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On the Structure of Transitive ω-limit Sets for Continuous Maps*

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For continuous maps in compact metric spaces, the admissible topological structure of attractors of single trajectories is discussed. For transitive ω-limit sets, the admissible topological structure and the dynamics on their arc components are described. Examples of sets with very simple structure, which fail to be ω-limit sets in $\mathbb{R}^2$, are suggested. It is proved that for a complete characterization of ω-limit sets in terms of arc components, we should take into consideration not only the number of arc components and their intersections but also the way in which convergence continua in the set are approximated.

Key Words: discrete dynamical system, ω-limit set.

1. INTRODUCTION

A *discrete dynamical system* consists of a compact space X and a discrete (semi)group of mappings of X into itself generated by the iteration of a continuous mapping f of X into itself. In this paper, the space X is supposed to be a metric space.

Let f^n denote the n th iteration of f and let x be a point of X. Then the set of points

$$\bigcup_{i=0}^{\infty} f^i x$$

is called the *orbit* of x or, strictly saying, the positive semi-orbit of x (for $i = 0$, we assume $f^0 x = x$). The set of all limit points of the sequence $x,\ fx,\ \ldots,\ f^n x,\ \ldots$ is called the *ω-limit set* of x and will be denoted by $\omega_f(x)$. For continuous maps on compact metric spaces, any ω-limit set is

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nonempty, closed and strictly invariant (i.e., $f(\omega_f(x)) = \omega_f(x)$, see e.g. [12]).

We say that a map $f : X \to X$ is *transitive* if, for every pair of open sets U and V, one can find $n \geq 1$ such that $f^n(U) \cap V \neq \varnothing$. For separable metric spaces, such a transitivity is equivalent to having a dense orbit.

From a general viewpoint, an ω-limit set can be considered as a kind of attractor for trajectories of a dynamical system. What sets can be attractors for discrete dynamical systems under the condition of transitivity on the attractor? At present it is well known that, in general, attractors may have very complicated structure. For example, there exists a transitive homeomorphism of the pseudoarc [10] and it is known that there are certain natural invariant subsets for the complex exponential family $E_\lambda(z) = \lambda e^z$, which are also topologically pseudoarcs [5]. Extremely pathological invariant sets could not be avoided even in smooth dynamics (see e.g. [7] where it was shown that the pseudocircle can be an invariant set for an area preserving C^∞ plane diffeomorphism).

Our goal in this paper is to discuss the admissible topological structure of ω-limit sets for continuous maps on compact metric spaces. In this direction the following results have been established earlier. For continuous maps of the interval, a complete topological characterization of ω-limit sets was obtained in [1]. It was shown that the admissible topological structure of ω-limit sets on the real line is exhausted by closed nowhere dense sets and finite collections of closed nondegenerate intervals. In higher dimensions, the problem is still open and only particular results were obtained. It was proved in particular that a (closed) totally disconnected compact set, a continuum with empty interior, and a union of finitely many disjoint nondegenerate Peano continua are ω-limit sets for continuous maps in $\mathbb{R}^n$, $n \geq 1$ [2, 3]. Some interesting results on ω-limit sets with interior for so called triangular maps was obtained in [9]. Note also that the problem of topological characterization of minimal attraction centers in $\mathbb{R}^n$ can be reduced to the problem of obtaining such a characterization for ω-limit sets [14].

In this paper, we consider the problem from the viewpoint of the continuum theory and describe the admissible topological structure of ω-limit sets using the notion of arc component of a set. The main result is represented by Theorem 4.

2. PRELIMINARIES

Let X be a compact metric space and f be a continuous map of X into itself. The system (X, f) is then said to be a discrete dynamical system.

It is said that X is *f-connected* [6] if for every closed proper subset A of X we have $f(A)\cap\overline{CA}\neq\varnothing$, where $\overline{CA}$ denotes the closure of the complement of A in X and $\varnothing$ is the empty set.

Let (X,f) be a discrete dynamical system. It is said that X is an ω-limit set if there exists a discrete dynamical system (Y,g) such that $X\subseteq Y$ and $g=f$ on X, and a point $x\in Y$ such that X is the ω-limit set of x under g, i.e., $X=\omega_g(x)$.

The following statement suggests a complete characterization of ω-limit sets. (It was proved in [6] for homeomorphisms and rediscovered in slightly different form in [13].)

Theorem Ω ([6], cf. also [13]). *A necessary and sufficient condition for X to be an ω-limit set is that X be f-connected.*

This statement indicates a dependence between the topological structure of ω-limit sets and the dynamics. It easily implies that finite ω-limit sets are cycles. Another consequence is the fact that isolated points of any infinite ω-limit set cannot be periodic.

The following two propositions provide some useful information about the dynamics on components of ω-limit sets (recall that for a space X, a *component* of X is a maximal connected subset of X).

Proposition A. If X is an ω-limit set and the number of components of X is finite, then each component is mapped by the function onto another in a cyclical manner.

Proof. It is an immediate consequence of Theorem Ω and the fact that X is invariant. ▮

The following proposition is an extension of the related one-dimensional result from [13].

Proposition B. If X is an ω-limit set consisting of infinitely many components, then X has empty interior (in Y) and each component of X contains at most one point of the orbit, ω-limit point of which is X.

Proof. Let $X=\omega_g(x)$. Suppose C is a component of X, which contains some $f^i(x)$ and $f^{i+m}(x)$. Then $g^m(C)\cap C\neq\varnothing$ and hence $g^m(C)\subseteq C$. For $j=0,\dots,m-1$, let K_j be the component that contains $g^{i+j}(C)$. Then, for all $k\geq m$, we have $g^k(x)\in\bigcup_{j=0}^{m-1}g^{i+j}(C)\subseteq\bigcup_{j=0}^{m-1}K_j$. Hence only finitely many points of the trajectory of x are outside $\bigcup_{j=0}^{m-1}K_j$. This contradicts the supposition that X has infinitely many components because any component can be separated from any finite number of other ones by an open set. ▮

3. MAIN RESULTS

We say that a set X is a *transitive ω-limit set* (or simply *transitive*) if $X = \omega_f(x)$ for a point $x \in X$.

By a *continuum* we mean a nonempty, compact, connected metric space. When referring to a space, the term *nondegenerate* means the space consists of more than one point. A topological space X is said to be *perfect* provided that every point of X is a limit point of X. By analogy with this notion we say that X is *C-perfect* if any of its component is a limit of other components, i.e., if for any connected component C of X, we have $\overline{X \backslash C} = X$.

The following simple theorem states general restrictions on the number and mutual location of components in transitive ω-limit sets.

THEOREM 1. *If X is transitive, then it is either a finite set, a union of finitely many disjoint nondegenerate continua, or a C-perfect set.*

Proof. If the number of components is finite, we apply Proposition 1. Otherwise Proposition 2 should be applied. ∎

It is clear that any finite collection of points in $\mathbb{R}^n$, $n \geq 1$, is an ω-limit set (it is a cycle for a continuous map). The following theorem states that any Cantor set is a transitive ω-limit set. (A *Cantor set* is defined to be a compact, totally disconnected, and perfect metric space. A topological space is said to be *totally disconnected* if each component of the space is a one-point set.)

THEOREM 2. *Any nonvoid totally disconnected perfect closed set is a transitive ω-limit set.*

Proof. It is known that a nonvoid closed subset X of the real line $\mathbb{R}^1$ is a transitive ω-limit set if and only if it is either a finite set, a Cantor set, or a union of finitely many disjoint nondegenerate closed intervals[1].

The theorem is implied by the well known fact that every nonvoid totally disconnected perfect set is a homeomorph of the Cantor set on the real line (see e.g. [11]): if we have a transitive continuous map f of the Cantor set C on the real line and a homeomorphism h between C and X, then the map $F = h \circ f \circ h^{-1}$ is a transitive continuous map of X (the indicated relation between maps F and f means the topological conjugacy). ∎

Recall that a *Peano continuum* can be defined as a continuous image of $[0, 1]$. If X has finitely many components, the following result takes place [3].

THEOREM 3. *If $X \subset \mathbb{R}^n$ consists of finitely many disjoint nondegenerate Peano continua, then X is a transitive ω-limit set.*

A set is said to be *arcwise connected* if each two points in the set belong to some homeomorph of the interval $[0,1]$ entirely contained in the set. For a space X, an *arc component* of X is a maximal arcwise connected subset of X. We say that an arc component E of X is *essential* if $\overline{X \setminus E} \neq X$, i.e., some points in E cannot be approximated by points from other arc components of the space X.

In the following theorem we take into account the internal topological structure of transitive ω-limit sets with finitely many components for continuous maps in locally arcwise connected spaces.

Theorem 4. *Let X be transitive. If an arc component E_0 of X is essential, then $f^n(E_0) \subset E_0$ for some n and*

$$X = \bigcup_{i=0}^{n-1} \overline{E_i}$$

where $E_1, \ldots, E_{n-1}$ are disjoint arc components containing $f(E_0), \ldots, f^{n-1}(E_0)$ respectively. For relatively open subsets $O_i = X \setminus \bigcup_{j \neq i} \overline{E_j}$, $i = 0, \ldots, n-1$, one has $O_i \neq \varnothing$, $O_{(i+1)\mathrm{mod} n} \subset f(O_i) \subset \overline{O_{(i+1)\mathrm{mod} n}}$, and

$$X = \bigcup_{i=0}^{n-1} \overline{O_i}\ .$$

At last, for the set $R = X \setminus \bigcup_{i=0}^{n-1} O_i$, which is nowhere dense and closed in X, one has

$$f(R) \subset R$$

(whenever $R \neq \varnothing$).

Proof. The case of finite X is trivial so we consider only the case of infinite X. Let $X = \omega_f(x)$ for a continuous f and a point $x \in X$. Let E_0 be an essential arc component of X. In this case, for some open set U, we have $U \cap E_0 \neq \varnothing$ and $U \cap (X \setminus E_0) = \varnothing$. Since the trajectory of x is dense in X, we can suppose without loss of generality that $x \in U \cap E_0$.

Recall that the f-image of any arcwise connected subset of X is arcwise connected. Since the trajectory of x must return to U, there exists $n \geq 1$ such that $f^n(E_0) \subset E_0$. For the sake of definiteness, we assume that $f^i(E_0) \cap E_0 = \varnothing$ for $0 < i < n$. For each $i = 1, 2, \ldots, n-1$, let E_i be the arc component of $f^i(E_0)$ in X, i.e., the maximal arcwise connected subset in X containing $f^i(E_0)$. Evidently, for any $i \in \{0, 1, \ldots, n-1\}$, we have $f(E_i) \subset E_{(i+1)\mathrm{mod} n}$.

Now the proof can be accomplished by proving the following claims.

Claim 5.

$$X = \bigcup_{i=0}^{n-1} \overline{E_i} \ .$$

Proof. It is immediately implied by the condition $X = \omega_f(x)$ and the supposition $x \in E_0$. Note only that each E_i, $0 \le i < n$, contains infinitely many points because otherwise the trajectory of x should be eventually periodic and hence the ω-limit set should be finite. ▮

Remark 6. Since the number of E_i is finite, the set X has finitely many components and can be represented in the form

$$X = \bigcup_{0 \le j < k} C_j,$$

where C_j are mutually disjoint continua. Here, k is a divisor of n and

$$C_j = \bigcup_{0 \le l < m} \overline{E_{j+kl}}$$

where $m = n/k$. For any $j \in \{0, \dots, k-1\}$, we have $f(C_j) = C_{(j+1) \bmod k}$.

For $i = 0, 1, \dots, n-1$, consider sets

$$O_i = X \setminus \bigcup_{j \ne i} \overline{E_j},$$

which are relatively open in X, and the set

$$R = X \setminus \bigcup_{i=0}^{n-1} O_i \ ,$$

which is closed.

Claim 7. If R is not empty, then $f(R) \subset R$.

Proof. The inclusion $f(R) \subset R$ is implied by the fact that R consists of points belonging simultaneously to closures of at least two sets E_i, E_j with different i, j and this property persists under iterations of f. ▮

Claim 8.

$$X = \bigcup_{i=0}^{n-1} \overline{O_i} \ .$$

Proof. Suppose by contradiction that the union of all sets $\overline{O_i}$ is not X. Then, for some open U, we have $U \cap X \neq \varnothing$ and $U \cap O_i = \varnothing$ for $i = 0, 1, \dots, n-1$. This means $U \cap X \subset R$. Since the trajectory of x is dense in X, we have $f^m(x) \in R$ for some m and hence by Claim 7 we obtain $f^i(x) \in R$ for all $i \geq m$. Since the arc component E_0 of X is essential, this contradicts the fact that the trajectory must return to O_0. ▮

Claim 9. For all $i \in \{0, 1, \dots, n-1\}$, one has $O_i \neq \varnothing$ and

$$O_{(i+1)\mathrm{mod}n} \subset f(O_i) \subset \overline{O_{(i+1)\mathrm{mod}n}}.$$

Proof. By Claim 5 $O_i \subset \overline{E_i}$ for $i = 0, 1, \dots, n-1$. Hence $f(O_i) \subset \overline{E_{(i+1)\mathrm{mod}n}}$. This inclusion implies $f(O_i) \cap \bigcup_{j \neq i+1} O_j = \varnothing$, i.e., $f(O_i) \subset O_{(i+1)\mathrm{mod}n} \cup R$.

If $f(O_i)$ is not contained in $\overline{O_{(i+1)\mathrm{mod}n}}$, then some open subset of O_i is mapped into the invariant set R. This contradicts the density of the trajectory of x in X. Hence $f(O_i) \subset \overline{O_{(i+1)\mathrm{mod}n}}$.

The inclusion $O_{(i+1)\mathrm{mod}n} \subset f(O_i)$ is implied by Claim 7, the above proved equalities $f(O_j) \cap O_{(i+1)\mathrm{mod}n} = \varnothing$ for $j \neq i$, and the fact that $f(X) = X$. ▮

The proof of Theorem 4 is completed. ▮

Remark 10. We can use Theorem 4 to obtain a similar representation of ω-limit sets with interior. Evidently, such ω-limit sets are transitive and have essential arc components (these are the arc components that have interior points).

Remark 11. Since the essential arc component E_0 in Theorem 4 is chosen arbitrarily, the statement of the theorem implies that the number of essential components in X is finite and all these arc components are included in the cycle of the arc component E_0.

4. EXAMPLES

Using the representation of ω-limit sets suggested by Theorem 4, one can construct a lot of sets that fail to be ω-limit sets for continuous maps in the Euclidean space $\mathbb{R}^n$ of any dimension $n > 1$. For instance, one can consider the union of two disjoint continua that contain finitely many essential arc components with nonvoid interiors and have distinct number of such components. By Theorem Ω these continua should be mapped each onto other but lack of arc components in one of these continua will lead to a contradiction.

In the following examples we use other properties of the structure of continua and continuous maps on continua (cf. also with the set of examples in [9]).

Example 12. Let

$$W = \{(x, w(x)) \in \mathbb{R}^2 \text{ s.t. } 0 < x \leq 1\}$$

where $w(x)$ is defined as follows. For any $n \geq 1$, we set $w(1/n) = 0$ if n is odd, and $w(1/n) = 1$ if n is even. On the components of $(0, 1] \backslash \{1/n\}_{n=1}^{\infty}$ the function w is extended continuously by linearity. If we consider the interval $I = \{(0, y) \in \mathbb{R}^2 \text{ s.t. } 0 \leq y \leq 1\}$, then the set $X = I \cup W$ is a continuum consisting of two arc components (see Fig. 1(a)). This continuum is sometimes called the $\sin(1/x)$-*continuum* because it is topologically equivalent to the continuum defined by using $\sin(1/x)$ instead of $w(x)$ in the definition of W above.

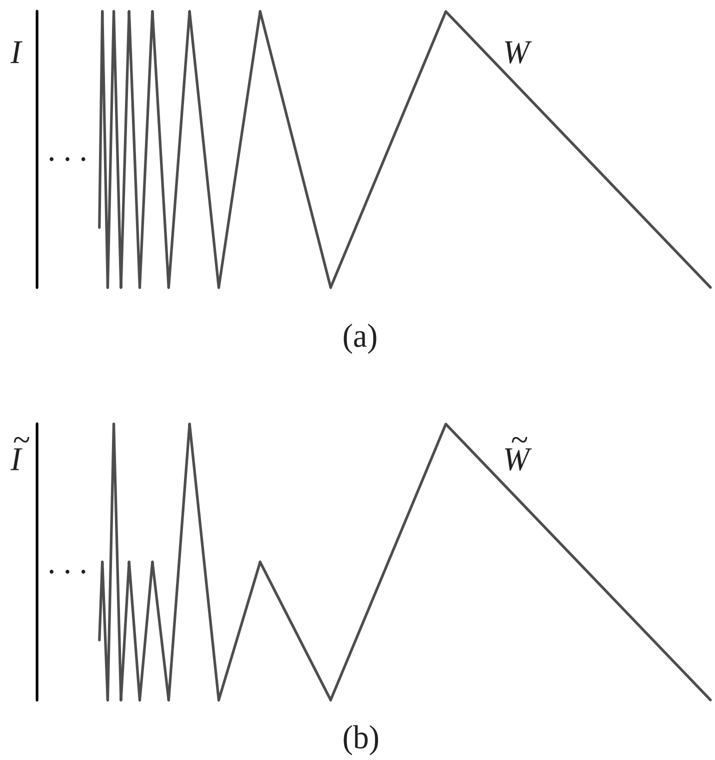

FIG. 1. (a) — the structure of $\sin(1/x)$ continuum; (b) — a $\sin(1/x)$-like continuum having a different limit behavior near the convergence interval

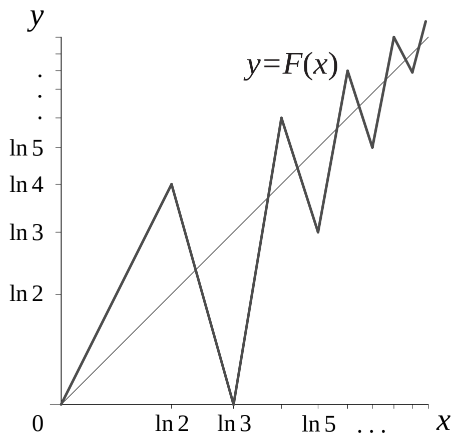

FIG. 2. The function $F(x)$

Let us prove that the continuum X is a transitive ω-limit set.[1] To this end, consider $\mathbb{R}^+ = [0, \infty)$ and define a transitive continuous map $F : \mathbb{R}^+ \to \mathbb{R}^+$ as follows. Set $F(0) = 0$. For integer $n > 1$, define $F(\ln n) = \ln(n+2)$, if n is even, and $F(\ln n) = \ln(n-2)$, if n is odd. Extend the map F on the intervals of $\mathbb{R}^+ \backslash \{\ln n\}_{n=1}^{\infty}$ continuously by linearity (see Fig. 2).

The map F is transitive in the following sense: for any open interval $U \subset \mathbb{R}^+$ and any positive real number T, one can find m such that $F^m(U) \supset [0, T]$. To prove this we consider the following possibilities.

(i) If the interval U contains $[\ln n, \ln(n+1)]$ for some $n > 1$, then $F(U)$ contains $[\ln(n-1), \ln(n+2)]$, and we obtain $F^{k+j}(U) \supset [0, \ln(n+k+j)]$ for some $k > 0$ and all $j \geq 0$.

(ii) If U contains either $[\ln n, \ln n + \varepsilon)]$ or $[\ln n - \varepsilon, \ln n]$ for some odd $n > 1$ and positive ε, but U does not contain neither $[\ln n, \ln(n+1)]$ nor $[\ln(n-1), \ln n]$, then $F(U)$ contains $[\ln(n-2), \ln(n-2) + \varepsilon]$ because $|F'(x)| > 1$ at any $x \neq \ln j$, $j = 1, 2, \ldots$. In this case, $F^k(U) \supset [0, \varepsilon]$ for some $k > 0$ and hence $F^{k+l}(U) \supset [0, \ln 2]$ for some $l > 0$, and therefore $F^{k+l+j}(U) \supset [0, \ln 2(j+1)]$ for all $j \geq 0$.

(iii) If U contains either $[\ln n, \ln n + \varepsilon)]$ or $[\ln n - \varepsilon, \ln n]$ for some even $n > 1$ and positive ε, but U does not contain neither $[\ln n, \ln(n+1)]$ nor $[\ln(n-1), \ln n]$, then $F(U)$ contains $[\ln(n+2), \ln(n+2) - \varepsilon]$ because $|F'(x)| > 1$ at any $x \neq \ln j$, $j = 1, 2, \ldots$. Thus, for some $k > 0$, we obtain $F^k(U) \supset [\ln(n+2k), \ln(n+2k-1)]$ (because the length of the interval

[1] As it was pointed in the referee's report, the space is also proved to be transitive in [4] and — as a particular case — in [8]; in the second paper the space suggested in Example 13 below is modified to provide a nontransitive arc connected continuum embeddable in $\mathbb{R}^2$.

$[\ln j, \ln(j+1)]$ tends to 0 as $j \to \infty$), and then we can apply the arguments of item (i) to the interval $F^k(U)$.

(iv) Let us prove that for any interval U there exists finite $k \geq 0$ such that $F^k(U)$ contains $\ln n$ for some $n \geq 1$, so we can apply the above described arguments (i)–(iii) to $F^k(U)$.

Suppose that the length of U is $\varepsilon > 0$ and $F^j(U) \cap \{\ln n\}_{n=1}^{\infty} = \emptyset$ for $j = 0, \ldots, k-1$. Since $|F'(x)| > 1$ for $x \neq \ln j$, $j = 1, 2, \ldots$, the length of $F^j(U)$ is at least ε for any $j \in \{0, \ldots, k-1\}$.

Fix $m \geq 1$ such that the distance between points $\ln m$ and $\ln(m+1)$ is less than ε. Since the length of $F^j(U)$ is at least ε for any $j \in \{0, \ldots, k-1\}$, we have $F^j(U) \subset [0, \ln m]$ for $j = 0, \ldots, k-1$.

The minimal absolute value of the slope of F on $[0, \ln m]$ is not less than $\lambda = (\ln(m+1) - \ln(m-2))/(\ln m - \ln(m-1)) > 1$. Therefore we have $|F^k(U)| \geq \lambda^k |U| = \lambda^k \varepsilon$ where $|\cdot|$ denotes the length of the interval. Since $\lambda^k \varepsilon \to \infty$ as $k \to \infty$, one cannot have $F^k(U) \cap \{\ln n\}_{n=1}^{\infty} = \emptyset$ for all $k \geq 0$.

Now let h be the homeomorphism of W onto $\mathbb{R}^+$ such that the image of each linear piece of W under h is a segment of unit length in $\mathbb{R}^+$. Then the map $f|_W = h^{-1} \circ F \circ h$ possesses similar transitive properties on W. It remains to extend f onto I by continuity. We can do this because $F(x) - x$ tends to zero uniformly as $x \to \infty$. Hence the map f should act like the identity on I.

Example 13. Let W and I be defined in Example 12. Consider $I^+ = \{(0, y) \in \mathbb{R}^2 \text{ s.t. } -0.5 \leq y \leq 0\}$ and let J be the interval that connects points $(0, -0.5)$ and $(1, 0)$ in $\mathbb{R}^2$ (see Fig. 3). The continuum $X = I \cup I^+ \cup J \cup W$ is called the *Warsaw circle.* Using the arguments from Example 12, it is not hard to prove in a similar way that the Warsaw circle can be a transitive ω-limit set too. (Hint: one can consider points $(1, 0)$ and $(0, 0)$ in Example 12 to be identical because these points are fixed points of the map).

However, if we remove the "arc" J from the Warsaw circle and consider the continuum $X' = I \cup I^+ \cup W$, then X' fails to be a transitive ω-limit set. In this case essential arc components are $I \cup I^+$ and W, $R = I$, and by using Theorem 4 we obtain a contradiction: we should satisfy $f(I) \subset I$ and $f(I \cup I^+) \subset W$ simultaneously that is not possible.

Example 14. Let

$$\widetilde{W} = \{(x, \widetilde{w}(x)) \in \mathbb{R}^2 \text{ s.t. } 0 < x \leq 1\}$$

where the function $\widetilde{w}$ is defined as follows. For all odd positive integer n, we set $\widetilde{w}(1/n) = 0$. For $n_i = i(i+1)$, $i = 1, 2, \ldots$, we define $\widetilde{w}(1/n_i) = 1$. For the rest of positive integer n, we set $\widetilde{w}(1/n) = 1/2$. On the components

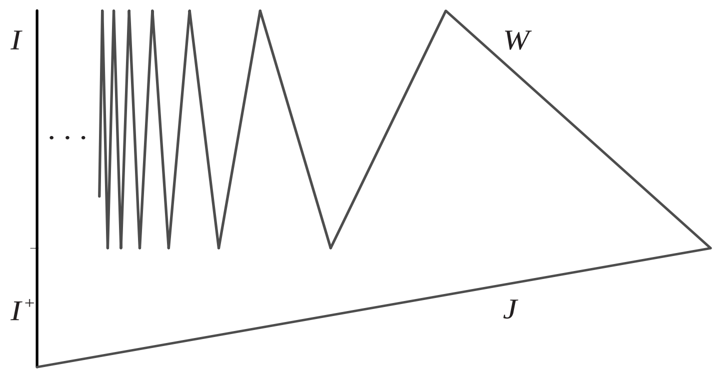

FIG. 3. If we remove the "arc" J from the Warsaw circle, then we obtain a prohibited structure from an admissible one

of $(0,1]\setminus\{1/n\}_{n=1}^{\infty}$, the function $\widetilde{w}$ is extended continuously by linearity (Fig. 1(b)).

Considering $\widetilde{I} = I = \{(0,y) \in \mathbb{R}^2 \text{ s.t. } 0 \le y \le 1\}$ and using arguments from Example 12, one can easily prove that the continuum $X = \widetilde{I} \cup \widetilde{W}$ is a transitive ω-limit set.

Let us prove that the continuum $I \cup W$ from Example 12 cannot be mapped continuously onto $\widetilde{I} \cup \widetilde{W}$. By contradiction let $h : I \cup W \to \widetilde{I} \cup \widetilde{W}$ be continuous and $h(I \cup W) = \widetilde{I} \cup \widetilde{W}$. It is not hard to prove that in this case we should have $h(I) = I$ and $h(W) = \widetilde{W}$. Let $(0,y)$ be a preimage of the point $(0,1)$ under h. Then for any lap L_i of W, which is sufficiently close to I, the point (x_i, y) with the ordinate y in the lap is mapped into a point, which is close to the point $(0,1)$. Hence the point (x_i, y) is mapped into a point of $\widetilde{W}$ that is located near a peak of the function $\widetilde{w}$. This implies that images of parts of the curve W between points (x_i, y) and (x_{i+1}, y) must cover longer and longer parts of the curve $\widetilde{W}$ between neighboring highest peaks of $\widetilde{w}$. Speaking formally, by continuity one can find a subsequence $\{L_{i_k}\}_{k=1}^{\infty}$ of laps of W such that each L_{i_k} contains a sequence of points $(x_1^{(k)}, y_1^{(k)}), (x_2^{(k)}, y_2^{(k)}), \ldots, (x_{2k}^{(k)}, y_{2k}^{(k)})$ such that $y_1^{(k)} < y_2^{(k)} < \ldots < y_{2k}^{(k)}$, $h((x_j^{(k)}, y_j^{(k)})) = (\widetilde{x}_j^{(k)}, 0)$ whenever j is odd, and $h((x_j^{(k)}, y_j^{(k)})) = (\widetilde{x}_j^{(k)}, 1/2)$ whenever j is even. In other words, the image of the lap L_{i_k} contains k successive "small saw teeth" of $\widetilde{W}$ with their successive interlacing minima and maxima. So we can assume also that $\widetilde{x}_1^{(k)} > \widetilde{x}_2^{(k)} > \ldots > \widetilde{x}_{2k}^{(k)}$. Since the y-lengths of the laps of W are bounded, under the above stated conditions we can find a sequence $(\xi^{(k)}, \zeta^{(k)})$ such that $\xi^{(k)} = x_j^{(k)}$, $\zeta^{(k)} = y_j^{(k)}$ for some odd $j = j(k) < 2k$ and $\lim_{k\to\infty} y_j^{(k)} = \lim_{k\to\infty} y_{j+1}^{(k)}$. We denote this limit by ζ because in this case we have also $\zeta = \lim_{k\to\infty} \zeta^{(k)}$. So both sequences of points $(x_j^{(k)}, y_j^{(k)})$ and $(x_{j+1}^{(k)}, y_{j+1}^{(k)})$ tend to the same point $(0, \zeta)$ as $k \to \infty$. On the other hand, we have $h((x_j^{(k)}, y_j^{(k)})) = 0$ and $h((x_{j+1}^{(k)}, y_{j+1}^{(k)})) = 1/2$. This implies a discontinuity of h on I and proves that $I \cup W$ cannot be mapped continuously onto $\widetilde{I} \cup \widetilde{W}$ (see Fig. 1).

The above represented reasoning proves, e.g., that the union of disjoint copies of $I \cup W$ and $\widetilde{I} \cup \widetilde{W}$ fails to be a transitive ω-limit set for a continuous map in $\mathbb{R}^2$.

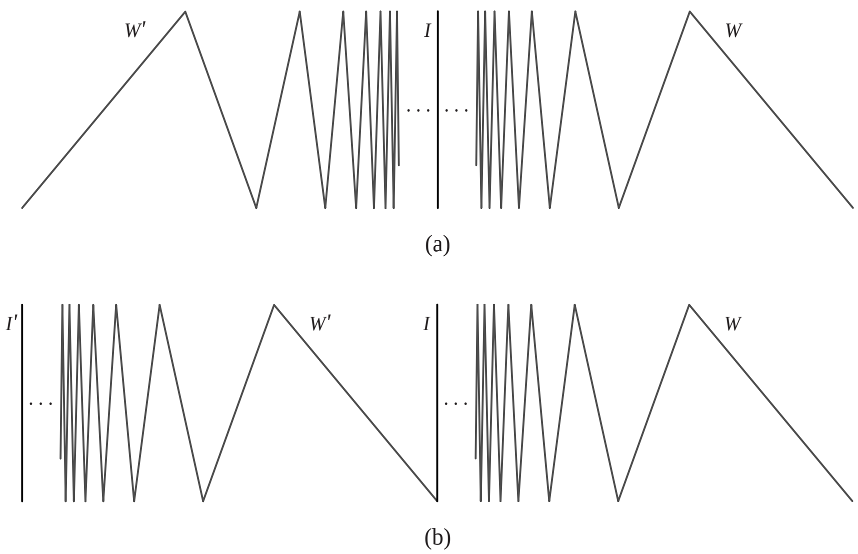

FIG. 4. (a) — this non-Peano continuum is a transitive ω-limit set for a continuous map; (b) — a different "union" of the same admissible structures that fails to be admissible

Example 15. Let us prove that a non-Peano continuum, which is the union of two admissible structures, can fail to be admissible. To this end, consider two different unions of two $\sin(1/x)$ continua (see Fig. 4). The continuum depicted in Fig. 4(a) is a transitive ω-limit set for a continuous map but the continuum in Fig. 4(b) is not admissible one.

In order to prove that the continuum in Fig. 4(a) is a transitive ω-limit set, we can use the transitive continuous map $f : W \cup I \to W \cup I$ that has been constructed in Example 12. Since continua $W \cup I$ and $W' \cup I$ are identical, there is a homeomorphism $h : W \cup I \to W' \cup I$ such that h is the identity on I. Now it is not hard to prove that the continuous map $\widetilde{f}$, which is equal to $f \circ h$ on $W \cup I$ and $f \circ h^{-1}$ on $W' \cup I$, is a transitive continuous map on $W \cup I \cup W'$.

In the case depicted in Fig. 4(b), we use Theorem 4. In this case, there are two essential arc components W and $W' \cup I$, and we have $R = I$. By Theorem 4 the conditions $f(W' \cup I) \subset W$ and $f(I) \subset I$ should be satisfied simultaneously for a continuous map f. Evidently, this is not possible.

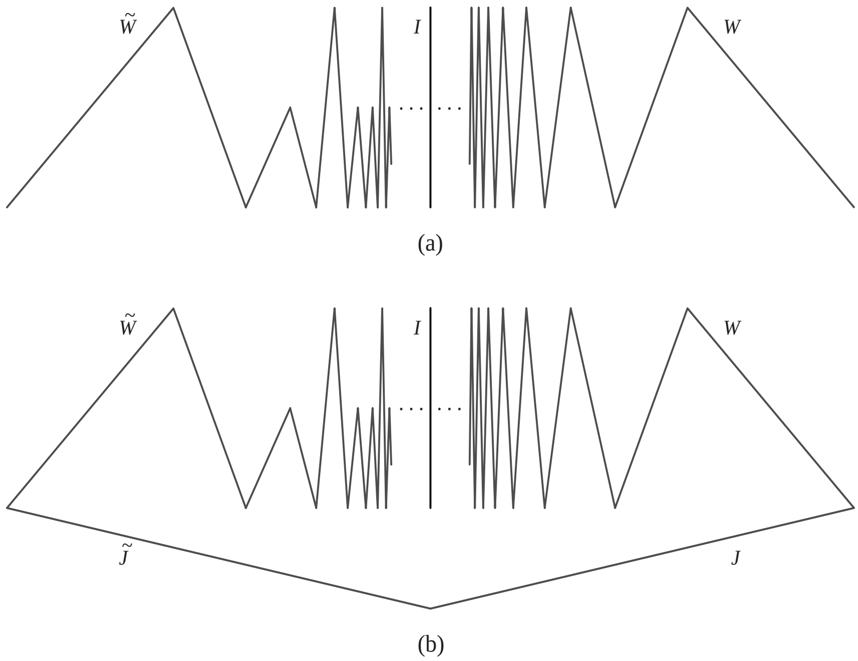

FIG. 5. (a) — this structure fails to be a transitive ω-limit set since there exists no map that translates W onto $\widetilde{W}$ continuously; (b) — by connecting endpoints of W and $\widetilde{W}$, we obtain an admissible structure from a prohibited one

EXAMPLE 16. Consider the non-Peano continuum $X = \widetilde{W} \cup I \cup W$, which is the union of W and $\widetilde{W}$ continua as it is shown in Fig. 5(a). Using Theorem 4 and arguments from Example 14, it is easy to prove that X is not a transitive ω-limit set for any continuous map. On the other hand, if we complete the "interval" X by an "arc" in order to obtain the "circle" $X' = \widetilde{W} \cup I \cup W \cup \widetilde{J} \cup J$ as it is shown in Fig. 5(b), we obtain a transitive ω-limit set. This can be proved by considering a transitive map $\widetilde{F} : \mathbb{R} \to \mathbb{R}$, which is similar to the map F from Example 12. On the half-line $\mathbb{R}^+$ the map $\widetilde{F}$ differs from F only on the interval $(\ln 2, \ln 4)$ where we define $\widetilde{F}(\ln 2) = -\ln 2$ and then extend $\widetilde{F}$ on the intervals $[\ln 2, \ln 3]$ and $[\ln 3, \ln 4]$ continuously by linearity (cf. Fig. 2). On the half-line $\mathbb{R}^-$ we define $\widetilde{F}(x) = \widetilde{F}(-x)$. These changes are used in order to provide transitivity of $\widetilde{F}$ on the whole $\mathbb{R}$. After this, similarly to Example 14, we can use a homeomorphism

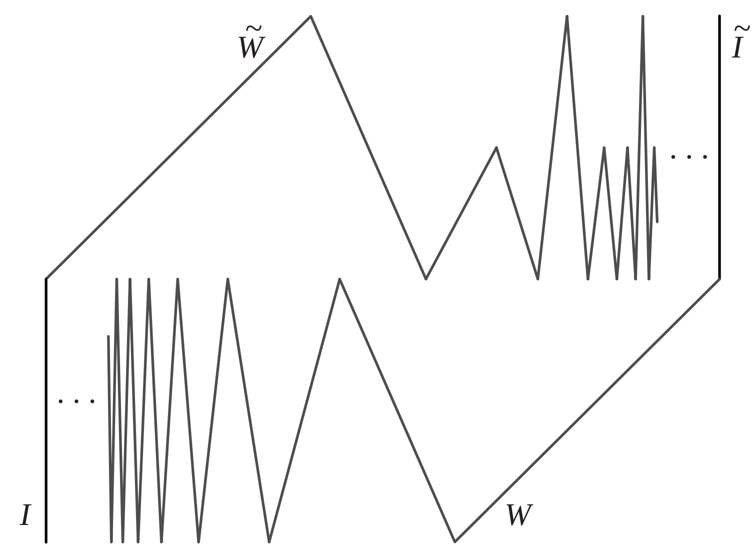

FIG. 6. Another "union" of W and $\widetilde{W}$, which is not a transitive ω-limit set

$\widetilde{h} : \widetilde{W} \cup \widetilde{J} \cup J \cup W \to \mathbb{R}$ such that $\widetilde{h}(J) = [0, \ln 2]$, $\widetilde{h}(\widetilde{J}) = [-\ln 2, 0]$, $\widetilde{h}(W) = [\ln 2, \infty]$, and $\widetilde{h}(\widetilde{W}) = [-\infty, -\ln 2]$, in order to obtain a transitive continuous map $\widetilde{f} = \widetilde{h}^{-1} \circ \widetilde{F} \circ \widetilde{h}$ on $\widetilde{W} \cup \widetilde{J} \cup J \cup W$.

Example 17. The continuum in Fig. 6 provides one more example of a union of admissible continua W and $\widetilde{W}$, which fails to be admissible. This non-Peano continuum is not admissible because by Theorem 4 two essential arc components W and $\widetilde{W}$ should be mapped one onto other continuously but it has been shown in Example 14 that it is not possible.

The above suggested examples show that any characterization of transitive ω-limit sets in terms of arc components must take into account not only the number of arc components and their intersections but also the way, in which convergence subcontinua are approximated by arc components in the set.

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