_1 : \mathbb{R}^2 \to \mathbb{R}$ and $p_2 : \mathbb{R}^2 \to \mathbb{R}$ be the standard projections, respectively in the $\widetilde{\phi}$ and $\widetilde{I}$ coordinates ($p_1(\widetilde{\phi}, \widetilde{I}) = \widetilde{\phi}$ and $p_2(\widetilde{\phi}, \widetilde{I}) = \widetilde{I}$). We also use $p_1$ and $p_2$ for the standard projections of the cylinder.

4) Given a measure $\mu$ on the cylinder that is positive on open sets, absolutely continuous with respect to the Lebesgue measure and a mapping $T \in Diff^1_r(S^1 \times \mathbb{R})$ we say that $T$ is $\mu$-exact if $\mu$ is invariant under $T$ and for any open set $A$ homeomorphic to the cylinder we have:

$$\mu(T(A) \backslash A) = \mu(A \backslash T(A)) \tag{4}$$

5) Let $D_{\log} \subset Diff^1_r(S^1 \times \mathbb{R})$ be such that, for all $T \in D_{\log}$ there exists an analytic circle diffeomorphism $f$ such that $T$ writes as in (1).

Every $T \in D_{\log}$ induces a mapping $\overline{T} : \mathrm{T}^2 \to \mathrm{T}^2$, where $\mathrm{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the 2-torus. Coordinates in the torus are denoted by $(\overline{\phi}, \overline{I})$.

6) Let $D_{Dehn} = \{T \in Diff^1_r(S^1 \times \mathbb{R}) : T$ induces a mapping $\overline{T} : \mathrm{T}^2 \to \mathrm{T}^2$ homotopic to the Dehn twist $(\phi, I) \to (\phi + I \ mod \ 1, I \ mod \ 1)\}$

7) Let $pro : S^1 \times \mathbb{R} \to \mathrm{T}^2$ be given by:

$$pro(\phi, I) = (\phi, I \ mod \ 1)$$

8) Given a point $\overline{x} \in \mathrm{T}^2$, we define its vertical rotation number as (when the limit exists):

$$\rho_V(\overline{x}) = \lim_{n \to \infty} \frac{p_2 \circ T^n(x) - p_2(x)}{n}, \text{ for any } x \in pro^{-1}(\overline{x}) \tag{5}$$

Now we are ready to state our main result:

THEOREM 1. *Let* $T \in D_{\log}$ *be such that* $\rho_V(T) = [\rho_V^-, \frac{p}{q}]$, *with* $\frac{p}{q} \in \mathbb{Q}$, $(p,q) = 1$ *and* $\rho_V^- < \frac{p}{q}$. *Then, if* $V \subset Diff_r^1(S^1 \times \mathbb{R})$ *is any sufficiently small neighborhood of* $T$ *in the* $C^1$ *topology, one and only one of the following possibilities holds:*
*1) for all* $G \in V$, $\rho_V^+(G) \leq p/q$
*2) for all* $G \in V$, $\rho_V^+(G) = p/q$
*3) for all* $G \in V$, $\rho_V^+(G) \geq p/q$

Remark:
Given an element $T$ of $D_{\log}$ as in (1), a simple computation shows that it is exact with respect to the following measure:

$$\mu_k(A) = \int_A e^{I/k} d\phi \wedge dI$$

So, theorem 3 of [2] implies that $\rho_V^- < 0 < \frac{p}{q}$ if and only if $T$ does not have rotational invariant curves. Thus, the above theorem may be applied to every mapping from $D_{\log}$ which does not have rotational invariant curves, something that necessarily happens if $k > 0$ is large enough.

Now let us present a "generic" version of the above result:

THEOREM 2. *For each rational number* $p/q$, *there exists an open and dense subset* $D_{gen}^{p/q} \subset \mathcal{W}_{p/q}^{crit} = \{T \in D_{Dehn} : \rho_V^-(T) < \rho_V^+(T) = p/q$ *and* $\rho_V^+(T_i) < p/q$, *for a certain sequence of twist mappings* $T_i \in D_{Dehn}$, $T_i \overset{i\to\infty}{\to} T$ *in the* $C^1$ *topology*$\}$, *such that for any* $T \in D_{gen}^{p/q}$ *there is a open neighborhood* $V \subset Diff_r^1(S^1 \times \mathbb{R})$ *of* $T$ *that satisfies:* $\rho_V^+(G) \leq p/q$, *for all* $G \in V$.

## 2. BASIC TOOLS

In this part of the paper we state all the results we use and present adequate references.

### 2.1. Results from [2] and [3]:

The first theorem we present asserts the existence of the vertical rotation interval and give some of its properties:

THEOREM 3. *To each mapping* $T \in D_{Dehn}$, *we can associate a closed interval* $\rho_V(T) = [\rho_V^-, \rho_V^+]$, *possibly degenerated to a single point, such that for every* $\omega \in ]\rho_V^-, \rho_V^+[$ *there is a compact* $\overline{T}$*-invariant set* $\overline{Q}_\omega \subset \mathrm{T}^2$ *with* $\rho_V(\overline{x}) = \omega$, *for all* $\overline{x} \in \overline{Q}_\omega$. *If* $\omega$ *is a rational number* $\frac{p}{q}$, *then* $\overline{Q}_\omega$ *is a q-periodic orbit. In fact, in this case there are at least 2 periodic orbits.*

*If the periodic orbits are finite, then at least one has positive index and another one has negative index.*

For a proof, see theorem 6 of the appendix of [5], theorem 5 of [2] and the results bellow from [4].

The following theorem is a combination of proposition 1 and theorem 3 of [5].

THEOREM 4. *Given any $T \in D_{\log}$, let $\overline{T}$ be the torus mapping induced by $T$. For every rational number $p/q \neq 0$, the $\overline{T}$-$q$-periodic points with vertical rotation number $p/q$ are finite. Moreover,* $\det[D\overline{T}^q \mid_P] = e^{-p\pi/k}$ *for every $q$-periodic point $P$ of $\overline{T}$ with $\rho_V = p/q$. In particular, if a periodic point $P$ is degenerate, then it must be a saddle-node (or a saddle-source).*

### 2.2. Results from [5]:

THEOREM 5. *The functions $\rho_V^+, \rho_V^- : D_{Dehn} \to \mathbb{R}$ are continuous in the $C^1$ topology.*

LEMMA 6. *Given $T \in D_{Dehn}$, let $f : \mathbb{R} \to \mathbb{R}$ be given by: $f(\alpha) = \rho_V^+(T_\alpha)$. Then $f$ is a non-decreasing function of $\alpha$.*

Remember that $T_\alpha : S^1 \times \mathbb{R} \to \mathrm{S}^1 \times \mathbb{R}$ is given by $T_\alpha(\phi, I) = T(\phi, I) + (0, \alpha)$.

### 2.3. Results from [6]:

LEMMA 7. *Let $T \in D_{Dehn}$ be such that $\rho_V(T) = [\rho_V^-, \frac{p}{q}]$, with $\rho_V^- < \frac{p}{q}$, $\frac{p}{q} \in \mathbb{Q}$ and $(p,q) = 1$. Suppose also that there is a neighborhood $\mathcal{U}$ of $T$ in $D_{Dehn}$ such that for any $T^* \in \mathcal{U}$, $\rho_V^+(T^*) \geq \frac{p}{q}$. Then $T$ has periodic orbits with vertical rotation number $\rho_V = \frac{p}{q}$ that can not be destroyed by arbitrarily small perturbations.*

LEMMA 8. *The set $(\rho_V^+)^{-1}(\omega) = \left\{T \in D_{Dehn} : \ \rho_V^+(T) = \omega\right\}$ is a path connected subset of $D_{Dehn}$ for any $\omega \in \mathbb{R}$.*

To conclude we present a lemma similar to the main result of [17], which says that we do not need to consider general perturbations in this setting. Vertical translations are enough for all applications.

LEMMA 9. *Let $T \in D_{Dehn}$ be such that $\rho_V(T) = [\omega^-, \omega^+]$. Suppose that by an arbitrarily $C^1$-small perturbation applied to $T$, we can change $\omega^+$, that is, there exists $T^*$ arbitrarily $C^1$-close to $T$, such that $\rho_V^+(T^*) \neq \omega^+$. Then for any given $\epsilon > 0$, at least one of the following inequalities must hold:*
*1) $\rho_V^+(T_\epsilon) \neq \omega^+$, or*
*2) $\rho_V^+(T_{-\epsilon}) \neq \omega^+$.*

*Moreover, given $T \in D_{Dehn}$ with $\rho_V(T) = [\omega^-, \omega^+]$, there exists a neighborhood $T \in \mathcal{U} \subset D_{Dehn}$ such that for any $T^* \in \mathcal{U}$, $\rho_V^+(T^*) = \omega^+$, if and only if, $\exists \epsilon > 0$ such that $\rho_V^+(T_\alpha) = \omega^+$, for all $\alpha \in [-\epsilon, \epsilon]$.*

## 3. PROOFS

### 3.1. Proof of theorem 1:

Our proof of theorem 1 is by contradiction. We suppose that there is a mapping $T_k$ satisfying the theorem hypothesis and such that the following holds:

$$\text{for every } \alpha \neq 0,\ \rho_V^+(T_{k,\alpha}) \neq p/q, \text{ where } T_{k,\alpha} : S^1 \times \mathbb{R} \to \mathrm{S}^1 \times \mathbb{R} \text{ is:}$$
$$T_{k,\alpha} : \begin{cases} \phi' = f(\phi) + I + k\log(1/f'(\phi))\ mod\ 1 \\ I' = I + k\log(1/f'(\phi)) + \alpha \end{cases} \tag{6}$$

Using lemma 9, we get that theorem 1 is true if and only if the above assertion is impossible. Thus we are left to show that (6) is impossible. As $\rho_V^- < p/q$ and $\rho_V^+(T_{k,\alpha}) > p/q$ for all $\alpha > 0$ (see lemma 6), we get from theorems 3 and 5 that $\overline{T}_{k,0}$ has $q$-periodic points with $\rho_V = p/q$.

Lemma 6 again implies that for any $\alpha < 0$, $\rho_V^+(T_{k,\alpha}) < p/q \Rightarrow$ the $q$-periodic points of $\overline{T}_{k,0}$ with $\rho_V = p/q$ are degenerate. Moreover, theorem 4 implies that these points are finite (so their topological index is zero) and they all have the same eigenvalues: $\{1, e^{-p\pi/k}\}$. As $p/q > 0$ (see the remark below theorem 1), they are all saddle-nodes.

Let us define $F_\alpha(\phi, I) \overset{def.}{=} T_{k,\alpha}^q(\phi, I) - (0, p)$. Clearly $F_0$ induces a torus diffeomorphism $\overline{F}_0$ with a finite number of fixed points $\{P_1, P_2, ..., P_N\}$ of zero vertical rotation number, which are all saddle-nodes.

Let us choose neighborhoods $\{V_1, V_2, ..., V_N\}$ of the $P_{i's}$ with $V_i \cap V_j = \emptyset$ $(i \neq j)$, $diam.(V_i) < d$, for $i = 1, 2, ..., N$ and an $\epsilon > 0$ such that:

$$\begin{array}{l} 1)\ \text{for every } \alpha \in [0, \epsilon],\ \text{the fixed points of } \overline{F}_\alpha \text{ with zero} \\ \text{vertical rotation number are contained in } V_1 \cup V_2 \cup ... \cup V_N \\ 2)\ q.\epsilon + d < p/2 \\ 3)\ \rho_V^-(T_{k,\alpha}) < p/q,\ \text{for any } \alpha \in [0, \epsilon] \end{array} \tag{7}$$

As we already said, using lemma 6 we get that for any fixed $0 < \overline{\alpha} < \epsilon$, $\rho_V^+(T_{k,\overline{\alpha}}) > (p.n_+ + 1)/q.n_+ > p/q$, for a sufficiently large $n_+ \in \mathbb{N}$, which implies that there exists a point $Q_+ \in S^1 \times \mathbb{R}$ such that $F_{\overline{\alpha}}^{n_+}(Q_+) = Q_+ + (0, 1)$. Clearly, condition 3 of (7) implies the existence of another point $Q_- \in S^1 \times \mathbb{R}$ such that $F_{\overline{\alpha}}^{n_-}(Q_-) = Q_- - (0, 1)$, for a sufficiently large $n_- \in \mathbb{N}$. Now we:

*Claim* : As $\alpha$ goes from 0 to $\overline{\alpha}$, each $P_i$ bifurcates into FINITELY many new fixed points, which are all inside $V_i$. At least one of these points is a topological saddle and one is a topological sink. Moreover, all these new fixed points are connected by the continuation of the center manifold of $P_i$ with respect to the parameter $\alpha$. We just may have to choose a smaller $\overline{\alpha} > 0$

For a proof of this claim, see lemma 1 and theorem 4 of [5] and [11]. The fact that EVERY $P_i$ bifurcates into at least 2 topologically non-degenerate new fixed points is a consequence of some results from [4], see for instance the ideas in the proof of corollary 2 of that paper. The only part of the claim that needs a proof, is the fact that each $P_i$ bifurcates into FINITELY many new fixed points as $\alpha$ goes from 0 to $\overline{\alpha}$. This result is not immediately implied by theorem 4 because if $\alpha \neq 0$, then $T_{k,\alpha} \notin D_{\log}$.

To prove it, first of all note that:

$$\det[DT_{k,\overline{\alpha}} \mid_{(\phi,I)}] = f'(\phi)$$
$$\Downarrow$$
$$\det[DT^q_{k,\overline{\alpha}} \mid_{(\phi,I)}] = f'(\phi).f'(p_1 \circ T_{k,\overline{\alpha}}(\phi,I))...f'(p_1 \circ T^{q-1}_{k,\overline{\alpha}}(\phi,I))$$

From expression (6), we get that

$$\begin{aligned} p_2 \circ T^q_{k,\overline{\alpha}}(\phi,I) &= k\log(1/[f'(\phi)...f'(p_1 \circ T^{q-1}_{k,\overline{\alpha}}(\phi,I))]) + q.\overline{\alpha} + I = \\ &= -k\log(\det[DT^q_{k,\overline{\alpha}} \mid_{(\phi,I)}]) + q.\overline{\alpha} + I \end{aligned} \tag{8}$$

Now if we suppose that $\overline{F_{\overline{\alpha}}}$ has infinitely many fixed points of zero vertical rotation number, then as it is an analytic diffeomorphism, by the same argument used in the proof of theorem 4, we get that there is a simple closed curve $\gamma \subset S^1 \times \mathbb{R}$ such that every $P \in \gamma$ is a fixed point of $F_{\overline{\alpha}}$. Clearly, from conditions 1 and 3 of (7) we get that the projection of $\gamma$ to the torus is contained in some $V_i$, thus it is homotopically trivial. Let $A$ be the bounded connected component of $\gamma^c$. From the previous remarks, $diam.(A) < d$. As $F_{\overline{\alpha}}(\gamma) = \gamma \Rightarrow F_{\overline{\alpha}}(A) = A \Leftrightarrow T^q_{k,\overline{\alpha}}(A) = A + (0,p)$. So given any point $x \in A$, we get that

$$p_2 \circ T^q_{k,\overline{\alpha}}(x) - p_2(x) > p - d \ \Rightarrow\ \det[DT^q_{k,\overline{\alpha}} \mid_x] < e^{-p/(2k)},$$

by expressions (7) and (8). But this contradicts the fact that, as $A$ is an open set, it has positive Lebesgue measure.

In Figure 1 we present an example of a possible dynamics in some $V_i$. In each $V_i$ we choose another open set $W_i = W_i^n \cup W_i^s$, as in figure 1. This is always possible due to some classical results in center manifold theory, again see for instance [11].

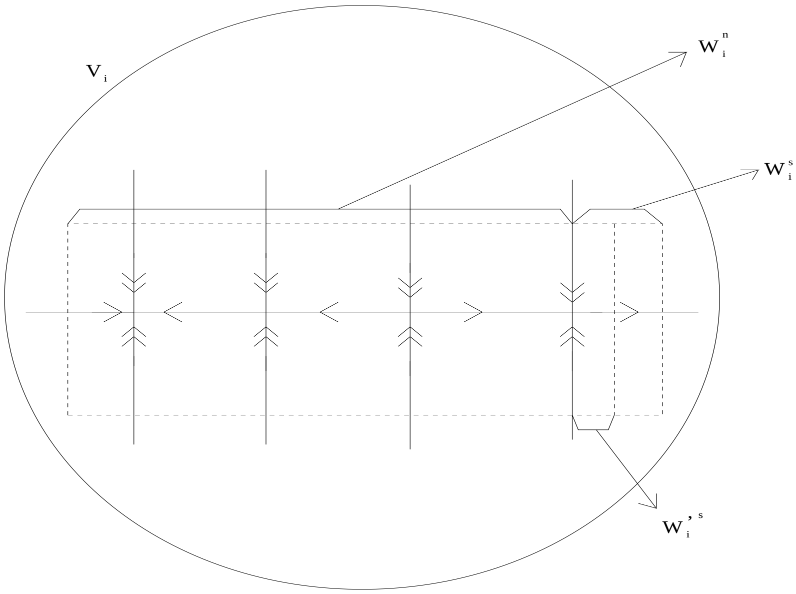


**FIG. 1.** Diagram showing the dynamics inside one $V_i$ after the bifurcation.

If we denote by $\overline{Q}_-$ and $\overline{Q}_+$ the projections of $Q_-$ and $Q_+$ to the torus, we get that the orbit of both these points by $\overline{F}_{\overline{\alpha}}$ can not fall inside any $W_i^n$, otherwise it would be attracted to a fixed point of $\overline{F}_{\overline{\alpha}}$ with zero vertical rotation number, something that contradicts the periodicity of $Q_-$ and $Q_+$. So if the orbit of any of these points fall inside some $W_i$, then it must belong to $W_i^s$. As $\overline{Q}_-$ and $\overline{Q}_+$ are periodic points for $\overline{F}_{\overline{\alpha}}$ (thus their orbits are finite), we can decrease the size of all the $W_i^s$, $i = 1, 2, ..., N$ in a way that

$$\left\{\left[\overline{Q}_- \cup .. \cup \overline{F}_{\overline{\alpha}}^{n_- -1}(\overline{Q}_-)\right] \cup \left[\overline{Q}_+ \cup .. \cup \overline{F}_{\overline{\alpha}}^{n_+ -1}(\overline{Q}_+)\right]\right\} \cap \{W_1' \cup .. \cup W_N'\} = \emptyset,$$

where $W_i' = W_i^n \cup W_i'^s$ and $W_i'^s$ is a new (possibly smaller) $W_i^s$. Again using results of the center manifold theory (see also the more general result in [22]), we can destroy all the fixed points inside $W_1' \cup ... \cup W_N'$ by a suitable deformation of $\overline{F}_{\overline{\alpha}}$, obtaining a diffeomorphism $\overline{F}_{\overline{\alpha}}^*$ which is equal $\overline{F}_{\overline{\alpha}}$ outside $W_1' \cup ... \cup W_N'$. This means that $\overline{F}_{\overline{\alpha}}^*$ does not have fixed points with zero vertical rotation number and $(F_{\overline{\alpha}}^*)^{n_+}(Q_+) = Q_+ + (0, 1)$ and $(F_{\overline{\alpha}}^*)^{n_-}(Q_-) = Q_- - (0, 1)$. But this contradicts a generalization of some results of [13] to this context. To be more precise, it contradicts the next theorem due to H.E.Doeff (see [12], theorem 5.3).

THEOREM 10. *If* $\overline{F}^*_{\overline{\alpha}} : \mathrm{T}^2 \to \mathrm{T}^2$ *is a homeomorphism homotopic to* $(\phi, I) \to (\phi + n.I \; mod\; 1, I \; mod\; 1)$, *for a certain* $n \in \mathbb{N}^*$ *and there are points* $Q_+, Q_- \in \mathrm{T}^2$ *such that* $\rho_V(Q_-) < 0 < \rho_V(Q_+)$, *then* $\overline{F}^*_{\overline{\alpha}}$ *has a fixed point of zero vertical rotation number.*

Remark: We have to apply Doeff's result because we do not know whether or not $\overline{F}^*_{\overline{\alpha}}$ is a tilt mapping (see [9]). If it has the tilt property, which in some sense is a generalization of the twist property, then the methods developed in [2] give the same result as above.

### 3.2. Proof of theorem 2:

Given a mapping $T^* \in \mathcal{W}^{crit}_{p/q}$, from [4] we have 2 possibilities:

i) $\overline{T}^*$ does not have $q$-periodic points with vertical rotation number $p/q$.

ii) $\overline{T}^*$ has a $q$-periodic point with vertical rotation number equal $p/q$.

The first possibility above implies that every nearby mapping also does not have $q$-periodic points with vertical rotation number equal $p/q$. So, theorem 3 implies $\rho^+_V(G) \leq p/q$, for all $G$ in a sufficiently $C^1$-small neighborhood of $T^*$.

If the second possibility holds, then the theorem hypothesis imply that the $q$-periodic points of $\overline{T}^*$ with vertical rotation number equal $p/q$ are degenerate, that is, they can be destroyed by $C^1$-arbitrarily small perturbations. In this case, the Kupka-Smale theorem and results of [10] say that for an open and dense subset of $\mathcal{W}^{crit}_{p/q}$ (that will be called $D^{p/q}_{gen}$), we can suppose that the $q$-periodic points of $\overline{T}^*$ with vertical rotation number $p/q$ consist of a finite number of saddle-nodes and saddle-sources.

So as in the main theorem, suppose that there exists a $T^* \in \mathcal{D}^{p/q}_{gen} \subset \mathcal{W}^{crit}_{p/q}$ such that the following holds:

$$\begin{array}{c}\text{for every } \alpha \neq 0,\ \rho^+_V(T^*_\alpha) \neq p/q,\ \text{where } T^*_\alpha : S^1 \times \mathbb{R} \to \mathrm{S}^1 \times \mathbb{R} \text{ is:} \\ T^*_\alpha(\phi, I) = T^*(\phi, I) + (0, \alpha)\end{array} \tag{9}$$

From the choice of $D^{p/q}_{gen}$ we can conclude the proof exactly as in the above theorem. So we get that $\rho^+_V(T^*_\alpha) = p/q$, for all sufficiently small $\alpha > 0$, that is, (9) is not true.

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