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On Existence of Noncompact Heteroclinic Curves*

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We consider Morse-Smale diffeomorphisms of three-manifolds for which the sphere S^3 is the universal covering. We prove that if the nonwandering set of a diffeomorphism f on such a manifold consists of exactly four fixed points, two of them being saddles, then the wandering set of f has at least one nonclosed heteroclinic curve.

Key Words: Morse-Smale diffeomorphisms, heteroclinic points, heteroclinic curves

1. INTRODUCTION

Let us recall that a diffeomorphism f of a smooth closed manifold M is *Morse-Smale* if the following holds: 1) its nonwandering set $\Omega(f)$ is finite and consists of hyperbolic periodic points, 2) the stable and unstable

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manifolds of saddle periodic points have only transversal intersection (see the survey [3] which contains many results and references on Morse-Smale diffeomorphisms).

Let $p, q \in \Omega(f)$ be saddle periodic points for which $W^u(p) \cap W^s(q) \neq \emptyset$, then following S. Smale's terminology we write $p \leq q$. We call a diffeomorphism f a *gradient-like* diffeomorphism, if the condition $p \leq q$ implies $dim\ W^s(p) < dim\ W^s(q)$. If $W^u(p) \cap W^s(q) \neq \emptyset$ and $dim\ W^s(p) = dim\ W^s(q)$, then from the transversality of the intersection of $W^u(p)$ with $W^s(q)$ it follows that $W^u(p) \cap W^s(q)$ is a countable set. Each point of this set is called a *heteroclinic point* of the diffeomorphism f.

From transversality of the intersection $W^u(p)$ with $W^s(q)$ it follows that $dim\ W^s(p) \leq dim\ W^s(q)$, hence any Morse-Smale diffeomorphism which does not contain heteroclinic points is a gradient-like diffeomorphism. If $W^u(p) \cap W^s(q) \neq \emptyset$ and $dim\ W^s(p) < dim\ W^s(q)$, then a connected component of the intersection $W^u(p) \cap W^s(q)$ is called a *heteroclinic submanifold*. If M is three-dimensional, then any heteroclinic submanifold is either a simple closed curve or a nonclosed curve without self-intersections. We call such curves *heteroclinic curves*.

One of the differences between diffeomorphisms and flows on three-dimensional manifolds is the possibility of nontrivial embedding for separatrices of saddle periodic points of a diffeomorphism in an ambient manifold. The first example of such a nontrivial embedding was constructed by Pixton [6]. Bonatti and Grines [2] considered a class of Morse-Smale diffeomorphisms on the sphere S^3 for which the nonwandering set consists of one saddle fixed point, two sinks and one source. The surprising fact here is that the classification (up to topological conjugacy) of this almost trivial dynamics is equivalent to the classification of knots in $S^2 \times S^1$ that are free-homotopy to the knot $\{x\} \times S^1$. Let us stress that due to [5], Morse-Smale flows with the similat nonwandering set on S^3 are topologically equivalent.

In [4], necessary and sufficient conditions were obtained for topological conjugacy of diffeomorphisms from the class of Morse-Smale diffeomorphisms on 3-manifolds which do not admit heteroclinic intersections (i.e., there are neither heteroclinic points, nor heteroclinic curves). In [1], it was proved that if M is a closed connected orientable three-dimensional manifold, then the following holds: for the existence of Morse-Smale diffeomorphism without heteroclinic curves on M with k saddle periodic points and l sinks and sources (in total) it is necessary and sufficient that if $k = l - 2$, then M is the 3-sphere and if $k \neq l - 2$ then M is the connected sum of $(k - l + 2)/2$ copies of $S^2 \times S^1$.

In the present paper we consider the class $MS(M^3, 4)$ of Morse-Smale diffeomorphisms whose nonwandering set consists of exactly four points two of which are saddles. In lemma 1 we show by using [1] that if a diffeomorphism f belongs to $MS(M^3, 4)$, then the Morse index of the two saddles are different i.e., their unstable manifolds have different dimensions. Thus the wandering set of such a diffeomorphism does not contain heteroclinic points. It follows immediately from [1] that if the wandering set of $f \in MS(M^3, 4)$ does not contain heteroclinic curves, then M^3 is $S^2 \times S^1$. Notice that one can easily construct an example of Morse-Smale diffeomorphism from the class $MS(S^2 \times S^1, 4)$ whose nonwandering set does not contain heteroclinic curves. Then it follows from [1] that if M^3 is different from $S^2 \times S^1$, then the wandering set of any diffeomorphism $f \in MS(M^3, 4)$ has heteroclinic curves. However it was still an open question whether such a curve is closed or nonclosed.

In this paper we prove that if the universal covering of the manifolds M^3 is the sphere S^3 then the wandering set of a diffeomorphism $f \in MS(M^3, 4)$ has at least one nonclosed heteroclinic curve (see theorem 4). Notice that there is a diffeomorphism $f \in MS(M^3, 4)$ whose wandering set does not contain closed heteroclinic curves.

2. PROOF OF THE THEOREM 4

Lemma 1. *Let $f \in MS(M^3, 4)$ has saddle fixed σ_1, σ_2. Then their Morse indeces are $u(\sigma_1) = 1$, $u(\sigma_2) = 2$ (or vice versa).*

Proof. It is enough to show that the saddles σ_1, σ_2 have different Morse indeces. Let us suppose by contradiction that the wandering set of the diffeomorphism f has no heteroclinic curves. As we have already mentioned in the Introduction, in this case the manifold M^3 is $S^2 \times S^1$ (this follows from [1]). Let us apply now the Lefschetz formula

$$L(f) = \sum_{p \in Fix\ f} ind\ (p, f),$$

where $L(f)$ is the Lefschetz number, which is calculated by the formula

$$L(f) = \sum_{i=0}^{n} (-1)^i Sp\ f_{i*},$$

and $Sp\ f_{i*}$ is the trace of the linear map $F_{i*} : H_i(M^3, \mathbb{R}) \to H_i(M^3, \mathbb{R})$ of the homology group $H_i(M^3, \mathbb{R})$ induced by f. Since $M^3 = S^2 \times S^1$ and f is orientation preserving, we have $L(f) = 0$, as $H_1(M^3, \mathbb{R}) = H_2(M^3, \mathbb{R}) = \mathbb{Z}$ and $Spf_{1*} = Spf_{2*} = 1$. On the other hand, the coincidence of Morse indeces of the saddles imply that the sum of indeces of fixed points cannot be zero. This contradiction proves the lemma. ∎

LEMMA 2. *If $f \in MS(M^3, 4)$ then the following inclusions take place:*

$$W^u(\sigma_1) - \sigma_1 \subset W^s(\omega), \quad W^s(\sigma_2) - \sigma_2 \subset W^u(\alpha).$$

Proof. . As f cannot have homoclinic points, $W^s(\sigma_i) \cap W^u(\sigma_i) = \emptyset$ $(i = 1, 2)$. Since f is a structurally stable diffeomorphism, it follows from transversality of stable and unstable manifolds of saddle points that $W^u(\sigma_1) \cap W^s(\sigma_2) = \emptyset$. As M^3 is the union of the stable (unstable) manifold of fixed points we get the result of the lemma. ∎

LEMMA 3. *Let $f \in MS(M^3, 4)$. Then the sets*

$$C_\omega \stackrel{\mathrm{def}}{=} \{\omega\} \cup W^u(\sigma_1) \text{ and } C_\alpha \stackrel{\mathrm{def}}{=} \{\alpha\} \cup W^s(\sigma_2)$$

are embeddings of the circle S^1.

Proof. . We need to prove that each of the sets C_ω and C_α is an embedding of the circle S^1 in M^3. We will prove this result only for C_ω because for C_α the proof is similar.

There is a C^1-immersion $\varphi : \mathbb{R} \to W^u(\sigma_1)$, where $\varphi(0) = \sigma_1$. Let us show that the immersion φ can be extended to a homeomorphism

$$\varphi : S^1 \cong \mathbb{R} \cup \{\infty\} \to W^u(\sigma_1) \cup \{\sigma_1\},$$

and we put $\varphi(\pm\infty) = \omega$. By lemma 2, $W^u(\sigma_1) - \sigma_1 \subset W^s\omega)$. Therefore the ω-limit set of each component of the set $W^u(\sigma_1) - \sigma_1$ consists of the single point ω. This imply the result of the lemma. ∎

THEOREM 4. *Let $f \in MS(M^3, 4)$ where M^3 is a closed orientable three-dimensional manifold for which the sphere S^3 is the universal covering. Then the wandering set of the diffeomorphism f contains at least one non-closed heteroclinic curve.*

Proof. Let us consider a fundamental domain F^s of the restriction $f|_{W^s(\sigma_1) - \sigma_1}$. As the point σ_1 is hyperbolic, we may assume that F^s is a closed annulus bounded by smooth closed curves C_1 and C_2 and these curve bound (in $W^s(\sigma_1)$) open disks containing the point σ_1. Let us take a simple closed curve C in F^s which is homotopic to the curves C_1 and C_2.

We shall divide the further proof into steps. The end of the proof of each step we shall denote by $\diamond$.

Step 0 *Let C be any closed curve which is homotopic to the curves C_1 and C_2, then $C \cap W^u(\sigma_2) \neq \emptyset$.*

Proof of step 0. Suppose the contrary, that is $C \cap W^u(\sigma_2) = \emptyset$. As $M^3 - \omega$ is the union of exactly three disjoint unstable manifolds $W^u(\alpha)$, $W^u(\sigma_2)$

and $W^u(\sigma_1)$, we have $C \subset W^u(\alpha)$. By compactness of $C_\omega = \{\omega\} \cup W^u(\sigma_1)$, there is a neighbourhood $U(\alpha)$ of the source α such that $U(\alpha) \cap C_\omega = \emptyset$. The inclusion $C \subset W^u(\alpha)$ and compactness of C imply the existence of a negative number n_0 such that $f^{n_0}(C) \subset U(\alpha)$.

Let us consider the universal covering $q : S^3 \to M^3$ and the covering diffeomorphism $\tilde{f} : S^3 \to S^3$. Let $\tilde{C}_\omega$ be a connected component of the set $q^{-1}(C_\omega)$. Since $q : S^3 \to M^3$ is finite-sheeted, $\tilde{C}_\omega$ is a simple closed curve containing at least one saddle $\tilde{\sigma}_1 \in q^{-1}(\sigma_1)$. It follows from the equality $u(\sigma_1) = 1$ that $W^s(\sigma_1)$ is homeomorphic to $\mathbb{R}^2$. As the curve $C \subset W^s(\sigma_1) - \sigma_1$ is nonhomotopic to zero in $W^s(\sigma_1) - \sigma_1$, it bounds (in $W^s(\sigma_1)$) a disk D which contains the point σ_1. As f has no homoclinic points, the intersection $D \cap C_\omega$ consists of exactly one point σ_1. Therefore there is a simple closed curve $\tilde{C} \in q^{-1}(C)$ which bounds a disk belonging to $W^s(\tilde{\sigma}_1)$ and intersecting the family of closed curves of the set $q^{-1}(C_\omega)$ in exactly one point $\tilde{\sigma}_1$.

Therefore $\tilde{C}$ and $q^{-1}(C_\omega)$ forms a nontrivial link with the linking coefficient -1 or $+1$ (depending on the orientation of the curves). Then $\tilde{f}^{n_0}(\tilde{C})$ and $\tilde{f}^{n_0}(q^{-1}(C_\omega))$ also forms nontrivial link with the linking coefficient -1 or $+1$. From the equality $f(C_\omega) = C_\omega$ it follows $\tilde{f}^{n_0}(q^{-1}(C_\omega)) = q^{-1}(C_\omega)$. Therefore $\tilde{f}^{n_0}(\tilde{C})$ and $q^{-1}(C_\omega)$ forms a nontrivial link.

On the other hand, $\tilde{f}^{n_0}(\tilde{C}) \in q^{-1}(U(\alpha))$. Without loss of generality we may assume that the set $q^{-1}(U(\alpha))$ is the union of disjoint connected components which are homeomorphic to $U(\alpha)$ and each of these components is homeomorphic to the three-dimensional disk. Therefore $\tilde{f}^{n_0}(\tilde{C})$ belongs entirely to one of these connected components, which we denote $U^{-1}(\alpha)$. As $U(\alpha) \cap C_\omega = \emptyset$ then $U^{-1}(\alpha) \cap q^{-1}(C_\omega) = \emptyset$. Therefore the linking coefficient of $\tilde{f}^{n_0}(\tilde{C})$ and $q^{-1}(C_\omega)$ is equal to zero. We get a contradiction. $\diamond$

Step 1 *For a compact (in the topology of the manifold $W^s(\sigma_1)$) subset $F \subset W^s(\sigma_1)$ and any point $m_0 \in int\ F$ there is a neighbourhood $U(m_0)$ which is homeomorphic to a disk and which has intersection with no more than one curve from the intersection $F \cap W^u(\sigma_2)$. Moreover if $U(m_0)$ intersects only one curve (denoted by l) then the intersection $U(m_0) \cap l$ consists of one component which is homeomorphic to a simple arc and divides $U(m_0)$.*

Proof of step 1. Suppose the contrary, i.e., for any neighbourhood $U(m_0)$ which is homeomorphic to a disk, the intersection $U(m_0) \cap (F \cap W^u(\sigma_2))$ consists of more than one curve. Then there is a sequence $m_k \in F \cap W^u(\sigma_2)$ which converges to a point $m_0 \in int\ F$ such that the points m_k lie in disjoint components of the intersection $F \cap W^u(\sigma_2)$. From this and transversality of the intersection $F \cap W^u(\sigma_2)$ it follows that the points m_k are isolated in the topology of the unstable manifold $W^u(\sigma_2)$. Therefore $m_0 \notin W^u(\sigma_2)$ (otherwise the unstable manifold $W^u(\sigma_2)$ would be self-limiting and there would be homoclinic points). As $M^3 - \omega = W^u(\sigma_2) \cup W^u(\alpha) \cup W^u(\sigma_1)$ then either $m_0 \in W^u(\alpha)$ or $m_0 \in W^u(\sigma_1)$. The inclusion $m_0 \in W^u(\alpha)$ is impossible because the unstable manifold $W^u(\alpha)$ is open and cannot contain accumulation points belonging to the unstable manifold $W^u(\sigma_2)$. The inclusion $m_0 \in W^u(\sigma_1)$ is also impossible because otherwise it would imply the existence of homoclinic points.

Thus there is a neighborhood $U(m_0)$ such that $U(m_0) \cap (F \cap W^u(\sigma_2))$ consists of one simple curve, which we denote by l. Let us show that l divides $U(m_0)$. Suppose the contrary. From the arguments above and transversality of the intersection F with $W^u(\sigma_2)$ it follows that the intersection of boundary points of l with the neighborhood $U(m_0)$ consists of exactly one point, which we denote by l^*. From the equality $M^3 - \omega = W^u(\sigma_2) \cup W^u(\alpha) \cup W^u(\sigma_1)$ it follows that the point l^* must belong to $W^u(\alpha) \cup W^u(\sigma_1)$. But it is impossible. The contradiction completes the proof of the step. $\diamond$

As by step 0, $W^u(\sigma_2)$ intersects an arbitrary curve C which is homotopic to the boundary components C_1 and C_2 of the annulus F^s, $F^s \cap W^u(\sigma_2)$ contains at least one arc d, with endpoints a_1, a_2 lying on the components C_1 and C_2 respectively.

Step 2 *There are finitely many arcs from the intersection $F^s \cap W^u(\sigma_2)$ whose endpoints lie on different boundary components of the annulus F^s.*

Proof of step 2. Let us suppose the contrary. Then there is a point $m_0 \in$ *int* F^s which is the topological limit of disjoint curves from $F^s \cap W^u(\sigma_2)$. But this contradicts to step 1. $\diamond$

Let us enumerate in cyclic order all arcs from the intersection $F^s \cap W^u(\sigma_2)$ whose endpoints lie on different boundary components of the annulus $F^s : d_1, \dots, d_k$. Let $\mathcal{D}_1, \dots, \mathcal{D}_k$ be curves from the $W^s(\sigma_1) \cap W^u(\sigma_2)$, containing arcs $d_1, \dots, d_k$ respectively. We notice that some of curves $\mathcal{D}_i$ may coincide.

Step 3 *There is at least one nonclosed curve among of the curves $\mathcal{D}_1$, $\dots, \mathcal{D}_k$.*

Proof of step 3. Suppose the contrary. Then using step 1 we can construct a closed curve which does not intersect the set $W^s(\sigma_1) \cap W^u(\sigma_2)$ and bounds (in the $W^s(\sigma_1)$) a disk which contains the point σ_1. But this contradicts step 0. $\diamond$

Step 4 *Let $\mathcal{D}$ be any nonclosed curve from the set $\mathcal{D}_1, \dots, \mathcal{D}_k$. Then at least one of the next conditions a), b) are fulfilled*

a) $f^i(F^s) \cap \mathcal{D} \neq \emptyset$ for all $i \geq 0$,

b) $f^{-i}(F^s) \cap \mathcal{D} \neq \emptyset$ for all $i \geq 0$.

Proof of step 4. Suppose the contrary. Then there is a number $i_0 > 0$ such that $\mathcal{D} \in$ *int* $(\bigcup\limits_{i=-i_0}^{i=i_0} f^i(F^s))$. Then there is a point $m_0 \in$ *int* $(\bigcup\limits_{i=-i_0}^{i=i_0} f^i(F^s))$ such that either any neighbourhood $U(m_0)$ contains an infinite set of connected components of the intersections $U(m_0) \cap \mathcal{D}$, or m_0 is one of two boundary points of the curve $\mathcal{D}$. But this contradicts step 1. $\diamond$

Step 5 *Each nonclosed curve from the intersection $W^s(\sigma_1) \cap W^u(\sigma_2)$ is invariant for some iterate of the diffeomorphism f.*

Proof of step 5. It is enough to prove the statement for any nonclosed curve $\mathcal{D}$ from the set $\mathcal{D}_1, \dots, \mathcal{D}_k$. Suppose the contrary, that is $f^i(\mathcal{D}) \neq \mathcal{D}$. By step 4 we may suppose for definiteness that $\mathcal{D} \cap f^i(F^s) \neq \emptyset$ for all $i \geq 0$. Then for any $i \geq 0$ there is an arc $A_i \subset (\mathcal{D} \cap f^i(F^s))$ whose endpoints belong to different boundary components $f^i(C_1)$ and $f^i(C_2)$ of the annulus $f^i(F^s)$. As $\mathcal{D}$ is not f^i-nvariant, the union $\bigcup\limits_{i \geq 0} f^{-i}(A_i)$ consists of finitely many disjoint arcs which belong to the annulus F^s and whose endpoints lie on different boundary components of F^s. But this contradicts to step 2. $\diamond$

Step 6 *Each nonclosed curve from the intersection $W^s(\sigma_1) \cap W^u(\sigma_2)$ has no self-intersections and has the points σ_1 and σ_2 as its boundary points.*

Proof of the step 6. Let us consider again any nonclosed curve $\mathcal{D}$ from the set $\mathcal{D}_1, \dots \mathcal{D}_k$. Without loss of generality we may assume that the curve $\mathcal{D}$ is f-invariant. Put $d = \mathcal{D} \cap F^s$. Then $\mathcal{D} = \bigcup\limits_{i \in \mathbb{Z}} f^i(d)$. By step 1, the topological limit of the sequence of arcs $f^i(d)$ is σ_1 as $i \to +\infty$ and similarly, the topological limit of the sequence of arcs $f^i(d)$ is σ_2 as $i \to -\infty$. So we get the result of step 6. $\diamond$

Step 6 completes the proof of the theorem. ▌

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