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Classification of Permutations and Cycles of Maximum Topological Entropy

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If f is a continuous self-map of a compact interval we can represent each finite fully invariant set of f by a permutation. We can then calculate the topological entropy of the permutation. This provides us with a numerical measure of complexity for any map which exhibits a given permutation type. In this paper we present cyclic and noncyclic permutations which have maximum topological entropy amongst all cyclic or noncyclic permutations of the same length.

Key Words: Topological entropy, dynamical system, permutation, cyclic permutation.

1. INTRODUCTION

A finite fully invariant set of a continuous map of a compact interval to itself induces a permutation in a natural way. If the invariant set is a periodic orbit the permutation is cyclic. We can calculate the topological entropy of any permutation θ and it is well known that this gives a lower bound for the topological entropy of any continuous self-map of the interval which exhibits a permutation of type θ. The notion of topological entropy

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of a permutation appeared in the literature in the 1980's but was first formalized in the paper *Combinatorial Patterns for Maps of the Interval* [11]. One of the questions Misiurewicz and Nitecki considered was which permutations or cycles achieve the maximum topological entropy amongst all permutations or cycles of the same cardinality. In their paper, Misiurewicz and Nitecki defined a family of cyclic permutations for $n = 4k+1$, $k \in \mathbb{N}$, and used this family to show that as $n \to \infty$ the maximum topological entropy of the set all n-cycles and of the set of all n-permutations both approach $\log(2n/\pi)$. This family was later shown to be maximal for $4k+1$-permutations (and hence for $4k+1$-cycles) [3].

Today we have an almost complete classification of maximum entropy cycles and permutations with only one case still unknown (that case being to identify those $4k+2$-cycles with maximum entropy). The most recent advance has been to show that those $4k$-cycles with maximum entropy defined in [9] are in fact the only ones with this property and was the subject of a paper given by the first author in Katsiveli. In this paper we give the definitions of all maximum entropy cycles and permutations identified to date.

2. PRELIMINARIES

In this section we will introduce the language and notation that we will use throughout the paper. We will state only those results which are essential for our purposes. Other results that we refer to have been stated fully in previous works as indicated.

For us f will always be a continuous map of a compact interval I into itself.

Definition 1. The *orbit* under f of a point $x \in I$ is the sequence $Orb(x) = \{x, f(x), f^2(x), \dots\}$. If $x = f^m(x)$ for some $m \in \mathbb{N}$, then $Orb(x)$ is finite and the orbit is *periodic*. The orbit has period n if n is the least positive integer for which $x = f^n(x)$.

Recall that a *permutation* on n letters is a bijective map $\theta : \{1, \dots, n\} \to \{1, \dots, n\}$. If θ has the property that for $1 \le p \le n$, $\theta^p(1) = 1$ if and only if $p = n$, then θ is a *cycle*.

Notation. We define P_n to be the set of all permutations on n letters and C_n to be the set of all cycles on n letters. We also let $P = \cup_{n \ge 1} P_n$ and $C = \cup_{n \ge 1} C_n$.

If x has a periodic orbit under f of period n we can write $Orb(x) = \{p_1, \dots, p_n\}$ with $p_1 < p_2 < \cdots < p_n$. This induces a cycle $\theta \in C_n$ in the following way:

$$\theta(i) = j \text{ if and only if } f(p_i) = p_j.$$

In general, if $S = \{p_1, \ldots, p_n\}$ (with $p_1 < p_2 < \cdots < p_n$) is any finite fully invariant set for f (that is, $f(S) = S$), we can define the permutation $\theta \in P_n$ by

$$\theta(i) = j \text{ if and only if } f(p_i) = p_j.$$

The permutation θ is called the *type* of S.

Definition 2. The *flip* permutation $\varphi \in P_n$ is defined as $\varphi(i) = n+1-i$, for $i \in \{1, \ldots, n\}$.

So for example, for $n = 7$ the flip permutation $\varphi \in P_7$ is

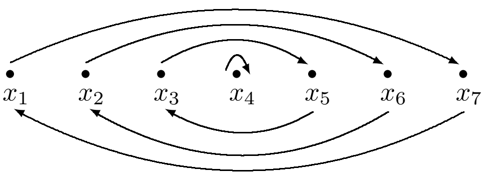


FIG. 1. The flip permutation φ_7.

There are permutations $\overline{\theta}$, $\widetilde{\theta}$ and θ^* associated to each permutation θ. These permutations are formulated in terms of the flip permutation φ as follows:

Definition 3. Let $\theta \in P_n$. Then

(1) The *reverse* of θ is the permutation $\widetilde{\theta} \in P_n$ where $\widetilde{\theta}(i) = \theta(\varphi(i))$.
(2) The *converse* of θ is the permutation $\theta^* \in P_n$ where $\theta^*(i) = \varphi(\theta(i))$.
(3) The *dual* of θ is the permutation $\overline{\theta} \in P_n$ where

$$\overline{\theta}(i) = \varphi(\theta(\varphi(i))) = \varphi(\widetilde{\theta}(i)) = \theta^*(\varphi(i)).$$

Corollary 4. *Let $\theta \in P_n$. Then*

(1) The reverse of θ is given by $\widetilde{\theta}(i) = \theta(n+1-i)$, for $i \in \{1, \ldots, n\}$.
(2) The converse of θ is given by $\theta^*(i) = n+1-\theta(i)$, for $i \in \{1, \ldots, n\}$.
(3) The dual of θ is given by $\overline{\theta}(i) = n+1-\theta(n+1-i)$, for $i \in \{1, \ldots, n\}$.

EXAMPLE 5. We illustrate the relationship between θ and $\overline{\theta}, \widetilde{\theta}$ and θ^* with an example. Let f be a map with a period 5 orbit such that $f(x_1) = x_2,\ f(x_2) = x_4,\ f(x_4) = x_5,\ f(x_5) = x_3$ and $f(x_3) = x_1$ (with $x_1 < x_2 < x_3 < x_4 < x_5$). The orbit is illustrated by the following diagram:

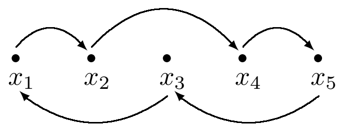


FIG. 2. The 5-cycle θ.

and is represented by the permutation $\theta \in P_5$ with $\theta(1) = 2,\ \theta(2) = 4,\ \theta(4) = 5,\ \theta(5) = 3$ and $\theta(3) = 1$. The permutation $\overline{\theta} \in P_5$ represents the period 5 orbit above with the orientation reversed as illustrated in Fig. 3 below.

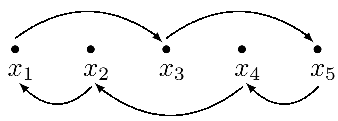


FIG. 3. The 5-cycle $\overline{\theta}$.

Clearly $\overline{\theta}$ is a cycle precisely when θ is a cycle. The permutation $\widetilde{\theta}$ is obtained from the permutation θ by first reversing the order on the elements of $Orb(x)$; that is, if $\theta(1) = 2,\ \theta(5) = 3, \ldots,$ we now consider $\widetilde{\theta}(5) = 2,\ \widetilde{\theta}(1) = 3$ and so on.

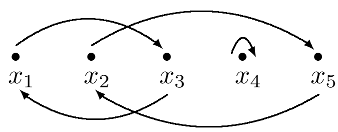


FIG. 4. The 5-permutation $\widetilde{\theta}$.

If θ is a cycle it is not necessarily true that $\widetilde{\theta}$ is a cycle. The permutation $\theta^* \in P_5$ represents the orbit above with the orientation reversed.

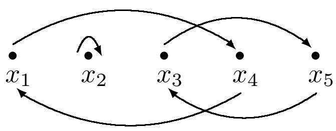


FIG. 5. The 5-permutation θ^*.

Of course θ^* is a cycle precisely when $\widetilde{\theta}$ is a cycle.

The idea of reversing the orientation of a permutation and of reversing the order on the points of a permutation can be represented by a composition of permutations.

For each permutation $\theta \in P_n$ we can define a unique map f_θ as follows:

DEFINITION 6. If S is a fully invariant set for f of type $\theta \in P_n$ then the map $f_\theta : [1, n] \to [1, n]$ satisfying

(i) $f_\theta(i) = \theta(i)$, for $i \in \{1, \ldots, n\}$,
(ii) f_θ is affine on each interval $I_i = \{x \in \mathbb{R} : i \leq x \leq i + 1\}$, for $i \in \{1, \ldots, n-1\}$,

is called the *linearisation* of f with respect to its invariant set of type θ.

DEFINITION 7. If, for each $i \in \{1, \ldots, n\}$, $f_\theta(i)$ is a local extremum of f_θ then f_θ is said to be *maximodal.* The permutation $\theta \in P_n$ is also said to be maximodal.

DEFINITION 8. The *entropy of a permutation* $\theta \in P$ is

$$h(\theta) = \inf_f \{h(f)\}$$

where f is any continuous self-map of I which has an invariant set of type θ and $h(f)$ is the topological entropy of f.

It has been shown (see [2]) that the map f_θ has the lowest topological entropy of any map which has an orbit of type θ, hence

PROPOSITION 9. *If $\theta \in P$ then $h(\theta) = h(f_\theta)$.*

We note that for each $\theta \in P$, $h(\overline{\theta}) = h(\theta)$.

DEFINITION 10. The *induced matrix* $M(\theta)$ of $\theta \in P_n$ is the $(n-1) \times (n-1)$ matrix with $i\,j^{th}$ entry given by

$$a_{i\,j} = \begin{cases} 1, & \text{if } f_\theta(I_i) \supset I_j, \\ 0, & \text{otherwise,} \end{cases}$$

where $I_i = \{x \in \mathbb{R} : i \le x \le i+1\}$ and $i, j \in [1, n-1]$.

The next proposition (due to [12], see [1] for a proof) allows us to calculate the entropy of a permutation directly from its induced matrix.

Proposition 11. *If $\theta \in P$ then $h(\theta) = \log \rho(M(\theta))$, where $\rho(M(\theta))$ is the spectral radius of the induced matrix of θ.*

So the combination of Propositions 9 and 11 describe a procedure for calculating the entropy of a permutation θ. We first construct the function f_θ, derive the matrix $M(\theta)$ and then calculate $\log\,\rho(M(\theta)) = h(\theta)$.

notation. For $n \in \mathbb{N}$ let

$$\mathcal{H}(C_n) = \max\{h(\theta) : \theta \in C_n\}$$

and

$$\mathcal{H}(P_n) = \max\{h(\theta) : \theta \in P_n\}.$$

3. PERMUTATIONS

As noted in the introduction, for $n = 4k+1$ the family of n-permutations defined by Misiurewicz and Nitecki does indeed attain maximum entropy amongst all n-permutations. In fact it turns out that if we generalize this family to the remaining odd periods, we have a family of n-permutations which is entropy maximal for any n odd [3]. Furthermore, it has been shown that this family is unique up to duality [4]. A striking feature of the family is that the permutations are cyclic so that $\mathcal{H}(C_n) = \mathcal{H}(P_n)$ for n odd. If we extend this generalization to the case where n is even we again obtain a family of permutations which achieve maximum entropy. It has been shown that this family is unique and that each permutation in the family is self-dual ($\vartheta_n = \overline{\vartheta_n}$) but is not cyclic [7, 8, 5]. So $\mathcal{H}(C_n) < \mathcal{H}(P_n)$ for n even. We describe these families in greater detail in Sections 3.1 and 3.2.

3.1. Maximum entropy n-permutations, n odd

In the early 1990's Geller and Tolosa [3] generalized the family of cycles defined by Misiurewicz and Nitecki [11] to the remaining odd periods obtaining the following definition:

DEFINITION 12. Let $n \in \mathbb{N}$ be odd and let $l = \lfloor \frac{n-1}{4} \rfloor$. The cyclic permutation θ_n is defined by

$$\theta_n : j \to \begin{cases} n - 2l - j, & \text{if } j \in O[1, n-2l-2] \\ j - n + 2l + 1, & \text{if } j \in O[n-2l, n] \\ n - 2l + j - 1, & \text{if } j \in E[2, 2l] \\ n + 2l - j + 2, & \text{if } j \in E[2l+2, n-1]. \end{cases}$$

It is easy to verify that this defines a family of cyclic permutations. We note the following general features of f_{θ_n}.

1. The map f_{θ_n} has a local minimum at $j = 1$ (and therefore also at $j = n$ since n is odd).
2. The map f_{θ_n} is maximodal and has all minimum values below all maximum values.
3. For $k = \frac{n+1}{2}$ the map f_{θ_n} has a global minimum at $j = k+1$ if k is even and $j = k$ if k is odd.
4. For $k = \frac{n+1}{2}$ the map f_{θ_n} has a global maximum at $j = k$ if k is even and $j = k+1$ if k is odd.

If $n = 4l + 1$ the cycle θ_n is of the form

$$\begin{gathered}(1 \;\; 2l \;\; 4l \;\; 2l+3 \;\; 3 \;\; 2l-2 \;\; 4l-2 \;\; 2l+5 \;\ldots\ldots\ldots \\ \ldots\ldots\ldots \underbrace{2i+1 \;\; 2l-2i \;\; 4l-2i \;\; 2l+2i+3}_{0 \le i \le l-1} \ldots\ldots\ldots \\ \ldots\ldots\ldots 2l-1 \;\; 2 \;\; 2l+2 \;\; 4l+1 \;\; 2l+1)\end{gathered}$$

and is illustrated below for the case $n = 9$.

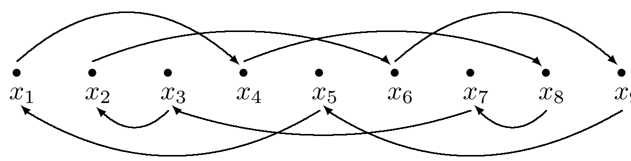
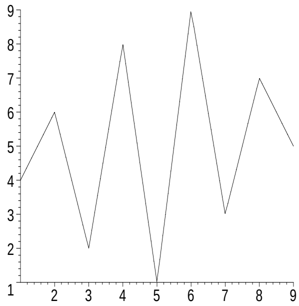


FIG. 6. The maximum entropy 9-permutation θ_9.

The general shape of the graph of f_{θ_n} together with the induced matrix $M(\theta_n)$ is illustrated for the cases $n = 9$ (Fig. 7) and $n = 7$ (Fig. 9).



$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 1 & 1 & 1 & \mathbf{0} & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{0} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \mathbf{0} & 1 & 1 & 1 & 1 & 1 & \mathbf{1} \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 1 & \mathbf{1} & 0 & 0 \end{bmatrix}$$

FIG. 7. The graph of f_{θ_9} and $M(\theta_9)$.

If $n = 4l + 3$ the cycle θ_n is of the form

$$\begin{aligned} &(1 \;\; 2l+2 \;\; n \;\; 2l+1 \;\; 2 \;\; 2l+4 \;\; n-2 \;\; 2l-1 \;\; 4 \ldots\ldots\ldots \\ &\qquad \ldots\ldots\ldots \underbrace{2l+2i \;\; n-2i+2 \;\; 2l-2i+3 \;\; 2i}_{1\le i\le l} \ldots\ldots\ldots \\ &\qquad\qquad \ldots\ldots\ldots n-3 \;\; 2l+5 \;\; 3 \;\; 2l \;\; n-1 \;\; n-2l) \end{aligned}$$

and is illustrated below for the case $n = 7$.

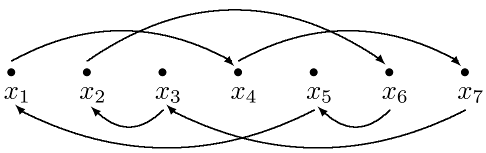
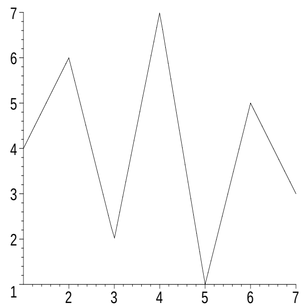


FIG. 8. The maximum entropy 7-permutation θ_7.



$$\begin{bmatrix} 0 & 0 & \mathbf{0} & 1 & \mathbf{1} & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ \mathbf{0} & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & 1 & 1 & 1 & \mathbf{0} & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

FIG. 9. The graph of f_{θ_7} and $M(\theta_7)$.

Of course, since for any permutation θ, $h(\theta) = h(\overline{\theta})$, the family of dual cycles also has maximum entropy. In this case $f_{\overline{\theta_n}}$ has a local maximum at $j = 1$ (and hence also at $j = n$).

It was shown in [11] that the induced matrix of an entropy maximal permutation has a characteristic shape which closely resembles a "diamond" pattern. Clearly a matrix which has a perfect diamond shape (that is, $\Diamond_{n-1}$ is the $0-1$ matrix such that for n odd and $k = (n-1)/2$, $d_{i\,j} = 1$ if and only if $j \in [k+1-i, k+i]$ for $i \leq k$, $j \in [i-k, 3k-i+1]$ for $i > k$) cannot be the induced matrix of any permutation. In the case where n is odd, the variation between the perfect diamond matrix and the matrix $M(\theta_n)$ is minimal. Indeed for $n = 4l+1$, $a_{i\,j} = 1$ if and only if $j \in [k-i, k+i-1]$ for even $i < k$, $j \in [i-k+1, 3k-i+2]$ for even $i > k$ and $j \in [1, n-2]$ for $i = k$, otherwise $a_{i\,j} = d_{i\,j}$, whilst for $n = 4l+3$, $a_{i\,j} = 1$ if and only if $j \in [k-i+2, k+i+1]$ for odd $i < k$, $j \in [i-k-1, 3k-i]$ for odd $i > k$ and $j \in [2, n-1]$ for $i = k$, otherwise $a_{i\,j} = d_{i\,j}$. The entries $a_{i\,j} \neq d_{i\,j}$ are marked in bold type in Fig. 7 and Fig. 9.

The idea of the Geller and Tolosa proof hinges on the fact that the entropy of any permutation θ can be defined in terms of the spectral radius of its induced matrix $M(\theta)$. So the inequality $\rho(M(\theta)) \leq \rho(M(\theta_n))$ is established for appropriate permutations $\theta \in P_n$.

3.2. Maximum entropy n-permutations, n even

We now turn our attention to the case where n is even. A straightforward generalization of the Misiurewicz-Nitecki orbit types to n even yields a family of n-permutations which has maximum topological entropy. The main difference in this case is that these permutations are not cyclic so that they are maximal in P_n but not in C_n. These results were obtained independently by King ([7], [8]), and Geller and Zhang [5].

Definition 13. Let n be even and $k = n/2$. The family of noncyclic n-permutations ϑ_n is defined by:

$$\vartheta_n : j \to \begin{cases} k - j + 1, & \text{if } j \in O[1, k] \\ j - k, & \text{if } j \in O[k+1, n-1] \\ k + j, & \text{if } j \in E[2, k] \\ 3k - j + 1, & \text{if } j \in E[k+1, n]. \end{cases}$$

1. The map f_{ϑ_n} has a local minimum at $j = 1$ (and therefore a local maximum at $j = n$ since n is even).
2. The map f_{ϑ_n} is maximodal and has all minimum values below all maximum values.
3. The map f_{ϑ_n} has a global minimum at $j = k$ if k is odd and $j = k+1$ if k is even.
4. The map f_{ϑ_n} has a global maximum at $j = k+1$ if k is odd and $j = k$ if k is even.

If $n = 4p + 2$ and p is odd then ϑ_n is a product of disjoint 2-cycles of the form

$$(1\ k)(3\ k-2)\ldots\ldots\left(\frac{k-1}{2}\ \frac{k+3}{2}\right)$$
$$(2\ k+2)(4\ k+4)\ldots\ldots(k-1\ n-1)$$
$$(k+1\ n)(k+3\ n-2)\ldots\ldots\left(\frac{3k-1}{2}\ \frac{3k+3}{2}\right).$$

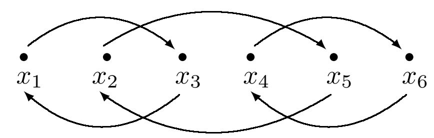
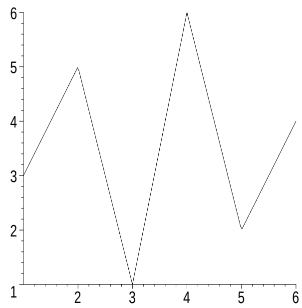


FIG. 10. The maximum entropy 6-permutation ϑ_6.

Again we give examples of the graphs of f_{ϑ_n} and the induced matrices $M(\vartheta_n)$ in the cases $n = 6$ (Fig. 11), $n = 10$ (Fig. 13) and $n = 8$ (Fig 15).



$$\begin{bmatrix} 0 & 0 & 1 & \mathbf{1} & 0 \\ \mathbf{1} & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & \mathbf{1} \\ 0 & \mathbf{1} & 1 & 0 & 0 \end{bmatrix}$$

FIG. 11. The graph of f_{ϑ_6} and $M(\vartheta_6)$.

If $n = 4p + 2$ and p is even then ϑ_n is a product of disjoint cycles of the form

$$(1\ k)(3\ k-2)\ldots\ldots\left(\frac{k-3}{2}\ \frac{k+5}{2}\right)\left(\frac{k+1}{2}\ \frac{k+1}{2}\right)$$
$$(2\ k+2)(4\ k+4)\ldots\ldots(k-1\ n-1)$$
$$(k+1\ n)(k+3\ n-2)\ldots\ldots\left(\frac{3k-3}{2}\ \frac{3k+5}{2}\right)\left(\frac{3k+1}{2}\ \frac{3k+1}{2}\right)$$

where each factor in the product is a 2-cycle except for the factors $\left(\frac{k+1}{2}\ \frac{k+1}{2}\right)$ and $\left(\frac{3k+1}{2}\ \frac{3k+1}{2}\right)$ which are both fixed points. An example is illustrated below for the case $n = 10$.

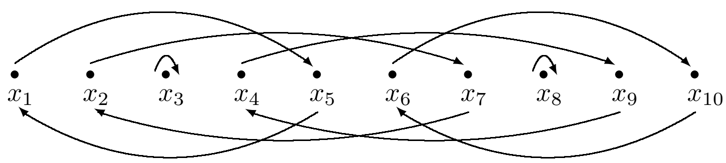
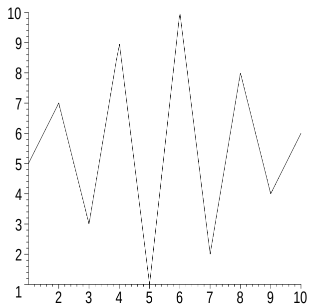


FIG. 12. The maximum entropy 10-permutation ϑ_{10}.



$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & \mathbf{1} & 0 \\ \mathbf{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{1} \\ 0 & \mathbf{1} & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

FIG. 13. The graph of $f_{\vartheta_{10}}$ and $M(\vartheta_{10})$.

If $n = 4p$ then ϑ_n is a product of disjoint 4-cycles of the form

$$(1\ k\ n\ k+1)(3\ k-2\ n-2\ k+3)\ldots\ldots(k-1\ 2\ k+2\ n-1).$$

The diagram below is for $n = 8$.

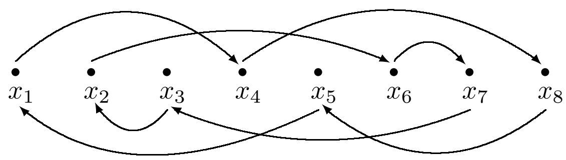
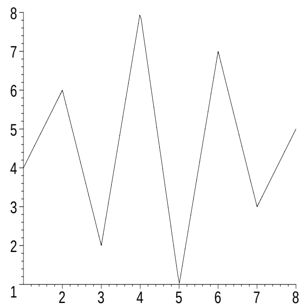


FIG. 14. The maximum entropy 8-permutation ϑ_8.



$$\begin{bmatrix} 0 & 0 & 0 & 1 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & \mathbf{1} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{1} & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & 1 & 0 & 0 & 0 \end{bmatrix}$$

FIG. 15. The graph of f_{ϑ_8} and $M(\vartheta_8)$.

The induced matrix of a permutation of even length will have odd dimension, hence the appropriate diamond matrix is $\Diamond_{n-1}$ such that $d_{i\,j} = 1$ if and only if $j \in [k+1-i, k-1+i]$ for $i \leq k$, $j \in [i-k+1, 3k-1-i]$ for $i > k$. Again, the matrix $\Diamond_{n-1}$ cannot represent any permutation but the matrix $M(\vartheta_n)$ is very close to it since $a_{i\,j} = 1$ if and only if $j \in [k+1-i, k+i]$ for j odd, $j < k$, $j \in [k-i, k-1+i]$ for j even, $j < k$, $j \in [i-k+1, 3k-i]$ for j even, $j > k$, $j \in [i-k, 3k-i-1]$ for j odd, $j > k$ and $j \in [1, n-1]$ for $j = k$. Again we indicate those entries $a_{i\,j} \neq d_{i\,j}$ in bold type.

A family of n-permutations associated to the family $\{\vartheta_n\}_{n\geq 2}$, n even is the family of *reverse* permutations defined below.

DEFINITION 14. Let n be even and $k = n/2$. The family of noncyclic n-permutations $\widetilde{\vartheta_n}$ is defined by:

$$\widetilde{\vartheta_n} : j \to \begin{cases} k+j, & \text{if } j \in O[1,k] \\ 3k-j+1, & \text{if } j \in O[k+1, n-1] \\ k-j+1, & \text{if } j \in E[2,k] \\ j-k, & \text{if } j \in E[k+1, n]. \end{cases}$$

In terms of the graph of f_{ϑ_n}, the graph of $f_{\widetilde{\vartheta_n}}$ is reflected through the line $j = \frac{n+1}{2}$. This family is also a family of self-dual permutations and for all n even, $h(\vartheta_n) = h(\widetilde{\vartheta_n})$ (see [8]). If a permutation ϕ is cyclic (respectively noncyclic) it is not true that $\widetilde{\phi}$ is necessarily cyclic (respectively noncyclic). In this case the family $\{\widetilde{\vartheta_n}\}_{n\geq 2}$ is also noncyclic. Thus we have two distinct n-permutations for each n even which attain maximum entropy in P_n.

1. The map $f_{\widetilde{\vartheta_n}}$ has a local maximum at $j = 1$ (and therefore a local minimum at $j = n$ since n is even).

2. The map $f_{\widetilde{\vartheta_n}}$ is maximodal and has all minimum values below all maximum values.

3. The map $f_{\widetilde{\vartheta_n}}$ has a global minimum at $j = k+1$ if k is odd and $j = k$ if k is even.

4. The map $f_{\widetilde{\vartheta_n}}$ has a global maximum at $j = k$ if k is odd and $j = k+1$ if k is even.

If $n = 4p + 2$ and p is odd then $\widetilde{\vartheta_n}$ is a product of disjoint 2-cycles of the form

$$\begin{aligned}
&(1\ k+1)(3\ k+3)\dots(k\ 2k)\\
&\quad(2\ k-1)(4\ k-3)\dots\left(\frac{k-3}{2}\ \frac{k+5}{2}\right)\left(\frac{k+1}{2}\ \frac{k+1}{2}\right)\\
&\qquad(k+2\ n-1)(k+4\ n-3)\dots\left(\frac{3k-3}{2}\ \frac{3k+5}{2}\right)\left(\frac{3k+1}{2}\ \frac{3k+1}{2}\right)
\end{aligned}$$

which is shown below for $n = 6$.

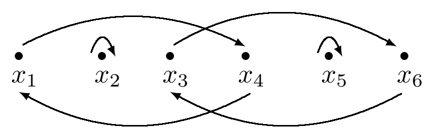


FIG. 16. The maximum entropy 6-permutation $\widetilde{\vartheta_6}$.

If $n = 4p + 2$ and p is even then $\widetilde{\vartheta_n}$ is a product of disjoint cycles of the form

$$\begin{aligned}
&(1\ k+1)(3\ k+3)\dots\dots(k\ 2k)\\
&\quad(2\ k-1)(4\ k-3)\dots\dots\left(\frac{k-1}{2}\ \frac{k+3}{2}\right)\\
&\qquad(k+2\ n-1)(k+4\ n-3)\dots\dots\left(\frac{3k-1}{2}\ \frac{3k+3}{2}\right).
\end{aligned}$$

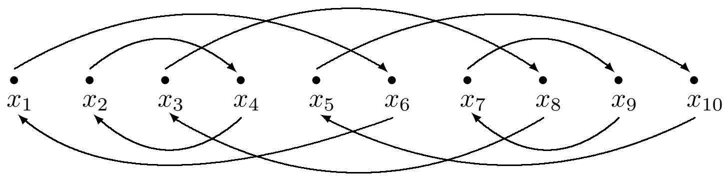


FIG. 17. The maximum entropy 10-permutation $\widetilde{\vartheta_{10}}$.

If $n = 4p$ and $k = 2p$ then ϑ_n is a product of disjoint 4-cycles of the form

$$(1\ k+1\ n\ k)(3\ k+3\ n-2\ k-2)\ldots\ldots(k-1\ n-1\ k+2\ 2).$$

The example illustrated is for $n = 8$.

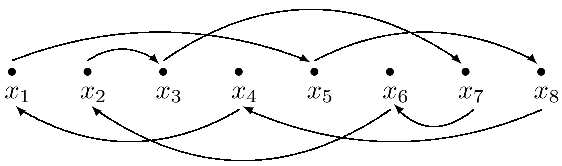


FIG. 18. The maximum entropy 8-permutation $\widetilde{\vartheta_8}$.

We leave it to the reader to construct examples of relevant graphs and matrices as desired.

To prove that these permutations are entropy maximal for n even an argument analogous to that used to prove the result for n odd suffices.

We can represent the maximum entropy n-permutations for any $n \in \mathbb{N}$ in a unified way.

DEFINITION 15. Let $n \in \mathbb{N}$, $k = n/2$ and let a be the fractional part of $n/2$. We define the n-permutation Θ_n as follows:

$$\Theta_n : j \to \begin{cases} k - j + 1 + (-1)^{((n+1)/2)}a, & \text{if } j \in O[1, k] \\ j - k + (-1)^{((n-1)/2)}a, & \text{if } j \in O[k + \frac{1}{2}, n] \\ k + j + (-1)^{((n+1)/2)}a, & \text{if } j \in E[2, k] \\ 3k - j + 1 + (-1)^{((n-1)/2)}a, & \text{if } j \in E[k + \frac{1}{2}, n]. \end{cases}$$

4. CYCLES

Having identified all maximum elements of P_n it is a natural question to consider the maximum elements of C_n. As we have already noted, the maximum entropy permutations are cyclic for n odd so both questions are answered simultaneously. Furthermore, for $n = 4$ the maximum entropy permutation is cyclic. So we now consider only the case where $n > 4$ is even.

Definition 16. Let $n = 4k$ for $k \in \mathbb{N} \setminus \{1\}$. We define the cyclic n-permutation ψ_n as follows:

$$\psi_n : j \to \begin{cases} 2k - j + 1, & \text{if } j \in O[1, k+1] \\ 2k - j + 2, & \text{if } j \in O[k+2, 2k+1] \\ j - 2k - 1, & \text{if } j \in O[2k+3, 3k] \\ j - 2k, & \text{if } j \in O[3k+1, n-1] \\ 2k + j, & \text{if } j \in E[2, k+1] \\ 2k + j - 1, & \text{if } j \in E[k+2, 2k] \\ 6k - j + 2, & \text{if } j \in E[2k+2, 3k] \\ 6k - j + 1, & \text{if } j \in E[3k+1, n]. \end{cases}$$

1. The map f_{ψ_n} has a local minimum at $j = 1$ (and hence a local maximum at $j = n$ since n is even).
2. The map f_{ψ_n} is maximodal and has all maximum values above all minimum values.
3. The map f_{ψ_n} has a global minimum at $j = 2k + 1$.
4. The map f_{ψ_n} has a global maximum at $j = 2k + 2$.

For example, for k odd, ψ_n is easily seen to be the cycle

$$\begin{aligned} &(2k+1 \;\; 1 \;\; 2k \;\; n-1 \;\ldots\; \overbrace{2k-2i+3 \;\; 2i-1 \;\; 2k-2i+2 \;\; n-2i+1}^{2 \le i \le \frac{k-3}{2}} \ldots \\ &\quad \ldots\; k+4 \;\; k-2 \;\; k+3 \;\; 3k+2 \;\; k+2 \;\; k \;\; k+1 \;\; 3k+1 \;\; 3k \;\; k-1 \\ &\qquad 3k-1 \;\; 3k+3 \;\ldots\; \overbrace{2k+2i+1 \;\; 2i \;\; 2k+2i \;\; n-2i+2}^{\frac{k-3}{2} \ge i \ge 2} \ldots \\ &\qquad\quad \ldots\; 2k+3 \;\; 2 \;\; 2k+2 \;\; n). \end{aligned}$$

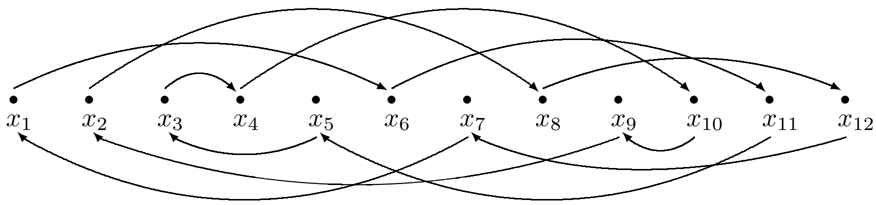
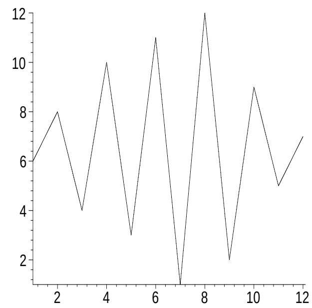


FIG. 19. The maximum entropy 12-cycle ψ_{12}.



$$\begin{bmatrix}
0&0&0&0&0&1&\mathbf{1}&0&0&0&0\\
0&0&0&\mathbf{1}&1&1&1&0&0&0&0\\
0&0&0&1&1&1&1&1&\mathbf{1}&0&0\\
0&0&1&1&1&1&1&1&1&0&0\\
0&\mathbf{0}&1&1&1&1&1&1&1&1&0\\
1&1&1&1&1&1&1&1&1&1&\mathbf{0}\\
\mathbf{1}&1&1&1&1&1&1&1&1&1&\mathbf{1}\\
0&\mathbf{1}&1&1&1&1&1&1&1&\mathbf{1}&\mathbf{1}\\
0&\mathbf{1}&\mathbf{1}&1&1&1&1&1&0&0&0\\
0&0&0&0&1&1&1&\mathbf{1}&0&0&0\\
0&0&0&0&\mathbf{1}&1&0&0&0&0&0
\end{bmatrix}$$

FIG. 20. The graph of $f_{\psi_{12}}$ and $M(\psi_{12})$.

Similarly, for k even, ψ_n is the cycle

$$\begin{aligned}
&(2k+1\ \ 1\ \ 2k\ \ n-1\ \ldots \overbrace{2k-2i+3\ \ 2i-1\ \ 2k-2i+2\ \ n-2i+1}^{2\le i\le \frac{k-2}{2}}\ \ldots\\
&\quad \ldots\ k+3\ \ k-1\ \ k+2\ \ 3k+1\ \ k+1\ \ k\ \ 3k\ \ 3k+2\ \ 3k-1\ \ k-2\\
&\qquad 3k-2\ \ 3k+4\ \ldots \overbrace{2k+2i+1\ \ 2i\ \ 2k+2i\ \ n-2i+2}^{\frac{k-4}{2}\ge i\ge 2}\ \ldots\\
&\qquad\quad \ldots\ \ 2k+3\ \ 2\ \ 2k+2\ \ n).
\end{aligned}$$

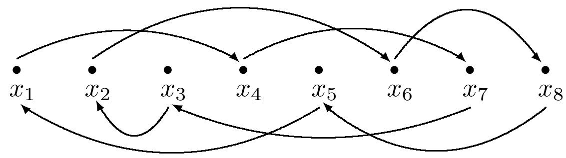
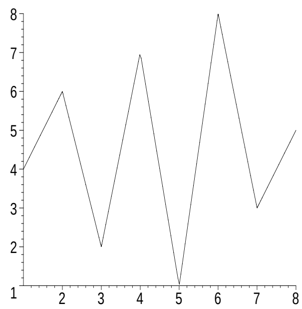


FIG. 21. The maximum entropy 8-cycle ψ_8.



$$\begin{bmatrix} 0 & 0 & 0 & 1 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{0} \\ \mathbf{1} & 1 & 1 & 1 & 1 & 1 & \mathbf{1} \\ 0 & 0 & 1 & 1 & 1 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 1 & 0 & 0 & 0 \end{bmatrix}$$

FIG. 22. The graph of f_{ψ_8} and $M(\psi_8)$.

If we compare the induced matrix of a maximum entropy n-cycle (n even) to the diamond matrix $\Diamond_{n-1}$ (that is, $d_{i\,j} = 1$ if and only if $j \in [2k+1-i, 2k-1+i]$, $i \leq 2k$, $j \in [i-2k+1, 6k-i-i]$, $i > 2k$) there is still little variation (although more than the variation between the maximum entropy n-permutation and $\Diamond_{n-1}$, as we would expect). Specifically for $a_{i\,j} \in M(\vartheta_n)$ we have $a_{i\,j} = 1$ if and only if $j \in [2k+1-i, 2k+i]$, i odd, $i < k+1$, $j \in [2k-i, 2k-1+i]$, i even, $i < k+1$, $j \in [2k+2-i, 2k+i-1]$, i odd, $k+1 < i \leq 2k$, $j \in [2k+1-i, 2k+i-2]$, i even, $k+1 < i \leq 2k$, $j \in [i-2k-1, 6k-i]$, i odd, $2k+1 < i < 3k$, $j \in [i-2k, 6k-i+1]$, i even, $2k+1 < i < 3k$, $j \in [i-2k, 6k-i-1]$, i odd, $i > 3k$, $j \in [i-2k+1, 6k-i]$, i even, $i > 3k$, $j \in [i-2k, 6k-1]$, $i = 2k+1$, $j \in [i-2k-1, 6k-i-1]$, $i = 3k$, i odd, $j \in [i-2k+1, 6k-i+1]$, $i = 3k$, i even, $a_{i\,j} = d_{i\,j}$ otherwise. For example, in Fig. 20 and Fig. 22 the entries of the matrix $M(\psi_8)$ which are in bold are precisely those $a_{i\,j} \neq d_{i\,j}$.

The cycle ψ_n is not self-dual however the dual $\overline{\psi_n}$ is automatically a cycle and has the same entropy as ψ_n. In the case of ψ_n the reverse permutation $\widetilde{\psi_n}$ is also a cycle and hence so is ψ_n^*.

Definition 17. Let $n = 4k$ for $k \in \mathbb{N} \setminus \{1\}$. We define the cyclic n-permutation $\widetilde{\psi_n}$ as follows:

$$\widetilde{\psi_n} : j \to \begin{cases} 2k + j, & \text{if } j \in O[1, k] \\ 2k + j + 1, & \text{if } j \in O[k + 1, 2k - 1] \\ 6k - j, & \text{if } j \in O[2k + 1, 3k - 1] \\ 6k - j + 1, & \text{if } j \in O[3k, n - 1] \\ 2k - j + 1, & \text{if } j \in E[2, k] \\ 2k - j, & \text{if } j \in E[k + 1, 2k - 2] \\ j - 2k + 1, & \text{if } j \in E[2k, 3k - 1] \\ j - 2k, & \text{if } j \in E[3k, n]. \end{cases}$$

1. The map f_{ψ_n} has a local maximum at $j = 1$ (and hence a local minimum at $j = n$ since n is even).

2. The map f_{ψ_n} is maximodal and has all maximum values above all minimum values.

3. The map f_{ψ_n} has a global minimum at $j = 2k$.

4. The map f_{ψ_n} has a global maximum at $j = 2k - 1$.

Specifically, $\widetilde{\psi_n}$ is the cycle

$$\begin{aligned} &(2k \;\; 1 \;\; 2k+1 \;\; n-1 \;\ldots\; \overbrace{2k+2i-2 \;\; 2i-1 \;\; 2k+2i-1 \;\; n-2i+1}^{2 \le i \le \frac{k-3}{2}} \;\ldots \\ &\quad \ldots\; 3k-3 \;\; k-2 \;\; 3k-2 \;\; 3k+2 \;\; 3k-1 \;\; k \;\; 3k \;\; 3k+1 \;\; k+1 \;\; k-1 \\ &\qquad k+2 \;\; 3k+3 \;\ldots\; \overbrace{2k-2i \;\; 2i \;\; 2k-2i+1 \;\; n-2i+2}^{\frac{k-3}{2} \ge i \ge 2} \;\ldots \\ &\qquad\quad \ldots\; 2k-2 \;\; 2 \;\; 2k-1 \;\; n) \end{aligned}$$

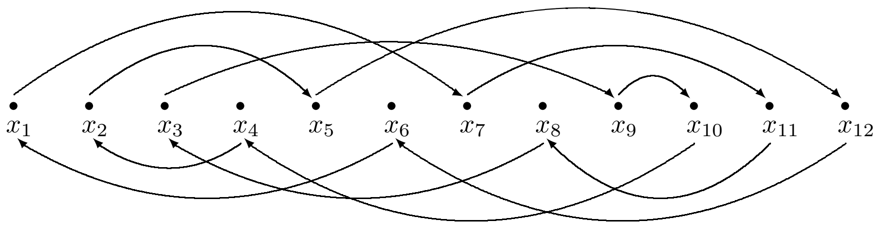


FIG. 23. The maximum entropy 12-permutation $\widetilde{\psi_{12}}$.

for k odd, and for k even, $\widetilde{\psi_n}$ is the cycle

$$\begin{aligned}
&(2k \;\; 1 \;\; 2k+1 \;\; n-1 \;\ldots\; \overbrace{2k+2i-2 \;\; 2i-1 \;\; 2k+2i-1 \;\; n-2i+1}^{2\le i\le \frac{k-2}{2}} \;\ldots \\
&\quad \ldots\; 3k-2 \;\; k-1 \;\; 3k-1 \;\; 3k+1 \;\; 3k \;\; k \;\; k+1 \;\; 3k+2 \;\; k+2 \;\; k-2 \\
&\qquad k+3 \;\; 3k+4 \;\ldots\; \overbrace{2k-2i \;\; 2i \;\; 2k-2i+1 \;\; n-2i+2}^{\frac{k-4}{2}\ge i\ge 2} \;\ldots \\
&\qquad\quad \ldots\; 2k-2 \;\; 2 \;\; 2k-1 \;\; n).
\end{aligned}$$

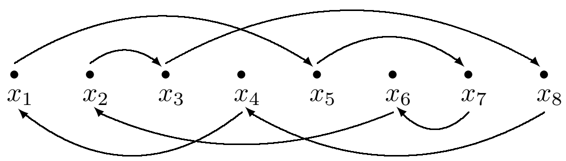


FIG. 24. The maximum entropy cycle $\widetilde{\psi_8}$.

THEOREM 18. *[9, 10] For $n = 4k$, $k \in \mathbb{N}$, $k > 1$, the cyclic permutations ψ_n, $\widetilde{\psi_n}$, $\overline{\psi_n}$ and ψ_n^* are the only cycles which have maximum entropy amongst all cycles of period n.*

To prove that these cycles have maximum entropy we need to consider the relationship between a permutation and its induced matrix. The key is to identify characteristics of the matrices which allow us to recognize those which represent permutations which have particular types of proper invariant subsets (which are modelled on the invariant subsets of ϑ_n). Any permutations with such subsets are clearly not cyclic, hence they need not be considered. It has been shown ([9, 10]) that $\rho(M(\psi_n))$ is maximal for a class of matrices which excludes these examples.

The authors are currently working with a family of $4k+2$-cycles which they believe to be entropy maximal.

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