QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **4**, 67–76 (2003)
ARTICLE NO. 53

Continuity of the Measure of Maximal Entropy for Unimodal Maps on the Interval

Peter Raith

Institut für Mathematik, Universität Wien
Strudlhofgasse 4, A 1090 Wien, Austria
E-mail: Peter.Raith@univie.ac.at

Let $T : [0,1] \to [0,1]$ be a unimodal map with positive topological entropy. Then T has a unique measure $\mu(T)$ of maximal entropy. It is proved that the map $T \mapsto \mu(T)$ is continuous with respect to the weak star–topology.

Key Words: unimodal map, perturbation, topological entropy, measure of maximal entropy, Markov diagram

1. INTRODUCTION

A map $T : [0,1] \to [0,1]$ is called a unimodal map, if T is continuous and there is a $c \in (0,1)$, such that $T|_{[0,c]}$ is strictly increasing and $T|_{[c,1]}$ is strictly decreasing (or $T|_{[0,c]}$ is strictly decreasing and $T|_{[c,1]}$ is strictly increasing). Consider the set of all unimodal maps endowed with the C^0–topology. If T is a unimodal map, then we assign a measure $\mu(T)$ of maximal entropy to T. In this paper continuity properties of this map $T \mapsto \mu(T)$ are investigated.

First we address the question, if the measure of maximal entropy is unique. Franz Hofbauer derived in [2] from his results in [1] that every unimodal map with positive entropy has a unique measure of maximal entropy. We will give a simpler proof of this result in Theorem 5.

It has been shown by Michał Misiurewicz in Theorem 2 of [4] that the map $T \mapsto h_{\mathrm{top}}(T)$, where $h_{\mathrm{top}}(T)$ is the topological entropy of T, is continuous on the set of all unimodal maps with positive entropy. On the other hand there are continuous functions $f : [0,1] \to \mathbf{R}$, such that $T \mapsto p(T,f)$, where $p(T,f)$ denotes the topological pressure, is not continuous (see [11]) on the set of all unimodal maps with positive entropy.

However, results on the continuity of the measure of maximal entropy with respect to the weak star–topology are known, if we consider another topology on a suitable subset of the collection of all unimodal maps. Consider the set of all C^1–unimodal maps satisfying $T'x \neq 0$ for all $x \neq c$ endowed with the C^1–topology. Then the subset of all maps with positive entropy is open by Theorem 5 of [6]. The map $T \mapsto \mu(T)$ is continuous on this subset by Theorem 3 in [9].

We will prove in Theorem 7 (and Corollary 8) that the map $T \mapsto \mu(T)$ is continuous with respect to the weak star–topology on the set of all unimodal maps with positive topological entropy (endowed with the C^0–topology). Obviously Theorem 7 implies the result on the C^1–continuity of the measure of maximal entropy mentioned above. For the proof two oriented graphs are introduced (one of this graphs, Hofbauer's Markov diagram, is used to prove the uniqueness of the measure of maximal entropy in Theorem 5), and to each graph a matrix is assigned. In Lemma 6 a relation between the spectral radii of these matrices is proved. This result implies the continuity of the measure of maximal entropy (and also the continuity of the topological entropy).

2. UNIMODAL MAPS

Suppose that X is a finite union of closed intervals. We call $\mathcal{Z}$ a *finite partition* of X, if $\mathcal{Z}$ consists of finitely many pairwise disjoint open intervals with $\bigcup_{Z \in \mathcal{Z}} \overline{Z} = X$. A map $T : X \to \mathbf{R}$ is called *piecewise monotonic* with respect to the finite partition $\mathcal{Z}$ of X, if $T|_Z$ is strictly monotonic, bounded and continuous for all $Z \in \mathcal{Z}$. If $T : X \to \mathbf{R}$ is a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$, then we call $\mathcal{Z}$ the *minimal partition* for T, if $\operatorname{card} \mathcal{Y} \geq \operatorname{card} \mathcal{Z}$ for every finite partition $\mathcal{Y}$ of X, such that T is piecewise monotonic with respect to $\mathcal{Y}$. Then every finite partition $\mathcal{Y}$ of X satisfying that T is piecewise monotonic with respect to $\mathcal{Y}$, is a refinement of $\mathcal{Z}$.

A continuous map $T : [0, 1] \to [0, 1]$ is called a *unimodal map*, if there exists a finite partition $\mathcal{Z}$ of $[0, 1]$ with $\operatorname{card} \mathcal{Z} = 2$, such that T is piecewise monotonic with respect to $\mathcal{Z}$ and $\mathcal{Z}$ is the minimal partition for T. Then there exists a $c \in (0, 1)$, such that either T is strictly increasing on $[0, c]$ and strictly decreasing on $[c, 1]$ (first type), or T is strictly decreasing on $[0, c]$ and strictly increasing on $[c, 1]$ (second type). If T is of the second type, then $h(x) = 1 - x$ conjugates T to a map of the first type. Therefore in the proofs we may always assume that T is of the first type.

In order to define a topology on piecewise monotonic maps (cf. [5] and [7]) we define the following notion. For $\varepsilon > 0$ two continuous functions $f : [a, b] \to \mathbf{R}$ and $\tilde{f} : [\tilde{a}, \tilde{b}] \to \mathbf{R}$ are called *ε–close*, if

• $|a - \tilde{a}| < \varepsilon$ and $|b - \tilde{b}| < \varepsilon$,
• $|f(x) - \tilde{f}(x)| < \varepsilon$ for all $x \in [a, b] \cap [\tilde{a}, \tilde{b}]$,
• $\sup_{x \in [a,\tilde{a}]} |f(x) - \tilde{f}(\tilde{a})| < \varepsilon$, if $a < \tilde{a}$, or $\sup_{x \in [\tilde{a},a]} |\tilde{f}(x) - f(a)| < \varepsilon$, if otherwise $\tilde{a} \le a$,
• $\sup_{x \in [\tilde{b},b]} |f(x) - \tilde{f}(\tilde{b})| < \varepsilon$, if $\tilde{b} < b$, or $\sup_{x \in [b,\tilde{b}]} |\tilde{f}(x) - f(b)| < \varepsilon$, if otherwise $b \le \tilde{b}$.

Observe that, if ε is small enough, then $(a, b) \cap (\tilde{a}, \tilde{b}) \neq \emptyset$.

Assume that X and $\tilde{X}$ are finite unions of closed intervals. Let $T : X \to \mathbf{R}$ be piecewise monotonic with respect to the finite partition $\mathcal{Z}$ of X, and let $\tilde{T} : \tilde{X} \to \mathbf{R}$ be piecewise monotonic with respect to the finite partition $\tilde{\mathcal{Z}}$ of $\tilde{X}$. We may assume that $\mathcal{Z} = \{Z_1, Z_2, \dots, Z_K\}$ with $Z_1 < Z_2 < \cdots < Z_K$ and $\tilde{\mathcal{Z}} = \{\tilde{Z}_1, \tilde{Z}_2, \dots, \tilde{Z}_{\tilde{K}}\}$ with $\tilde{Z}_1 < \tilde{Z}_2 < \cdots < \tilde{Z}_{\tilde{K}}$. For $j \in \{1, 2, \dots, K\}$ let $T_j : \overline{Z_j} \to \mathbf{R}$ be the unique continuous function with $T_j|_{Z_j} = T|_{Z_j}$, and for $j \in \{1, 2, \dots, \tilde{K}\}$ let $\tilde{T}_j : \overline{\tilde{Z}_j} \to \mathbf{R}$ be the unique continuous function with $\tilde{T}_j|_{\tilde{Z}_j} = \tilde{T}|_{\tilde{Z}_j}$. Suppose that $\varepsilon > 0$. Then $(T, \mathcal{Z})$ and $(\tilde{T}, \tilde{\mathcal{Z}})$ are said to be *ε–close in the R^0–topology*, if

• $\operatorname{card} \mathcal{Z} = \operatorname{card} \tilde{\mathcal{Z}}$, and
• T_j and $\tilde{T}_j$ are ε–close in the sense defined above for $j = 1, 2, \dots, K$.

Let $T : [0, 1] \to [0, 1]$ be a unimodal map, and suppose that $\mathcal{Z}$ is the minimal partition for T. Then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that every unimodal map $\tilde{T} : [0, 1] \to [0, 1]$ with minimal partition $\tilde{\mathcal{Z}}$ and $\|\tilde{T} - T\|_\infty < \delta$ satisfies that $(T, \mathcal{Z})$ and $(\tilde{T}, \tilde{\mathcal{Z}})$ are ε–close in the R^0–topology, where $\|f\|_\infty := \sup_{x\in[0,1]} \bigl|f(x)\bigr|$. Moreover, $\tilde{T}$ is of the first type, if and only if T is of the first type.

A *topological dynamical system* (X, T) is a continuous map T of a compact metric space X into itself. The definition of the topological entropy $h_{\text{top}}(T)$ can be found in § 7.1 and § 7.2 of [12], and the definition of the measure–theoretic entropy $h_\mu(T)$ of a T–invariant Borel probability measure μ on X can be found in § 4.4 of [12]. According to the variational principle (see e.g. Theorem 8.6 of [12]) we have $h_{\text{top}}(T) = \sup_\mu h_\mu(T)$, where the supremum is taken over all T–invariant Borel probability measures μ on X. A T–invariant Borel probability measure μ on X is called a *measure of maximal entropy* of T, if $h_\mu(T) = h_{\text{top}}(T)$.

Although unimodal maps T are topological dynamical systems, this need not be true for small perturbations $\tilde{T}$ of T in the R^0–topology, because $\tilde{T}$ need not be continuous. In this case the definition of the topological entropy $h_{\text{top}}(\tilde{T})$ can be found in [7], and the definition of a measure of maximal entropy of $\tilde{T}$ can be found in [8].

3. THE MARKOV DIAGRAM AND THE GRAPH $(\mathcal{G}, \to)$

We will describe two at most countable oriented graphs associated to a unimodal map T. The first one is the Markov diagram $(\mathcal{D}, \to)$, which was introduced in [2] (see also [3]) in order to describe the orbit structure of T. Another oriented graph $(\mathcal{G}, \to)$ has been introduced in [7] (cf. also [5]) in order to describe the behaviour of the dynamics of small perturbations of T.

At first we define the Markov diagram. Let $T : [0,1] \to [0,1]$ be a unimodal map, and let $\mathcal{Z}$ be the minimal partition for T. Assume that $Z_0 \in \mathcal{Z}$ and that D is an open subinterval of Z_0. We call a nonempty $C \subseteq [0,1]$ *successor* of D, if there exists a $Z \in \mathcal{Z}$ with $C = TD \cap Z$, and we write $D \to C$. Every successor C of D is again an open subinterval of an element of $\mathcal{Z}$. Let $\mathcal{D}$ be the smallest set with $\mathcal{Z} \subseteq \mathcal{D}$ and such that $D \in \mathcal{D}$ and $D \to C$ imply $C \in \mathcal{D}$. Then $(\mathcal{D}, \to)$ is called the *Markov diagram* of T. The set $\mathcal{D}$ is at most countable and its elements are open subintervals of elements of $\mathcal{Z}$.

In [7] an oriented graph $(\mathcal{G}, \to)$ is introduced in order to describe the jumps up of the topological entropy (see also [5]). Again suppose that $T : [0,1] \to [0,1]$ is a unimodal map. Define $\mathcal{G} := \big(\{T^n c : n \in \mathbf{N}\} \setminus \{c\}\big) \cup \{c^-, c^+\}$, where we assume that $c^- \neq c^+$. For $a, b \in \mathcal{G}$ we introduce an arrow $a \to b$, if and only if $b = Ta$, or $b = Tc$ and $a \in \{c^-, c^+\}$, or $c = Ta$ and $b \in \{c^-, c^+\}$, or $Tc = c$ and $a, b \in \{c^-, c^+\}$.

Let $(\mathcal{H}, \to)$ be an oriented graph. For $n \in \mathbf{N}$ we call $c_0 \to c_1 \to \cdots \to c_n$ a *path of length* n in $\mathcal{H}$, if $c_j \in \mathcal{H}$ for $j \in \{0, 1, \ldots, n\}$ and $c_{j-1} \to c_j$ for $j \in \{1, 2, \ldots, n\}$. We call $c_0 \to c_1 \to c_2 \to \cdots$ an *infinite path* in $\mathcal{H}$, if $c_j \in \mathcal{H}$ for all $j \in \mathbf{N}_0$ and $c_{j-1} \to c_j$ for all $j \in \mathbf{N}$. The oriented graph $\mathcal{H}$ is called *irreducible*, if for every $c, d \in \mathcal{H}$ there exists a finite path $c_0 \to c_1 \to \cdots \to c_n$ in $\mathcal{H}$ with $c_0 = c$ and $c_n = d$. An irreducible subset $\mathcal{C}$ of $\mathcal{H}$ is called *maximal irreducible* in $\mathcal{H}$, if every $\mathcal{C}'$ with $\mathcal{C} \subset \mathcal{C}' \subseteq \mathcal{H}$ is not irreducible.

We define the $\mathcal{H} \times \mathcal{H}$–matrix $M_{\mathcal{H}} := \big(M_{c,d}\big)_{c,d \in \mathcal{H}}$ by

$$M_{c,d} := \begin{cases} 1 & \text{if } c \to d, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

If $u \mapsto uM_{\mathcal{H}}$ is a continuous linear operator on $\ell^1(\mathcal{H})$, then denote by $\|M_{\mathcal{H}}\|$ its norm and by $r(M_{\mathcal{H}})$ its spectral radius. Observe that $u \mapsto uM_{\mathcal{H}}$ is a continuous linear operator on $\ell^1(\mathcal{H})$, if $\mathcal{H}$ is finite (this is obvious) or $\mathcal{H}$ is a subset of the Markov diagram of a piecewise monotonic map (in this case see p. 371 of [3]) or $\mathcal{H}$ is a subset of the oriented graph $(\mathcal{G}, \to)$ associated to a piecewise monotonic map in [7] (in this case see p. 105 of [7]). We have

$$\|M_{\mathcal{H}}{}^n\| = \sup_{c\in\mathcal{H}} \operatorname{card}\{c_0 = c \to c_1 \to \cdots \to c_n \text{ is a path of length } n \text{ in } \mathcal{H}\} \tag{2}$$

and

$$r(M_{\mathcal{H}}) = \lim_{n\to\infty} \|M_{\mathcal{H}}{}^n\|^{\frac{1}{n}} = \inf_{n\in\mathbf{N}} \|M_{\mathcal{H}}{}^n\|^{\frac{1}{n}} \ , \tag{3}$$

whenever $u \mapsto uM_{\mathcal{H}}$ is a continuous linear operator on $\ell^1(\mathcal{H})$.

The following result gives a lower bound for $r(M_{\mathcal{C}})$, if $\mathcal{C}$ is a finite and irreducible oriented graph. We omit the proof, because it is a simple exercise for zero-one-matrices (the result is also a simple consequence of results in [4]).

Lemma 1. *Suppose that* $(\mathcal{C}, \to)$ *is an irreducible oriented graph with* $\operatorname{card}\mathcal{C} = n$, *such that there exists a* $c \in \mathcal{C}$ *having at least two different successors in* $\mathcal{C}$. *Then*

$$r(M_{\mathcal{C}}) \geq \lambda_n > \sqrt[n]{2} \ ,$$

where λ_n *is the largest root of the polynomial* $x^n - x - 1$.

Remark. Observe that the property, that there exists a $c \in \mathcal{C}$ having at least two different successors in $\mathcal{C}$, is equivalent to $r(M_{\mathcal{C}}) > 1$.

Now we calculate $r(M_{\mathcal{G}})$ for a unimodal map, where $(\mathcal{G}, \to)$ is the oriented graph introduced above. Recall that c is the unique real number, such that $\{(0,c),(c,1)\}$ is the minimal partition for T. Define $p(c) := \min\{n \in \mathbf{N} : T^n c = c\}$, where we set $p(c) := \infty$, if $T^n c \neq c$ for all $n \in \mathbf{N}$.

Lemma 2. *Let* $T : [0,1] \to [0,1]$ *be a unimodal map.*

- *If* $p(c) = \infty$, *then* $r(M_{\mathcal{G}}) = 1$.
- *Suppose that* $p(c) = n \in \mathbf{N}$. *Then* $r(M_{\mathcal{G}}) = \sqrt[n]{2}$.

Proof. If $p(c) := \infty$, then every element of $\mathcal{G}$ has exactly one successor. Hence (2) and (3) imply $r(M_{\mathcal{G}}) = 1$. In the case $p(c) = n \in \mathbf{N}$ the characteristic polynomial of $M_{\mathcal{G}}$ equals $(-1)^{n+1}(x^{n+1} - 2x)$, and therefore $r(M_{\mathcal{G}}) = \sqrt[n]{2}$. ▮

Suppose that $T : [0,1] \to [0,1]$ is a unimodal map. Let $(\mathcal{D}, \to)$ be the Markov diagram of T. By Theorem 7 in [3] we get

$$h_{\text{top}}(T) = \log r(M_{\mathcal{D}}) \ . \tag{4}$$

We will need the following result, which calculates the Markov diagram in a certain special case of a unimodal map.

Lemma 3. *Let $T : [0,1] \to [0,1]$ be a unimodal map of the first type, and suppose that $T^3c < T^2c < c < Tc$. Then $p(c) = \infty$. Define $B_1 := (c, Tc)$ and for $n \in \mathbf{N}$, $n > 1$ set $B_n := (T^n c, c)$. Then we have $B_n \to B_1$ and $B_n \to B_{n+1}$ for every $n \in \mathbf{N}$. Moreover, the Markov diagram $(\mathcal{D}, \to)$ of T has a unique maximal irreducible $\mathcal{C} \subseteq \mathcal{D}$ with $r(M_{\mathcal{C}}) > 1$. We have $\mathcal{C} = \{B_n : n \in \mathbf{N}\}$ and $r(M_{\mathcal{C}}) = r(M_{\mathcal{D}}) = 2$.*

Proof. As $T^3c < T^2c < c$ we get by induction that $(T^n c)_{n\geq 2}$ is strictly decreasing. This implies that $T^n c < c$ for all $n \geq 2$, and therefore $p(c) = \infty$. Since $0 < T^2c$ and $T1 \leq T^2c$ (because $Tc \leq 1$) we obtain $T^n 0 < T^2c$ for all $n \geq 0$ and $T^n 1 < T^2c$ for all $n \geq 2$. By its definition $TB_1 := (T^2c, Tc)$, and $TB_n := (T^{n+1}c, Tc)$ for $n > 1$. Hence $B_n \to B_1$ and $B_n \to B_{n+1}$ for every n, and $\mathcal{C} = \{B_n : n \in \mathbf{N}\}$ is irreducible. An easy induction shows that for every $D \in \mathcal{D}$ there is a $j \geq 0$, such that $T^j c$ is an endpoint of D. This shows that if D has two different successors, then at least one of these successors must be in $\mathcal{C}$. Therefore $r(M_{\mathcal{C}'}) = 1$ for every maximal irreducible $\mathcal{C}' \subseteq \mathcal{D}$ with $\mathcal{C}' \neq \mathcal{C}$. Fix $C \in \mathcal{C}$ and $n \in \mathbf{N}$. Since every $D \in \mathcal{C}$ has exactly two successors in $\mathcal{C}$, there are exactly 2^n different paths $C_0 \to C_1 \to \cdots \to C_n$ of length n in $\mathcal{C}$ with $C_0 = C$. Now (2) and (3) imply $r(M_{\mathcal{C}}) = 2$. ▮

Next assume that T is of the first type and that $h_{\text{top}}(T) > 0$. We claim that $T^2c < c < Tc$. If $Tc \leq c$, then $T[0,1] \subseteq [0,c]$. Therefore every $C \in \mathcal{D}$ has exactly one successor. Now (2) and (3) imply $r(M_{\mathcal{D}}) = 1$, which contradicts (4). If $T^2c \geq c$, then $c \leq Tc$ and $T[c,Tc] \subseteq [c,Tc]$. Since $T[0,1] \subseteq [0,Tc]$ for every $C \in \mathcal{D}$ there are at most $n+1$ different paths $C_0 \to C_1 \to \cdots \to C_n$ of length n in $\mathcal{D}$ with $C_0 = C$. Again (2) and (3) imply $r(M_{\mathcal{D}}) = 1$ contradicting (4).

Moreover, observe that $T[T^2c, Tc] \subseteq [T^2c, Tc]$, if $T^3c \geq T^2c$ holds. If $x \notin \bigcup_{j=0}^{\infty} T^{-j}[T^2c, Tc]$, then $T^n x \in [0,c]$ for all $n \geq 1$. Hence the sequence $(T^n x)_{n\geq 1}$ is monotonic, and therefore it converges to a fixed point of T. This implies $h_{\text{top}}(T) = h_{\text{top}}(T|_{[T^2c,Tc]})$ and every ergodic T–invariant Borel probability measure μ with $h_\mu(T) > 0$ is concentrated on $[T^2c, Tc]$. The map $h(x) = \frac{x - T^2c}{c - T^2c}$ conjugates $T|_{[T^2c,Tc]}$ to a unimodal map $\tilde{T} : [0,1] \to [0,1]$ (let $\{(0,\tilde{c}), (\tilde{c},1)\}$ be the minimal partition for $\tilde{T}$) of the first type with $\tilde{T}\tilde{c} = 1$ and $\tilde{T}^2\tilde{c} = 0$. Hence in the proofs we may always assume that T is of the first type and either $T^3c < T^2c < c < Tc$ or T satisfies $Tc = 1$ and $T^2c = 0$.

Our next result describes the Markov diagram of T. It can also be found in [2]. As our proof is shorter and simpler than the proof of the (more detailled) result in [2], which is mainly given in [1], we give the proof here.

LEMMA 4. *Let $T : [0,1] \to [0,1]$ be a unimodal map of the first type with $Tc = 1$ and $T^2c = 0$. Define $A_1 := (c,1) = (c,Tc)$ and $A_2 := (0,c) = (T^2c,c)$. For $n \in \mathbf{N}$, $n > 2$ let A_n be the successor of A_{n-1}, which satisfies $T^nc \in \overline{A_n}$. Then for every $n \in \mathbf{N}$ there exists a $j \in \{0,1,\dots,n-1\}$, such that A_n is an open interval with the endpoints T^nc and T^jc. If $j \geq 1$, then $A_n \subseteq A_j$. Moreover, the Markov diagram $(\mathcal{D},\to)$ of T satisfies $\mathcal{D} = \{A_n : n \in \mathbf{N}\}$. If $n \in \mathbf{N}$, then $A_n \to A_{n+1}$. Furthermore $A_1 \to A_1$. Suppose that $n \in \mathbf{N}$, $n > 1$, assume that A_n has more than one successor in $(\mathcal{D},\to)$, and let $k \in \mathbf{N}$ be the smallest number with $A_n \to A_k$. Then $k < n$, and A_{n-k+j} has only one successor in $(\mathcal{D},\to)$ for all $j \in \{1,2,\dots,k-1\}$.*

Proof. A simple induction shows that for every $n \in \mathbf{N}$ there exists a $j < n$, such that A_n is an open interval with the endpoints T^nc and T^jc, and that $A_n \subseteq A_j$, if $j \geq 1$. Moreover, simple calculations give $A_1 \to A_1$, and $A_n \to A_{n+1}$ for every $n \in \mathbf{N}$. Finally, let $n \in \mathbf{N}$, $n > 1$, assume that A_n has more than one successor, let k be the smallest number with $A_n \to A_k$, and let $u < n$ be the largest number, such that A_u has more than one successor. Then A_{u+1} is an open interval with the endpoints $T^{u+1}c$ and c. By induction we obtain for each $j \leq n-u$ that A_{u+j} is an open interval with the endpoints $T^{u+j}c$ and $T^{j-1}c$. If $u = n-1$, this implies $A_n \to A_1$. Otherwise we have $A_{u+j} \subseteq A_{j-1}$ for $j \in \{2,3,\dots,n-u\}$, and therefore $A_n \to A_{n-u}$. ∎

4. CONTINUITY OF THE MEASURE OF MAXIMAL ENTROPY

Consider a unimodal map T with positive entropy. In [2] it is proved that T has a unique measure of maximal entropy. We present a proof for this fact, which is much simpler than the proof in [1] and [2]. Then we prove our main result, which states that the measure of maximal entropy is continuous. The continuity of the topological entropy has been obtained in [4]. However, our proof also shows that the topological entropy is continuous (without using the results of [4]).

THEOREM 5. *Let $T : [0,1] \to [0,1]$ be a unimodal map with $h_{\text{top}}(T) > 0$. Then T has a unique measure $\mu(T)$ of maximal entropy.*

Proof. By Theorem 11 of [3] and Theorem 2 of [2] it suffices to show that there exists a unique maximal irreducible $\mathcal{C} \subseteq \mathcal{D}$ with $r(M_{\mathcal{C}}) = r(M_{\mathcal{D}})$, where $(\mathcal{D},\to)$ is the Markov diagram of T.

We may assume that T is of the first type and either $T^3c < T^2c < c < Tc$ or T satisfies $Tc = 1$ and $T^2c = 0$. Since Lemma 3 implies the desired result in the first case, it remains to consider the case $Tc = 1$ and $T^2c = 0$. Let $A_1, A_2, \dots$ be as in Lemma 4. As $h_{\text{top}}(T) > 0$, Theorem 11 of [3] implies the existence of a maximal irreducible $\mathcal{C} \subseteq \mathcal{D}$ with $r(M_\mathcal{C}) > 1$. Now let k be the smallest natural number, such that A_k is contained in a maximal irreducible $\mathcal{C} \subseteq \mathcal{D}$ with $r(M_\mathcal{C}) > 1$. If $\mathcal{C} = \{A_j : j \geq k\}$, then every maximal irreducible $\mathcal{C}' \subseteq \mathcal{D}$ with $\mathcal{C}' \neq \mathcal{C}$ satisfies $r(M_{\mathcal{C}'}) = 1$. Otherwise, Lemma 4 implies that $\mathcal{C}$ is finite. Let n be the largest natural number with $A_n \in \mathcal{C}$. If $\mathcal{C}' \subseteq \mathcal{D}$ is maximal irreducible, $\mathcal{C}' \neq \mathcal{C}$ and $r(M_{\mathcal{C}'}) > 1$, then $\mathcal{C}' \subseteq \mathcal{D} \setminus \{A_1, A_2, \ldots, A_n\}$. By Lemma 4, for every $C \in \mathcal{C}'$ and every $u \in \mathbf{N}$ there are at most $2^{\frac{u}{n+1}+1}$ different paths $C_0 = C \to C_1 \to \cdots \to C_u$ of length u in $\mathcal{C}'$ with $C_0 = C$. Now (2) and (3) imply $r(M_{\mathcal{C}'}) \leq \sqrt[n+1]{2}$. On the other hand $\mathcal{C} \subseteq \{A_1, A_2, \ldots, A_n\}$. Hence Lemma 1 gives $r(M_\mathcal{C}) > \sqrt[n]{2} > \sqrt[n+1]{2} \geq r(M_{\mathcal{C}'})$. ∎

In order to prove the continuity of the measure of maximal entropy we prove first that $r(M_\mathcal{D}) > r(M_\mathcal{G})$. This result has been obtained in a slightly different (and less explicit) form in [4]. For the convenience of the reader we give a proof.

Lemma 6. *Let $T : [0,1] \to [0,1]$ be a unimodal map with $h_{\text{top}}(T) > 0$. Then $r(M_\mathcal{D}) > r(M_\mathcal{G})$, where $(\mathcal{D}, \to)$ denotes the Markov diagram of T.*

Proof. We may assume that T is of the first type and either $T^3c < T^2c < c < Tc$ or T satisfies $Tc = 1$ and $T^2c = 0$. If $T^3c < T^2c < c < Tc$, then Lemma 2 and Lemma 3 imply $r(M_\mathcal{G}) = 1 < 2 = r(M_\mathcal{D})$.

It remains to consider the case $Tc = 1$ and $T^2c = 0$. Let $A_1, A_2, \ldots$ be as in Lemma 4. As $h_{\text{top}}(T) > 0$ we get $r(M_\mathcal{D}) > 1$ by (4). If $p(c) = \infty$, then Lemma 2 implies $r(M_\mathcal{G}) = 1 < r(M_\mathcal{D})$. Finally, suppose that $p(c) = n \in \mathbf{N}$. By Lemma 4 A_n is an open interval with the endpoints $T^nc = c$ and T^jc for some $j < n$. Since A_n is an open interval we must have $j \geq 1$, and $A_n \subseteq A_j$ by Lemma 4. This implies $A_n = A_j$. Therefore Lemma 4 gives $\mathcal{D} = \{A_1, A_2, \ldots, A_{n-1}\}$. Now Lemma 1 gives $r(M_\mathcal{D}) > \sqrt[n-1]{2}$. Using Lemma 2 we obtain $r(M_\mathcal{G}) = \sqrt[n]{2} < \sqrt[n-1]{2} < r(M_\mathcal{D})$. ∎

Now we prove the main result of this paper on the continuity of the measure of maximal entropy. If we write $(\tilde{T}, \tilde{\mathcal{Z}}) \to (T, \mathcal{Z})$ we mean $(\tilde{T}, \tilde{\mathcal{Z}})$ converges to $(T, \mathcal{Z})$ in the R^0–topology. For Borel probability measures $\tilde{\mu} \to \mu$ means $\tilde{\mu}$ converges to μ in the weak star–topology. The continuity of the topological entropy has been obtained by Michał Misiurewicz in [4].

Theorem 7. *Let $T : [0,1] \to [0,1]$ be a unimodal map with $h_{\text{top}}(T) > 0$. Suppose that $\mathcal{Z}$ is the minimal partition for T. Then there exists a $\delta > 0$, such that every $(\tilde{T}, \tilde{\mathcal{Z}})$, which is δ–close to $(T, \mathcal{Z})$ in the R^0–topology, has a unique measure $\mu(\tilde{T})$ of maximal entropy. Furthermore we have*

$$\lim_{(\tilde{T}, \tilde{\mathcal{Z}}) \to (T, \mathcal{Z})} \mu(\tilde{T}) = \mu(T) ,$$

$$\lim_{(\tilde{T},\tilde{\mathcal{Z}})\to(T,\mathcal{Z})} h_{\mathrm{top}}(\tilde{T}) = h_{\mathrm{top}}(T) \quad \textit{and}$$
$$\lim_{(\tilde{T},\tilde{\mathcal{Z}})\to(T,\mathcal{Z})} h_{\mu(\tilde{T})}(\tilde{T}) = h_{\mu(T)}(T) \; .$$

Proof. As $h_{\mathrm{top}}(T) > 0$, T has a unique measure $\mu(T)$ of maximal entropy by Theorem 5. Using (4) and Lemma 6 we obtain that $h_{\mathrm{top}}(T) > \log r(M_{\mathcal{G}})$. By Theorem 1 in [10] there exists a $\delta > 0$, such that every $(\tilde{T}, \tilde{\mathcal{Z}})$, which is δ–close to $(T, \mathcal{Z})$ in the R^0–topology, has a unique measure $\mu(\tilde{T})$ of maximal entropy. Moreover, Theorem 1 in [10] also shows that $\lim_{(\tilde{T},\tilde{\mathcal{Z}})\to(T,\mathcal{Z})} \mu(\tilde{T}) = \mu(T)$. From Theorem 1 and Theorem 2 in [7] (or directly from Theorem 2 in [4]) we get $\lim_{(\tilde{T},\tilde{\mathcal{Z}})\to(T,\mathcal{Z})} h_{\mathrm{top}}(\tilde{T}) = h_{\mathrm{top}}(T)$. Finally, the formula $\lim_{(\tilde{T},\tilde{\mathcal{Z}})\to(T,\mathcal{Z})} h_{\mu(\tilde{T})}(\tilde{T}) = h_{\mu(T)}(T)$ follows from the property $h_{\mu(\tilde{T})}(\tilde{T}) = h_{\mathrm{top}}(\tilde{T})$. ▮

The following result follows immediately from Theorem 7 and Theorem 5, because on the collection of unimodal maps convergence in the maximum norm implies convergence in the R^0–topology.

Corollary 8. *Let $T : [0,1] \to [0,1]$ be a unimodal map with $h_{\mathrm{top}}(T) > 0$. Then there exists a $\delta > 0$, such that every unimodal map $\tilde{T} : [0,1] \to [0,1]$ with $\|\tilde{T} - T\|_\infty < \delta$ satisfies $h_{\mathrm{top}}(\tilde{T}) > 0$, and therefore it has a unique measure $\mu(\tilde{T})$ of maximal entropy. Moreover we have*

$$\lim_{\substack{\|\tilde{T}-T\|_\infty \to 0 \\ \tilde{T} \text{ is a unimodal map}}} \mu(\tilde{T}) = \mu(T) \; .$$

Remark. Observe that for unimodal maps the topological pressure is not continuous in general. By Theorem 7 in [11] the topological pressure is upper semi–continuous for every continuous function $f : [0,1] \to \mathbf{R}$, if and only if c is not periodic. Moreover, Theorem 7 in [11] implies that there exist continuous functions $f : [0,1] \to \mathbf{R}$, such that the map $T \mapsto p(T, f)$ on the set of all unimodal maps with positive topological entropy endowed with the C^0–topology is not upper semi–continuous. On the other hand, if the set of all C^1–unimodal maps is endowed with the C^1–topology, then the map $T \mapsto p(T, f)$ is upper semi–continuous by Theorem 2 in [9]. However, the example given in Section 4 of [9] shows, that also in this case we do not have lower semi–continuity of the topological pressure in general.

REFERENCES

1. F. Hofbauer, *Maximal measures for simple piecewise monotonic transformations,* Z. Wahrsch. Verw. Gebiete **52** (1980), 289–300.
2. F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy 2,* Israel J. Math. **38** (1981), 107–115.
3. F. Hofbauer, *Piecewise invertible dynamical systems,* Probab. Theory Related Fields **72** (1986), 359–386.
4. M. Misiurewicz, *Jumps of entropy in one dimension,* Fund. Math. **132** (1989), 215–226.
5. M. Misiurewicz, S. V. Shlyachkov, *Entropy of piecewise continuous interval maps*, in: *European conference on iteration theory (ECIT 89), Batschuns, 1989,* pp. 239–245, (eds.: Ch. Mira, N. Netzer, C. Simó, Gy. Targoński), World Scientific, Singapore, (1991).
6. M. Misiurewicz, W. Szlenk, *Entropy of piecewise monotone mappings,* Studia Math. **67** (1980), 45–63.
7. P. Raith, *Continuity of the Hausdorff dimension for piecewise monotonic maps,* Israel J. Math. **80** (1992), 97–133.
8. P. Raith, *Stability of the maximal measure for piecewise monotonic interval maps,* Ergodic Theory Dynam. Systems **17** (1997), 1419–1436.
9. P. Raith, *Stability of the topological pressure for piecewise monotonic maps under C^1–perturbations,* J. Anal. Math. **78** (1999), 117–142.
10. P. Raith, *Perturbations of a topologically transitive piecewise monotonic map on the interval,* in: *Proceedings of the European conference on iteration theory (ECIT 1996), Urbino, 1996,* Grazer Math. Ber. **339** (1999), pp. 301–312, (eds.: L. Gardini, G. L. Forti, D. Gronau, L. Paganoni).
11. P. Raith, *Discontinuities of the pressure for piecewise monotonic interval maps,* Ergodic Theory Dynam. Systems **21** (2001), 197–232.
12. P. Walters, *An introduction to ergodic theory,* Graduate Texts in Mathematics 79, Springer, New York, (1982).