QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **4**, 39–57 (2003)
ARTICLE NO. 51

Weak Equivalence of Cocycles and Mackey Action in Generic Dynamics *

Valentin Golodets

School of Mathematics, University of N.S.W., Sydney NSW 2052, Australia
E-mail: golodets@maths.unsw.edu.au

and

Vyacheslav Kulagin

Institute for Low Temperature Physics & Engineering, Ukrainian National Academy of Sciences, 47 Lenin Ave., 61103 Kharkov, Ukraine
E-mail: kulagin@ilt.kharkov.ua

Let $\mathcal{R}$ be an equivalence relation generated by a countable ergodic homeomorphism group of a perfect Polish space X. We consider cocycles taking values in Polish groups on $\mathcal{R}$ modulo meager subsets of X. Two cocycles are called weakly equivalent if they are cohomologous up to an automorphism of $\mathcal{R}$. The notion of generic associated Mackey action is introduced, which is an invariant of weak equivalence for cocycles. Regular cocycles with values in an arbitrary Polish group and transient cocycles with values in an arbitrary countable group are completely classified up to weak equivalence.

Key Words: Cocycle, generic dynamics, generic Mackey action.

1. INTRODUCTION

Generic dynamical system arises from the action of a homeomorphism group on a Baire topological space, but unlike the usual topological dynamics the properties of such systems are studied modulo meager sets (by analogy with null-sets in measurable dynamics). One of the sharp distinctions of the situation in generic theory from the measurable one appears in the orbit theory: it was proved by D.Sullivan, B.Weiss and J.Wright that any two ergodic actions of countable groups by homeomorphisms on a

* Supported in part by a CRDF grant number UM1-2092.

perfect Polish space are orbit equivalent modulo meager sets ([24]). Our investigation is devoted to the classification problem for cocycles of countable homeomorphism groups of a perfect Polish space up to weak equivalence and is closely related to the orbit theory. The concept of weak equivalence for cocycles was introduced in the measure-theoretic setting in [9] as a weakening of the notion of cohomological equivalence and, on the other hand, as a generalization of the notion of weak equivalence for dynamical systems in the sense of [5]. In our context it has the same definition: two cocycles of a homeomorphism group with values in a given Polish group are called weakly equivalent if they are cohomologous up to an element from the normalizer of this dynamical system. In measurable dynamics the structure of cocycles has been researched mainly for amenable transformation groups ([26], [27], [9], [3],[4], [10], [22], see also a survey of S.Bezugly [2]). Presence of very good generic orbit theory for arbitrary (ergodic) countable homeomorphism groups of a Polish space ([24], [25]) permits us to suggest that a worthwhile theory of cocycles of such dynamical systems could be developed. The first step into this direction was made in [8], where the uniqueness theorem for ergodic cocycles with values in an arbitrary Polish group was proved. A solution of the outer conjugacy problem for homeomorphism groups and the theory of subrelations of generic equivalence relations are applications of generic theory of cocycles (see [8]). It is interesting to note that in generic dynamics any Polish group can be represented as the image of an ergodic cocycle of a $\mathbb{Z}$-action ([8]), which constitutes a striking difference from the measurable situation (in fact, a locally-compact group which is the image of an ergodic cocycle of an automorphism of Lebesgue space must be amenable [26]).

The Mackey's construction of a group action associated with a cocycle of the measurable dynamical system (the image of a cocycle) has appeared in [18] and is a generalization of the concept of flow built under a function ([1]). It is a well known that a Mackey action technique is an essential tool in the studying of cocycles in ergodic theory. Remind that the Krieger's classification of automorphisms of a Lebesgue space with quasi-invariant measures up to orbit equivalence ([15]), which is equivalent to the classification of Radon-Nikodym cocycle's of dynamical systems, is reduced to the problem of conjugating of Poincare flows. At the same time the Poincare flow is nothing else than the Mackey actions associated with the Radon-Nikodym cocycle. One should mention that the classification of general cocycles of amenable automorphism groups of a Lebesgue space with values in locally-compact amenable groups was also realized in terms of Mackey actions ([10]).

We introduce the notion of generic Mackey action. In more detail, let $\mathcal{R} = \mathcal{R}_\Gamma$ be an equivalence relation generated by an action of countable group Γ by homeomorphisms on a perfect Polish space X. Let $Z^1(\mathcal{R}_\Gamma, G)$

denotes the set of (Borel) cocycles of $\mathcal{R}$ with values in a Polish group G. We associate with any cocycle $\alpha \in Z^1(\mathcal{R}, G)$ an action $W_\alpha(G)$ of the group G on the Baire space of the generic ergodic decomposition of the skew product $X \times_\alpha G$. This action is an invariant of weak equivalence for cocycles, i.e. weak equivalent cocycles have (generically) conjugate Mackey actions (prop. 11).

For any $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$ there are two possibilities: 1). $W_\alpha(G)$ is transitive modulo a meager set (essential transitivity). 2). Every $W_\alpha(G)$-orbit is meager (proper ergodicity). The first case corresponds exactly to the property of regularity of α, that is, α is cohomologous to some ergodic cocycle with values in a closed subgroup of G. This fact combining with the uniqueness theorem for ergodic cocycles lead us to the classification of regular cocycles (up to weak equivalence) with values in an arbitrary Polish group (Theorem 18).

The next class of cocycles for which we apply the developed generic Mackey action approach is transient cocycles (the terminology is inherited from measurable dynamics). The skew product action $X \times_\alpha G$ for a transient cocycle $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$ is of discrete type (this is an analogue of type I action in measurable theory). We show that for transient cocycles with values in an arbitrary countable group the generic Mackey action is a complete invariant of weak equivalence (Theorem 22). Both cases of regular and transient cocycles demonstrate the effectiveness of associated action approach to use in generic dynamics. It is still open to apply this methods to the problem of classification of general cocycles.

The organization of the paper is as follows. Section 2 provides some preliminary information. Section 3 contains results on ergodic decomposition in generic dynamics and on some properties of nonergodic homeomorphism groups of a perfect Polish space. This results are needed for our study of generic Mackey action, however they may have also an independent interest. In Section 4 the generic Mackey action technique is developed, and the classifications of regular (4.1) and transient (4.2) cocycles are obtained.

2. PRELIMINARIES

Recall that a topological space X is called *Baire* if it satisfies one of the following equivalent conditions: 1. *Any nonempty open subset of X is nonmeager.* 2. *The intersection of a countable many dense open subsets is dense in X.* 3. *Any comeager subset of X is dense.* ([13]). Every Polish or locally compact Hausdorff space is Baire. Note that it follows from the definition that any G_δ-subset U of the Baire space X is Baire in the relative topology. When X is Polish then U will be Polish again ([16]). A subset $A \subset X$ of the topological space X is said to have the *Baire property* if there exists a meager $M \subset X$ with $A \triangle M$ being open. Every Borel subset of X has the Baire property ([16]).

PROPOSITION 1 ([8]). *Let X be a Baire topological space and $A_n \subset X$ ($n \in \mathbb{N}$) be a countable family of sets having the Baire property. Then there exists a dense G_δ-subset Y of X, such that $A_n|_Y$ is open in Y for all $n \in \mathbb{N}$.*

A Borel bijection Θ of a Polish space X is said to be a *pseudo-homeomorphism* if for some dense G_δ-subset $Y \subset X$ $\Theta|_Y$ is a homeomorphism of Y. Call two Baire spaces Y_1 and Y_2 to be pseudo-homeomorphic if there exist meager sets $M_1 \subset Y_1$, $M_2 \subset Y_2$ such that $Y_1 \setminus M_1$ and $Y_2 \setminus M_2$ are homeomorphic. Any two perfect Polish spaces are pseudo-homeomorphic (see [24]).

PROPOSITION 2. *Let G be a homeomorphism group of a second countable Baire space X. Then the following conditions are equivalent: (i) There exists $x_0 \in X$ with the orbit Gx_0 is dense in X. (ii) There exists a dense G_δ-subset $X' \subset X$ such that for every $x \in X'$ the orbit Gx is dense in X. (iii) Any G-invariant set B with the Baire property is either meager or comeager.*

Proof. (ii) $\Rightarrow$ (iii) is shown in [13]. (i) $\Rightarrow$ (ii) ([25]): Let $\{U_n\}_{n=1}^{\infty}$ be a countable basis for the topology of X. A G_δ-set $X' = \bigcap_{n=1}^{\infty} G[U_n]$ is nonempty, because it contains Gx_0. Then one easily checks that every $x \in X'$ has a dense G-orbit in X. (iii) $\Rightarrow$ (i): Every $G[U_n]$ is open and dense, hence $X' = \bigcap_{n=1}^{\infty} G[U_n]$ is dense, now as above. ∎

DEFINITION. A homeomorphism group G of a Baire second countable space X is called (generically) *ergodic* if one of the equivalent conditions from 2 is valid.

For $\mathcal{R} \subset X \times X$ an equivalence relation on X and $A \subset X$ the *saturation* $\mathcal{R}[A]$ of A by $\mathcal{R}$ is a set $\{x \in X : x\mathcal{R}y \text{ for some } y \in Y\}$. A subset $T \subset X$ is a *transversal* for $\mathcal{R}$ if T meets each equivalence class of $\mathcal{R}$ at exactly one point.

In the sequel let X denotes a perfect Polish space. A countable Borel equivalence relation $\mathcal{R}$ on X is called *generic* if for any meager $A \subset X$ its saturation $\mathcal{R}[A]$ is meager too ([24]). If Γ is a countable homeomorphism group of X then the equivalence relations $\mathcal{R} = \mathcal{R}_\Gamma$ is generic. It was shown in [24] that, modulo meager sets, any generic countable equivalence relation on X is generated by a countable homeomorphism group of X. Two actions a_i ($i = 1, 2$) of countable groups Γ_i by homeomorphisms of X are (generically) *orbit equivalent* if there exists a pseudo-homeomorphism

θ of X with $\theta(\mathcal{R}_{\Gamma_1}[x]) = \mathcal{R}_{\Gamma_2}[\theta x]$ for all $x \in X$ modulo a meager subset of X. This is equivalent to say that the equivalence relations $\mathcal{R}_{\Gamma_1}$ and $\mathcal{R}_{\Gamma_2}$ are isomorphic.

From now on $\mathcal{R}$ stands for a countable generic equivalence relation on X. A set Aut$\mathcal{R}$ of automorphisms of $\mathcal{R}$ consists of such pseudo-homeomorphisms Θ of X for which $\Theta(\mathcal{R}[x]) = \mathcal{R}[\Theta x]$ for all x from some comeager subset of X. We say that Θ is an *inner* automorphism of $\mathcal{R}$ ($\Theta \in \mathrm{Int}\mathcal{R}$) if $\Theta \in \mathrm{Aut}\mathcal{R}$ and $(\Theta x, x) \in \mathcal{R}$ for all $x \in X$ outside of a meager subset. $\mathcal{R}$ is called *ergodic* if it is generated by an ergodic countable homeomorphism group of X.

THEOREM 3 (Sullivan–Weiss–Wright, [24]). *Let Γ_1, Γ_2 be countable ergodic homeomorphism groups of a perfect Polish space X. Then, modulo a meager subset of X, $\mathcal{R}_{\Gamma_1}$ and $\mathcal{R}_{\Gamma_2}$ are isomorphic (in other terms Γ_1 and Γ_2 are orbit equivalent).*

Let G a Polish group. A Borel map $\alpha : \mathcal{R} \to G$ is called a *cocycle* of $\mathcal{R}$ with values in G if for some $\mathcal{R}$-invariant dense G_δ-subset Y of X $\phi(x,y)\phi(y,z) = \phi(x,z)$ for all $(x,y),(y,z) \in \mathcal{R}|_{Y\times Y}$. Let $Z^1(\mathcal{R},G)$ denotes the set of all cocycles of $\mathcal{R}$ with values in G (with the identification of cocycles which agree modulo meager subsets of X). Two cocycles $\alpha,\beta \in Z^1(\mathcal{R},G)$ are said to be *cohomologous* ($\alpha \approx \beta$), if there exists a Borel function $f : Y \to G$ such that $\alpha(x,y) = f(x)\beta(x,y)f(y)^{-1}$ for all $(x,y),(y,z) \in \mathcal{R}$ modulo a meager subset of X.

Here are some traditional examples of cocycles.

1. Let Θ be a homeomorphism of X without fixed points, G be a Polish group, and $f : X \to G$ be an arbitrary Borel function. Θ defines a free $\mathbb{Z}$-action on X, let $\mathcal{R} = \mathcal{R}_{\mathbb{Z}}$ be a an equivalence relation generated by this action. Define a cocycle $\varphi_f : \mathcal{R} \to G$ by setting:

$$\varphi_f(x,\Theta^n x) = \begin{cases} f(x)\cdot\ldots\cdot f(\Theta^{n-1}x), & n \geq 1 \\ e, & n = 0 \\ f(\Theta^{-1}x)^{-1}\cdot\ldots\cdot f(\Theta^n x)^{-1}, & n \leq -1 \end{cases} \tag{1}$$

Note that any cocycle ρ of a $\mathbb{Z}$-action can be represented in the form (1), it suffices to put $f(x) = \rho(x,\Theta x)$. Furthermore one may assume f to be continuous.

2. Let $X = \prod_{n=1}^{\infty} X_n$, where each X_n is a finite set, card $X_n = k_n$, and X is equipped with the product topology. The group $\Gamma = \bigoplus_{n=1}^{\infty} \mathbb{Z}_{k_n}$ acts on X in a natural way. Let $\mathcal{R} = \mathcal{R}_\Gamma$, then $\mathcal{R}$ may be represented in the following form: $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n$, where $\mathcal{R}_n = \{(x,y) \in X \times X : x_m = y_m \; for \text{ all } m > n\}$

is a finite equivalence relation, and $\mathcal{R}_n \subset \mathcal{R}_{n+1}$ for all $n \in \mathbb{N}$. Suppose that for every $n \in \mathbb{N}$ we are given a function $f_n : X_n \to G$. Lets define a *product* cocycle $\pi : \mathcal{R} \to G$ by: $\pi(x, y) = f_1(x_1) \cdot \ldots \cdot f_n(x_n) f_n(y_n)^{-1} \cdot \ldots \cdot f_1(y_1)^{-1}$.

Suppose that $\mathcal{R}$ is generated by a homeomorphism group Γ of X. For $\alpha \in Z^1(\mathcal{R}, G)$ a *skew product action* $\Gamma(\alpha)$ is the action of Γ on the space $X \times G$ given by $\gamma(x, g) = (\gamma x, \alpha(\gamma x, x)g)$ (see [8]). Modulo meager subsets of X, one may view the skew product action as an action by homeomorphisms on $X \times G$ ([8]). A skew product equivalence relation $\mathcal{R}(\alpha)$ (or $X \times_\alpha G$) on $X \times G$ may be also defined as follows: $(x, g_1)\ \mathcal{R}(\alpha)\ (y, g_2)$ iff $x\ \mathcal{R}\ y$ and $\alpha(y, x) = g_2 {g_1}^{-1}$. Evidently $\mathcal{R}(\alpha)$ is generated by $\Gamma(\alpha)$. A cocycle $\alpha \in Z^1(\mathcal{R}, G)$ is *ergodic* if the skew product $\mathcal{R}(\alpha)$ is ergodic.

Cocycles $\alpha, \beta \in Z^1(\mathcal{R}, G)$ are called *weakly equivalent* if there exists $\Theta \in \mathrm{Aut}\mathcal{R}$ such that $\alpha \approx \beta \circ (\Theta \times \Theta)$. It is not difficult to see that weakly equivalent cocycles $\alpha, \beta \in Z^1(\mathcal{R}, G)$ generate orbit equivalent skew product actions (i.e. $\mathcal{R}(\alpha) \cong \mathcal{R}(\beta)$).

Assume we are given two actions $W_1(G)$ and $W_2(G)$ of a group G on Baire spaces Ω_1 and Ω_2 respectively. This actions are called (generically) isomorphic if there exists a $W_i(G)$-invariant comeager subset $Y_i \subset \Omega_i$ $(i = 1, 2)$, and there exists a homeomorphism θ from Y_1 onto Y_2 such that $\theta W_1(g) \theta^{-1} y = W_2(g) y$ for all $y \in Y_2$, $g \in G$.

Recall that an action of a Polish group G on a topological (resp. Borel) space X is called continuous (resp. Borel) if the mapping $(g, x) \to gx$ from $G \times X \to X$ is continuous (resp. Borel) ([13]).

3. PROPERTIES OF GENERIC ERGODIC DECOMPOSITION

Ergodic decomposition is a necessary tool in the study of general group actions and it is a well researched in measurable ergodic theory. As remarked in [25] there is a nice decomposition of general dynamical systems in generic dynamics (in contrast to the classical topological setting). We recall the main idea of this decomposition.

Let Γ be an arbitrary group of homeomorphisms of a perfect Polish space X and $\mathcal{E} = \mathcal{E}_\Gamma$ be an equivalence relation on X generated by this group. Let's consider the following equivalence relation $\widetilde{\mathcal{E}}$ on X, which is called the generic ergodic decomposition ([25], [14]):

$$x \widetilde{\mathcal{E}} y \Leftrightarrow \overline{\mathcal{E}[x]} = \overline{\mathcal{E}[y]}$$

Then $\mathcal{E} \subset \widetilde{\mathcal{E}}$, $\widetilde{\mathcal{E}}$ is a G_δ- subset of $X \times X$, every $\widetilde{\mathcal{E}}[x]$ is a G_δ-set of X (so is a Polish space ([16])), and the action of Γ on every $\widetilde{\mathcal{E}}[x]$ is minimal. (see [13], [25]). Thus, the equivalence class $\widetilde{\mathcal{E}}[x]$ $(x \in X)$ may be qualified as an ergodic component of x with respect to the Γ-action. Generally speaking, $\widetilde{\mathcal{E}}[x] \neq \overline{\mathcal{E}[x]}$.

For our further purposes we need a more detail research of such a decomposition.

Proposition 4. *Let Ω be the topological factor-space $X/\widetilde{\mathcal{E}}$ and $\phi : X \to \Omega$ be the factor-map. Then the following is true:*

(i)ϕ is open.

(ii)Ω is a Baire, second countable T_0-space.

(iii)If $S \subset \Omega$ is meager then $\phi^{-1}(S)$ is meager, and if $L \subset X$ is meager and $\widetilde{\mathcal{E}}$-invariant then $\phi(L)$ is meager.

(iv)A Borel structure on Ω generated by its topology is standard.

Proof. *(i)* is an immediate consequence of the following

Lemma 5. *Let $O \subset X$ be open. Then $\widetilde{\mathcal{E}}[O] = \mathcal{E}[O]$.*

Proof. Suppose $x \in O$ and $t \in \widetilde{\mathcal{E}}[x]$. Let $Z = \overline{\mathcal{E}[x]}(\supset \widetilde{\mathcal{E}}[x])$. Since the action of Γ on Z is ergodic, the set $\mathcal{E}[O] \cap Z$ is dense and open in Z. If $t \notin \mathcal{E}[O]$ then $\widetilde{\mathcal{E}}[t]$ is meager in Z, which contradicts $\widetilde{\mathcal{E}}[t] = \widetilde{\mathcal{E}}[x]$. ▮

(ii) It follows from *(i)* that Ω is Baire and second countable. T_0-property is a consequence of the fact that for a G_δ-equivalence relation $\widetilde{\mathcal{E}}$ the family $\{\widetilde{\mathcal{E}}[B_n]\}_{n=1}^{\infty}$, where $\{B_n\}_{n=1}^{\infty}$ is a basis for the topology of X, is separating (see [12]). By *(i)* this family is open, so $\phi(\widetilde{\mathcal{E}}[B_n])_{n=1}^{\infty}$ separates points of Ω.

(iii) Suppose that S is meager in Ω then $S = \bigcup\limits_{n=1}^{\infty} S_n$, where S_n is nowhere dense. One has $\overline{S_n} \supset \phi(\overline{\phi^{-1}(S_n)})$ and if $\phi^{-1}(S_k)$ is nonmeager in X for some k, then the openness of ϕ implies that $\overline{S}_k$ has a nonempty interior, which is a contradiction.

(iv) By [23, 5.1] the equivalence relation $\widetilde{\mathcal{E}}$ has a Borel selector and, hence, a Borel transversal $F \subset X$. It follows that the bijective map $\phi : F \to \Omega$ is Borel, and now the Borel structure on Ω is standard by [17, th. 3.2]. ▮

We will call the factor-space Ω from the above statement a generic space of the ergodic decomposition of the system (X, Γ). One sees from (iii) that Ω is defined modulo meager sets. Suppose $F \subset X$ is meager and Γ-invariant, $X' = X \setminus F$. How related then Ω and $\Omega' = X'/\widetilde{\mathcal{E}}_\Gamma$? Let $F_1 = \{x \in F : \widetilde{\mathcal{E}}_\Gamma[x] \cap F = \widetilde{\mathcal{E}}_\Gamma[x]\}$. Then $\phi(F_1)$ – meager in Ω, and the factor-space $(X \setminus F_1)/\widetilde{\mathcal{E}}_\Gamma(\subset \Omega)$ is homeomorphic to Ω'. Hence Ω and Ω' are pseudo-homeomorphic, that agrees with the generic point of view.

Note also that for any Γ-invariant subset Y of X the generic ergodic decomposition $\widetilde{\mathcal{E}}_Y$ corresponding to the action of Γ on Y coincides with $\widetilde{\mathcal{E}} \cap (Y \times Y)$.

The following proposition shows that there exists a Polish space among the collection of generic factor-spaces of a given dynamical system (X, Γ).

Proposition 6. *Let Ω, ϕ be as above. Then there exist a dense $\widetilde{\mathcal{E}}$-invariant G_δ-set $X' \subset X$ and a dense G_δ-set $\Omega' \subset \Omega$ such that $\phi(X') = \Omega'$ and Ω' is Polish.*

Proof. Let $\{B_n\}_{n=1}^{\infty}$ be a countable basis for the topology of Ω. By virtue of 1 there exists a dense G_δ-subset $\Omega' \subset \Omega$ such that for all n $B_n' = B_n \cap \Omega'$ is clopen in Ω. Because Ω' is a T_0-space and $\{B_n'\}_{n=1}^{\infty}$, of course, forms the basis for the topology of Ω', any two points $a, b \in \Omega'$ can be separated by a clopen subset of Ω', so Ω' is Hausdorff. For any closed $C \subset \Omega'$ one has $C = \bigcap\limits_{k=1}^{\infty} coB_{n_k}'$ – the intersection of clopen sets, so if $c \notin C$, one can separate c from C by a clopen subset of Ω', hence Ω' is regular. Thus, by Urysohn theorem, Ω' is metrizable. Let $X' = \phi^{-1}(\Omega')$, then X' is a dense $\widetilde{\mathcal{E}}$-invariant G_δ-subset of X and $\phi : X' \to \Omega'$ is open. But a continuous open metrizable image of a Polish space is Polish again (Sierpinsky, see also [13]). ▮

We now show that after elimination from X a meager subset, one can find a transversal for $\widetilde{\mathcal{E}}$, which plays the role of a Polish factor-space of the ergodic decomposition.

Proposition 7. *Let X, $\widetilde{\mathcal{E}}$ be as above. Then there exists a dense $\widetilde{\mathcal{E}}$-invariant G_δ-set $X' \subset X$, a closed in X' transversal T' for $\widetilde{\mathcal{E}}$ on X' and an open, continuous map $\pi : X' \to T'$, where T' is taken with the relative topology, such that $x\widetilde{\mathcal{E}}y \Leftrightarrow \pi(x) = \pi(y)$.*

Proof. By virtue of 6 one may think that the factor-space Ω is Polish and totally disconnected, and, of course, every $\widetilde{\mathcal{E}}$-equivalence class is closed in X. Hence we are in the conditions of [21, 5.1], which implies that there exists a closed $\widetilde{\mathcal{E}}$-transversal $T \subset X$. Let $p = \phi|_T : T \to \Omega$. This is the bijective continuous map, so the inverse $p^{-1} : \Omega \to T$ is Borel (see [17]). Use 1 to find a dense G_δ-subset $\Omega' \subset \Omega$ such that $p^{-1}|_{\Omega'} : \Omega' \to T$ is continuous. Then $X' = \phi^{-1}(\Omega')$, $T' = p^{-1}(\Omega')$ and $\pi = p^{-1}|_{\Omega'}$ work. ▮

It should be noted that all of the properties from prop. 4 holds for T. Further, an isolated point of the space T corresponds to the second category ergodic component of a Γ-action in X. The cardinality of the set of isolated points of T is no more than countable.

Suppose now that T is a perfect space and Γ is countable. It is easy to see that every $\widetilde{\mathcal{E}}$-orbit is either discrete or a perfect Polish space in the relative topology. If $\widetilde{\mathcal{E}}[x]$ is discrete then it is no more than countable and coincides with $\mathcal{E}[x]$. Our intention now to decompose the space X of a Γ-action into a discrete type part and a purely continuous part. This corresponds to the extraction of a type I part from the space of dynamical system in measurable dynamics (cf. [7]).

PROPOSITION 8. *Let Γ be a countable group of homeomorphisms of a perfect Polish space X, and suppose that $T \subset X$ is a closed $\widetilde{\mathcal{E}}_\Gamma$-transversal, which is perfect as a topological space. Then there exists a dense Γ-invariant G_δ-set $Y \subset X$ and a partition of Y into clopen $\widetilde{\mathcal{E}}_\Gamma$-invariant subsets $Y = Y_d \cup Y_c$ such that every $\widetilde{\mathcal{E}}_\Gamma$-orbit in Y_d is discrete and every $\widetilde{\mathcal{E}}_\Gamma$-orbit in Y_c is a perfect Polish space.*

Proof. Note that if $\widetilde{\mathcal{E}}_\Gamma[x]$ $(x \in X)$ is no more than countable then it is discrete. This follows from the fact that a perfect Polish space is always uncountable ([13, 6.3]). Let $D = \{t \in T : \widetilde{\mathcal{E}}_\Gamma[t]$ is no more than countable $\}$. We let $X_d = \widetilde{\mathcal{E}}[D]$.

LEMMA 9. *D is an F_σ-subset of T.*

Proof. Let ρ be a metric on X. Set, for every $n \in \mathbb{N}$, $C^n = \{t \in D : \rho(t, y) \geq 1/n,$ for all $y \in \widetilde{\mathcal{E}}_\Gamma[t]\}$. Then $D = \bigcup_{n\in\mathbb{N}} C^n$. Show that C^n is closed in T. Suppose that $t_k \to t$ $(k \to \infty)$, where $t_k \in C^n$, $t \in T$. Then $\rho(t_k, \gamma t_k) \geq 1/n$ for all $\gamma \in \Gamma$, $k \in \mathbb{N}$. Fix $\gamma \in \Gamma$. Given any $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for any $k \geq M$ $\rho(t_k, t) < \varepsilon$ and $\rho(\gamma t_k, \gamma t) < \varepsilon$. Then $\rho(\gamma t, t) \geq \rho(\gamma t_k, t_k) - \rho(t_k, t) - \rho(\gamma t, \gamma t_k) > 1/n - 2\varepsilon$. It follows that $\rho(\gamma t, t) \geq 1/n$ for all $\gamma \in \Gamma$. Thus t is an isolated point of the space $\mathcal{E}_\Gamma[t](= \Gamma t)$, which implies that $\mathcal{E}[t]$ is a discrete space, $\mathcal{E}[t] = \widetilde{\mathcal{E}}[t]$ and $t \in C^n$. ▌

Discard from T a meager F_σ-set F such that $D \setminus F$ is clopen in $T \setminus F$ (1). Put $Y = X \setminus \widetilde{\mathcal{E}}[F]$, $Y_d = \widetilde{\mathcal{E}}[D \setminus F]$, $Y_c = Y \setminus Y_d$ to complete the proof. ▌

If, for an action of countable group Γ by homeomorphisms of X, the space Y_c from the above proposition is empty we will call this action to have a *discrete type*. If Y_d is empty, the action of Γ is said to have a *purely continuous type*. Obviously, the type of action is an invariant of orbit equivalence. Using standard arguments one may state the following criterion of the discreteness: the Γ-action is of discrete type iff the equivalence relation $\mathcal{E}_\Gamma$ is generically smooth, i.e. $\mathcal{E}_\Gamma$ has, modulo a meager subset of X, a Borel transversal.

Note also without proof that using 7 and 8 one may obtain a complete classification of non-ergodic actions of countable groups by homeomorphisms of X up to orbit equivalence (the ergodic case is theorem 3). We write here the precise formulation only for the case of infinite discrete type action, in which we will need below.

PROPOSITION 10. *Suppose that an action of countable group Γ on X is of discrete type, and each Γ-orbit is infinite (modulo a meager subset of X). Then the equivalence relation $\mathcal{R}_\Gamma$ is isomorphic to an equivalence relation $\mathcal{F}_\infty$ on the space $Y \times \mathbb{N}$, where Y is a perfect Polish space, and* $(y_1, n_1)\, \mathcal{F}_\infty\, (y_2, n_2) \Leftrightarrow y_1 = y_2$.

4. GENERIC MACKEY ACTION

Throughout this section Γ will be a countable homeomorphism group of X and $\mathcal{R} = \mathcal{R}_\Gamma$ the corresponding equivalence relation on X.

Let G be a Polish group, $\alpha \in Z^1(\mathcal{R}, G)$. Let $V(G)$ be an action of the group G on $X \times G$ defined by $V(g)(x,h) = (x, hg^{-1})$. This action is continuous and commutes with the skew product action $\Gamma(\alpha)$.

Let $\mathcal{E}_\alpha$ denotes the skew product equivalence relation on $X \times G$ generated by $\Gamma(\alpha)$, and let $\widetilde{\mathcal{E}}_\alpha$ be the corresponding generic ergodic decomposition equivalence relation. Let $\Omega = (X \times G)/\widetilde{\mathcal{E}}_\alpha$ be the topological factor-space and $\phi : X \times G \to \Omega$ be the factor-map.

DEFINITION. An action $W(G)$ of the group G on the space Ω defined by

$$W(g)v = \phi(V(g)y),$$

where $y \in \phi^{-1}(v)$, $v \in \Omega, g \in G$, is called the *generic Mackey action* associated with the cocycle α.

Every $V(g)$ $(g \in G)$ is a homeomorphism, commuting with $\Gamma(\alpha)$. It follows that $z_1 \widetilde{\mathcal{E}}_\alpha z_2$ is equivalent $w(g)z_1 \widetilde{\mathcal{E}}_\alpha w(g)z_2$, therefore our definition is correct.

Further, suppose that X' is a dense Γ-invariant G_δ- subset of X, $F = X \setminus X'$. The set of points of $X \times G$ whose $\widetilde{\mathcal{E}}_\alpha$-orbits belong to $F \times G$ is $V(G)$-invariant and $\widetilde{\mathcal{E}}_\alpha$-invariant. So the G-spaces $\Omega = (X \times G)/\widetilde{\mathcal{E}}_\alpha$ and $\Omega' = (X' \times G)/\widetilde{\mathcal{E}}_\alpha$ are essentially the same (4, (iii)). The latter means that a generic Mackey action is independent on discarding from X a meager subset. Conversely, if we discard a meager G-invariant F_σ-set $M \subset \Omega$ from Ω, then the set $\phi^{-1}(M)$ is of the form $F \times G$ and meager. Now the action $W(G)$ on $\Omega' = \Omega \setminus M$ is the generic Mackey action associated with the cocycle $\alpha \in Z^1(\mathcal{R} \cap (X' \times X'), G)$, where $X' = X \setminus F$.

PROPOSITION 11. *Let $\alpha, \beta \in Z^1(\mathcal{R}, G)$ be weakly equivalent cocycles, then the Mackey actions $W_\alpha(G)$ and $W_\beta(G)$ are isomorphic.*

Proof. One has $\beta(x,y) = \varphi(x)\alpha(\theta x, \theta y)\varphi(y)^{-1}$ for all $(x,y) \in \mathcal{R}$, where $\theta \in \mathrm{Aut}\mathcal{R}$ can be assumed without loss of generality to be a homeomorphism, and $\varphi : X \to G$ to be continuous (1). Lets define a homeomorphism $\Phi : X \times G \to X \times G$ by $\Phi(x,h) = (\theta^{-1}x, \varphi(x)h)$. It is not difficult to see that Φ realizes an orbit isomorphism of the actions $\Gamma(\alpha)$ and $\Gamma(\beta)$. Since Φ commutes with the action $V(G)$, one concludes that $z_1 \widetilde{\mathcal{E}}_\alpha z_2 \Leftrightarrow \Phi z_1 \widetilde{\mathcal{E}}_\beta \Phi z_2$.

Let $\Omega_1 = (X \times G)/\widetilde{\mathcal{E}}_\alpha$, $\Omega_2 = (X \times G)/\widetilde{\mathcal{E}}_\beta$ and let $\phi_i : X \times G \to \Omega_i$ $(i = 1,2)$ be factor-maps. Then a map $\widetilde{\Phi} : \Omega_1 \to \Omega_2$ defined by $\widetilde{\Phi}\omega = \phi_2(\Phi({\phi_1}^{-1}(\omega)$ is a homeomorphism between Ω_1 and Ω_2, and $\widetilde{\Phi} W_\alpha(g) \widetilde{\Phi}^{-1} = W_\beta(g)$ for all $g \in G$. ▮

Lemma 12. *The Mackey action $W_\alpha(G)$, associated with a cocycle $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$ is ergodic if and only if $\mathcal{R}_\Gamma$ is ergodic.*

Proof. Suppose that $\mathcal{R}_\Gamma$ is ergodic. Then the group generated by $\Gamma(\alpha)$ and $V(G)$ acts ergodically on $X \times G$. Now the image of a dense orbit under the factor-map $\phi : X \times G \to \Omega$ is a dense $W_\alpha(G)$-orbit in Ω. Conversely, assume that $W_\alpha(G)$ is an ergodic action on Ω. If $\mathcal{R}_\Gamma$ is not ergodic then one may think that there exists a nonempty clopen $\mathcal{R}_\Gamma$-invariant $O \subsetneqq X$. Hence $O \times G$ is a $\Gamma(\alpha) \times V(G)$-invariant clopen subset of $X \times G$. This implies that $O \times G$ is $\widetilde{\mathcal{E}}_\alpha$-invariant (5) and $\phi^{-1}(\phi(O \times G)) = O \times G$. Since $\phi(O \times G)$ is open in Ω and $W_\alpha(G)$-invariant, it is comeager. Hence $O \times G$ is comeager in $X \times G$, which is a contradiction. ▮

It should be noted that the Mackey action $W_\alpha(G)$ on Ω is continuous. Lets consider a Borel structure on Ω generated by its topology (which is standard by 4, (iv)). Then $W_\alpha(G)$ is automatically a Borel action. Apply [19, theorem 2'] to conclude that for every $\omega \in \Omega$ the orbit $W(G)\omega$ is Borel in Ω and the stabilizer $H_\omega = \{g \in G : W(g)\omega = \omega\}$ is closed in G. For each $\omega \in \Omega$ lets consider a map $q_\omega : G/H_\omega \to G\omega$ given by $q_\omega(gH_\omega) = W(g)\omega$. It is easy to verify that q_ω is bijective and continuous.

The following statement was proved in [6] for the case of a Polish space. The application of the same arguments allows one to conclude that a variant for a Baire space also holds:

Proposition 13. *Let Y be a Baire T_0-space, which is standard in the Borel structure generated by its topology. Let G be a Polish group acting continuously on Y. Then the orbit Gy $(y \in Y)$ is nonmeager in itself if and only if the map $q_y : G/H_y \to Gy$ is a homeomorphism.*

From now on suppose that Γ acts ergodically on X. By virtue of 2 and 12 one may assume always that the generic Mackey action $W_\alpha(G)$ is minimal. Then using 13 we obtain the following:

Proposition 14. *Let $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$, where G is a Polish group and Γ is an ergodic countable homeomorphism group of X. Let $W_\alpha(G)$ be the associated with α generic Mackey action on a space Ω. Then exactly one of the following holds:*

(1) There exists an orbit $W_\alpha(G)\omega_0$ which is comeager in Ω, and then $W_\alpha(G)\omega_0$ is homeomorphic to a Polish space G/H_{ω_0} (essential transitivity of the generic Mackey action).

(2) Each $W_\alpha(G)$-orbit is meager in itself (proper ergodicity of the generic Mackey action).

Now turn to the case of countable G. By virtue of 6 one may assume that the generic Mackey action associated with $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$ is an action on a Polish space. Our purpose now is to represent an arbitrary countable ergodic homeomorphism group of a perfect Polish space as a generic Mackey action associated with a cocycle of a generic ergodic countable equivalence relation. Remind that in measurable dynamics only actions of amenable groups can be representable as an image of a cocycle of an ergodic automorphism ([26], [11]).

Proposition 15. *Let $W(G)$ be an ergodic continuous action of a countable (discrete) group G on a perfect Polish space S. Then there exists a cocycle $\alpha \in Z^1(\mathcal{R}, G)$, where $\mathcal{R}$ is a generic ergodic countable equivalence relation, such that $W(G)$ is isomorphic to the generic Mackey action associated with α.*

Proof. Let $X = \{0,1\}^G$ be a G-space. Let $Y = X^{\mathbb{Z}}$. There are two commuting actions on Y: (1) the ergodic $\mathbb{Z}$-action by powers of homeomorphism Θ: $(\Theta y)_i = y_{i+1}$; (2) The action of G: $(gy)_i = gy_i$. Set $Z = S \times Y$, $\Gamma = \mathbb{Z} \times G$. Lets consider the following action of Γ on Z: $(n,g)(s,y) = (W(g)\, s, g\, \Theta^n\, y)$, where $(n,g) \in \mathbb{Z} \times G$, $(s,y) \in Z$. Let $\mathcal{R}_\Gamma$ be the equivalence relation on Z generated by Γ. Define $\varphi \in Z^1(\mathcal{R}_\Gamma, G)$ by setting: $\varphi((W(g)\, s, g\, \Theta^n\, y), (s,y)) = g$. Let $\widetilde{\mathcal{E}}_\varphi \subset (Z \times G) \times (Z \times G)$ be the generic ergodic decomposition corresponding to the skew product action $\Gamma(\varphi)$. One may assume that the $\mathbb{Z}$-action on Y is minimal. It is routine to verify that for every $s \in S$, $y \in Y$, $h \in G$ $\widetilde{\mathcal{E}}_\varphi[((s,y),h)] = \bigcup_{g' \in G} \{W(g')s\} \times Y \times \{g'h\}$. Note also that the sets $\{W(g')s\} \times Y \times \{g'h\}$ for different g' are clopen pairwise disjoint subsets of $\widetilde{\mathcal{E}}_\varphi[((s,y),h)]$. Let $\widetilde{S} = (Z \times G)/\widetilde{\mathcal{E}}_\varphi$ be the topological factor-space. Define a map $J : \widetilde{S} \to S$ by: $J(\widetilde{\mathcal{E}}_\varphi[((s,y),h)]) = W(h^{-1})s$. It is not difficult to check that J is bijective. So S is identified with the factor-space $(Z \times G)/\widetilde{\mathcal{E}}_\varphi$. Show that the topology of S is the factor-topology. Indeed, let $\pi : Z \to S$ denote the factor-map. If $O \subset S$ is open then $\pi^{-1}(O)$ is representable in the form of the union of open sets: $\pi^{-1}(O) = \bigcup_{g \in G} W(g)O \times Y \times \{g\}$. Similarly if $M \subset S$ is not open then

$\pi^{-1}(M)$ is not open in Z. Let $W_\varphi(G)$ denotes the generic Mackey action associated with φ, then $W_\varphi(\widehat{h})\widetilde{\mathcal{E}}_\varphi[((s,y),h)] = \widetilde{\mathcal{E}}_\varphi[((s,y),h\widehat{h}^{-1})]$ and hence $J((W_\varphi(\widehat{h})\widetilde{\mathcal{E}}_\varphi[((s,y),h)]) = W(\widehat{h})J(\widetilde{\mathcal{E}}_\varphi[((s,y),h])$ for all $(s,y,h) \in Z \times G$, $\widehat{h} \in G$. This means that $W(G)$ and $W_\varphi(G)$ are isomorphic. ▌

4.1. Regular cocycles

Obviously, a cocycle $\alpha \in Z^1(\mathcal{R}, G)$ is ergodic if and only if the generic Mackey action $W_\alpha(G)$ is a trivial action on the one-point space. Ergodic cocycles have been studied in [8], where the uniqueness theorem was obtained:

Theorem 16 ([8]). *Any two ergodic cocycles with values in a given Polish group G are weakly equivalent.*

Our purpose here to extend the classification to a wider class of cocycles defined as follows:

Definition. A cocycle $\alpha \in Z^1(\mathcal{R}, G)$ is called *regular* if it is cohomologous to an ergodic cocycle with values in a closed subgroup $H \subset G$. The group H is called *determinative* for α.

Proposition 17. *A cocycle $\alpha \in Z^1(\mathcal{R}, G)$ is regular if and only if the generic Mackey action $W_\alpha(G)$ is essentially transitive.*

Proof. Suppose $W_\alpha(G)$ is essentially transitive. Let $\widetilde{\mathcal{E}}_{\Gamma(\alpha)}$ be a generic ergodic decomposition on $X \times G$ corresponding to $\Gamma(\alpha)$, $\Omega = (X \times G)/\widetilde{\mathcal{E}}_{\Gamma(\alpha)}$, $\phi : X \times G \to \Omega$ be the factor-map.

Without loss of generality one may assume that $W_\alpha(G)$ is a transitive action on the topological space G/H, where $H = H_{\phi(x_0,e)}$ is the stabilizer of the point $\phi(x_0,e)$, $x_0 \in X$, e is the identity of G (14). One has $\phi(\gamma x, \alpha(\gamma x, x)) = \phi(x,e)$, so

$$W(\alpha(x,\gamma x))\phi(\gamma x, e) = \phi(x,e) \tag{2}$$

Let $\mathrm{s} : G/H \to G$ be a Borel section ([13, 12.17]). Define a Borel map $f : X \to G$ by $f(x) = \mathrm{s}(\phi(x,e))$, and a cocycle $\beta \in Z^1(\mathcal{R}, G)$ by $\beta(x,\gamma x) = f(x)^{-1}\alpha(x,\gamma x)f(\gamma x)$.

It follows from 2 that for each $\gamma \in \Gamma$, $x \in X$, $\alpha(x,\gamma x)f(\gamma x) = f(x)h$, for some $h = h(x,\gamma) \in H$. Thus, β takes all its values in the subgroup H. We claim that the cocycle β as an element of $Z^1(\mathcal{R}, H)$ is ergodic.

Indeed, one can suppose that f is continuous. Then a map $\rho : X \times G \to X \times G$ given by $\rho(x,g) = (x, f(x)g)$ is a homeomorphism, which transfers any $\widetilde{\mathcal{E}}_{\Gamma(\alpha)}$-orbit to a $\widetilde{\mathcal{E}}_{\Gamma(\beta)}$-orbit and commutes with the action $V(G)$. Let $M = \phi^{-1}(x_0, e)$ be a $\widetilde{\mathcal{E}}_{\Gamma(\alpha)}$-orbit of the point (x_0, e). Then

$\rho(M) = \widetilde{\mathcal{E}}_{\Gamma(\beta)}[(x_0, f(x_0))]$. Since $f(x_0) \in H$ the orbit $\Gamma(\beta)(x_0, f(x_0)) \subset X \times H$, and the closedness of $X \times H$ implies that $\rho(M) \subset X \times H$. Since $V(H)M = M$, $\rho(M) = V(H)\rho(M)$ is dense in $X \times H$. The latter means that β is ergodic.

Conversely, suppose that α is regular, i.e. $\alpha \approx \beta$ with an ergodic $\beta \in Z^1(\mathcal{R}, K)$, where K is a closed subgroup of G. Hence the generic Mackey actions $W_\alpha(G)$ and $W_\beta(G)$ are isomorphic (β is considered here as an element of $Z^1(\mathcal{R}, G)$), and it is enough to prove the transitivity of $W_\beta(G)$.

Without loss generality one may assume that any $\Gamma(\beta)$-orbit is dense in $X \times K$. Then $\widetilde{\mathcal{E}}_{\Gamma(\beta)}$-orbit of (x, e) is $X \times K$ (we consider the skew product action $\Gamma(\beta)$ on $X \times G$). Since $V(G)(X \times K) = X \times G$ one easily sees that the action $W_\beta(G)$ is transitive on $\Omega_\beta = (X \times G)/\widetilde{\mathcal{E}}_{\Gamma(\beta)}$. ▌

Theorem 18. *Let $\mathcal{R}$ be a generic ergodic countable equivalence relation on X, G be a Polish group. Suppose that $\alpha_1, \alpha_2 \in Z^1(\mathcal{R}, G)$ are regular. Then α_1 and α_2 are weakly equivalent if and only if their determinative groups are conjugate in G.*

Proof. For a closed subgroup F of G denote by $\langle F \rangle$ the conjugacy class of F in G. Let H_i be a determinative group for α_i ($i = 1, 2$). Suppose that α_1 and α_2 are weakly equivalent. Then an isomorphism of the generic Mackey actions $W_{\alpha_1}(G)$ and $W_{\alpha_2}(G)$ implies that the homogeneous G-spaces G/H_1 and G/H_2 are isomorphic, so H_1 and H_2 are conjugate.

Suppose now that $\langle H_1 \rangle = \langle H_2 \rangle$. Then there exist cocycles $\beta_i \approx \alpha_i$ ($i = 1, 2$) and $g \in G$ such that β_1 is an ergodic cocycle with values in H_1, β_2 is an ergodic cocycle with values in gH_1g^{-1}. Thus the cocycle $\beta_3 = g^{-1}\beta_2 g$ ($\in Z^1(\mathcal{R}, H_1)$) is ergodic. By virtue of 16, β_1 is weakly equivalent to β_3, which yields the weak equivalence of α_1 and α_2. ▌

The question of existence of a regular cocycles with a given determinative subgroup is reduced to the existence theorem for ergodic cocycles, which has been obtained in [8]: given any Polish group G there exists an ergodic cocycle of a generic countable equivalence relation with values in G.

We end this subsection with showing that any cocycle α taking values in a compact (Polish) group K is regular. We need the following:

Lemma 19. *Let K be a topological Hausdorff group and let Y be a homogeneous topological continuous K-space with the T_0-axiom. Suppose that the stabilizer $H = H_{y_0}$ is closed for some $y_0 \in Y$ and the factor- space K/H is compact. Then the space Y is nonmeager in itself.*

Proof. One has $Y = Ky_0$. Assume the contrary: $Y = \bigcup_{n=1}^{\infty} S_n$, with S_n is nowhere dense in Y for any n. Denote by q a continuous K-map from K/H

onto Ky_0: $q(kH) = ky_0$. Let $F_n = \overline{S_n}$. Then $K/H = \bigcup_{n=1}^{\infty} q^{-1}(F_n)$. As K/H is a Baire space there exists $N \in \mathbb{N}$ with int $q^{-1}(F_N)$ being nonempty. Let $C_N = q^{-1}(F_N)$. Then $\bigcup_{k\in K} k\ \mathrm{int}C_N = K/H$ and there exists a finite subcover $\bigcup_{i=1}^{m} k_i \mathrm{int}C_N = K/H$ $(k_i \in K, k_1 = e)$. Hence $\bigcup_{i=1}^{m} k_i F_N = Ky_0$. For every $1 \le i \le m$ $k_i F_N$ is closed and its interior $\mathrm{int}k_i F_N = \varnothing$. Let $U_0 = Y$, $U_i = U_{i-1} \backslash k_i F_N$ for $1 \le i \le m$. Let i be a minimal with $U_i = \varnothing$. Then $k_i F_N \supset U_{i-1}$ that impossible. ▌

Now, let $W_\alpha(K)$ be the generic Mackey action on the space Ω. For any $\omega \in \Omega$ the orbit $W_\alpha(K)\omega$ is of second category in itself and, hence, it is comeager. The latter exactly means the essential transitivity of $W_\alpha(K)$.

Proposition 20. *Let $\alpha \in Z^1(\mathcal{R}, K)$, where K is a compact Polish group. Then α is regular.*

Proof. Let $W_\alpha(K)$ be an associated with α generic Mackey action on a Baire space Ω. We know that each stabilizer H_ω is closed. By virtue of 19 the orbit $W_\alpha(K)\omega$ is nonmeager in itself, and, hence, it is comeager in Ω (14). Thus one may think that $\Omega = W_\alpha(K)\omega = K/H_\omega$ and $W_\alpha(K)$ is transitive. ▌

4.2. Transient cocycles

Although the main theorem of this subsection is proved for cocycles taking values in countable groups, the basic properties of transient cocycles we state for Polish group valued cocycles.

Definition. A cocycle $\alpha \in Z^1(\mathcal{R}, G)$ is called *transient* if there exist a nonmeager Borel set $B \subset X$ and a neighborhood W of the identity in G such that $\alpha(x, y) \notin W$ for all $(x, y) \in \mathcal{R} \cap (B \times B)$, $x \neq y$.

The reader is referred to [22], [2] for the original definition of transient cocycle in measurable dynamics.

Proposition 21. *A cocycle $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$ is transient if and only if the skew product action $\Gamma(\alpha)$ is of discrete type.*

Proof. Firstly note that for any $\alpha \in Z^1(\mathcal{R}_\Gamma, G)$ there exists a comeager G_δ-subset $X' \subset X$ such that every map $\alpha_\gamma = \alpha(\gamma\cdot, \cdot) : X' \to G$ $(\gamma \in \Gamma)$ is continuous (see 1). Suppose that α is transient. One may assume that the set $B \subset X$ from the definition is clopen. Lets show that every $\Gamma(\alpha)$-orbit is discrete. Suppose the contrary: $\gamma_n(\alpha)(x, g_1) \to (y, g_2)$ $(n \to \infty)$, i.e. $\gamma_n x \to y$ and $\alpha(\gamma_n x, x) \to g$ $(g = g_2 {g_1}^{-1})$. It follows from the ergodicity of $\mathcal{R}$ the existence of $\gamma_0 \in \Gamma$ with $\gamma_0 y \in B$. Let $\alpha(\gamma_0 y, y) = g_0$. Choose a neighborhood V of the identity in G with $V = V^{-1}$ and $g_0 V \cdot V \cdot V {g_0}^{-1} \subset W$, where W is a neighborhood of the identity from the definition of the transient cocycle α. There exist a neighborhood O of y and $N \in \mathbb{N}$ such that $\gamma_n x \in O$ and $\alpha(\gamma_n x, \gamma_m x) \in V$ is valid for all $m, n > N$. Let $O' \subset O$ be such a neighborhood of y that $\gamma_0 O' \subset B$ and $\alpha(\gamma_0 y', y') \in g_0 V$ for all $y' \in O'$. Then for any $\gamma_n x$, $\gamma_m x \in O'$ one has: $\alpha(\gamma_0\gamma_m x, \gamma_0\gamma_n x) = \alpha(\gamma_0\gamma_m x, \gamma_m x)\alpha(\gamma_m x, \gamma_n x)\alpha(\gamma_n x, \gamma_0\gamma_n x) \subset g_0 V \cdot V \cdot V^{-1} {g_0}^{-1} \subset W$, which is a contradiction.

Conversely, suppose that $\Gamma(\alpha)$ is an action of discrete type. Then the set of points of $X \times G$ with nondiscrete $\Gamma(\alpha)$-orbits is $V(G)$-invariant and meager in $X \times G$. Hence one may discard it. Assume the contrary: for each Borel nonmeager $B \subset X$ and each neighborhood of the identity $W = W(e)$ in G there exists $(x, y) \in \mathcal{R} \cap (B \times B)$, $x \neq y$ with $\alpha(x, y) \in W$. Let $\mathcal{E} = \mathcal{E}_{\Gamma(\alpha)}$ be a skew product equivalence relation and $\widetilde{\mathcal{E}}$ be a generic ergodic decomposition corresponding to $\mathcal{E}$. It follows from the discreteness of all $\mathcal{E}$-orbits that $\widetilde{\mathcal{E}} = \mathcal{E}$. Since $(X \times G)/\widetilde{\mathcal{E}}$ is a standard Borel space (proposition 4(iv)), there exists a Borel $\mathcal{E}$-transversal $T \subset X \times G$ ([13]). Because $\mathcal{E}$ is countable, T is nonmeager in $X \times G$. Let Z' be a $\Gamma(\alpha)$-invariant dense G_δ-subset in $X \times G$ with $T' = T \cap Z'$ being open in Z'. Then $T' = O \cap Z'$, where O is open in $X \times G$. There exists a set of the form $K \times W$, where K is open in X, W is open in G, such that $K \times W \subset O$ and $(K \times W) \cap T'$ is nonmeager in $X \times G$. Find $g_0 \in G$ such that $B = (K \times W) \cap T' \cap (X \times \{g_0\})$ is nonmeager in $X \times \{g_0\}$ (i.e. in X) (the existence of such a point is provided by [20, th. 15.4]). By our assumption there exist $x, y \in B (\subset X)$ with $x \neq y$, $x \mathcal{R}_\Gamma y$ and $\alpha(y, x) \in W {g_0}^{-1}$. Then $(x, g_0)\ \mathcal{E}\ (y, \alpha(y, x) g_0) \in (y, W) \subset K \times W \subset O$. On the other hand, $(y, \alpha(y, x) g_0) \in Z'$ because Z' is $\mathcal{E}$-invariant. So $(y, \alpha(y, x) g_0) \in T'$, which contradicts that T is a $\mathcal{E}$-transversal. ▮

Note that the generic Mackey action associated with a transient cocycle $\alpha \in Z^1(\mathcal{R}, G)$ is free (modulo meager sets). Indeed, it is shown above that there exists a Borel $\mathcal{E}_{\Gamma(\alpha)}$-transversal $T \subset X \times G$. Let $\phi : X \times G \to T = (X \times G)/\mathcal{E}$ be a factor-map. One may assume that the generic Mackey action $W_\alpha(G)$ is an action on T, where T is considered with the factor-topology. Suppose the contrary: there exist $g \in G$ $(g \neq e)$ and $t_0 \in T$: $W(g)t_0 = t_0$. Since $\phi^{-1}(t_0) = \Gamma(\alpha)t_0$, $V(g)\Gamma(\alpha)t_0 = \Gamma(\alpha)t_0$, in particular, $V(g)t_0 = \gamma(\alpha)t_0$ for some $\gamma \in \Gamma$. The latter is true only if $g = e$.

Example. Suppose that in the above example 1 a group $G = \mathbb{R}$ and $f(x) > 0$ for all $x \in X$. Then the cocycle φ_f is transient.

THEOREM 22. *Let Γ be an ergodic countable homeomorphism group of X and G be a countable group. Two transient cocycles α_1, $\alpha_2 \in Z^1(\mathcal{R}_\Gamma, G)$ are weakly equivalent if and only if the generic Mackey actions $W_{\alpha_1}(G)$, $W_{\alpha_2}(G)$ are isomorphic.*

Proof. Suppose that the generic Mackey actions $W_{\alpha_1}(G)$ and $W_{\alpha_2}(G)$ are isomorphic. We assume that Γ is a minimal group on X. Let $\mathcal{E}_i$ be an equivalence relation on $X\times G$, generated by the skew product action $\Gamma(\alpha_i)$ $(i=1,2)$. $\mathcal{E}_i$ is of discrete type by 21. Since each Γ-orbit is infinite in X, $\mathcal{E}_i$ is of infinite discrete type. Then, in view of G is countable, one may assume that $X\times G$ is homeomorphic to $T_i\times\mathbb{N}$, where $T_i\subset X\times G$ is a (clopen) transversal for $\mathcal{E}_i$, and $z\mathcal{E}_i\widetilde{z} \iff z=(t,n_1)$, $\widetilde{z}=(t,n_2)$, $t\in T_i$, $W_{\alpha_i}(G)$ is an action on T_i. Let $\Phi: T_1\to T_2$ be a homeomorphism conjugating the generic Mackey actions $W_{\alpha_1}(G)$ and $W_{\alpha_2}(G)$, i.e. $\Phi W_{\alpha_2}(g)\Phi^{-1}=W_{\alpha_1}(g)$ for all $g\in G$. The standard exhausting argument allows one to assert that there exist homeomorphisms $h_n: T_1\times\{1\}\to T_1\times\{n\}$, $\widetilde{h}_n: T_2\times\{1\}\to T_2\times\{n\}$ with $h_n\in \mathrm{Int}\mathcal{E}_1$, $\widetilde{h}_n\in\mathrm{Int}\mathcal{E}_2$ $(n\in\mathbb{N})$. Let $V(G)$ be an action of G on $X\times G$ given by: $V(g)(x,h)=(x,hg^{-1})$. One easily checks that if $z\in T_1\times\{n\}$ and $V(g)z\in T_1\times\{m\}$, then $V(g)z=h_mW_{\alpha_1}(g)h_n^{-1}z$. Similarly if $z\in T_2\times\{\widetilde{n}\}$ and $V(g)z\in T_2\times\{\widetilde{m}\}$, $V(g)z=\widetilde{h}_mW_{\alpha_2}(g)\widetilde{h}_n^{-1}z$. Extend Φ to a homeomorphism of $X\times G$ by setting: $\Phi z=\widetilde{h}_n\Phi h_n^{-1}z$, if $z\in T_1\times\{n\}$. Then Φ realizes an orbit isomorphism of the actions $\Gamma(\alpha_1)$ and $\Gamma(\alpha_2)$. Besides, for $g\in G$, $z\in X\times G$ one has: $\Phi V(g)z=\Phi h_mW_{\alpha_1}(g)h_n^{-1}z=\widetilde{h}_mW_{\alpha_2}(g)\widetilde{h}_n^{-1}\Phi z=\tau V(g)\Phi z$ for some $m,n\in\mathbb{N}$, and $\tau=\tau(g)\in\mathrm{Int}\mathcal{E}_2$. Let $\mathcal{R}_{i,G}$ be an equivalence relation generated by $\Gamma(\alpha_i)\times V(G)$ on $X\times G$ $(i=1,2)$. Lets consider a cocycle $\varphi_i\in Z^1(\mathcal{R}_{i,G},G)$ given by:

$$\varphi_i((x',h'),(x,h))={h'}^{-1}\alpha_i(x',x)h$$

where $(x',h')\ \mathcal{R}_{i,G}\ (x,h)$. Then

$$\varphi_1(V(g)(x,h),(x,h))=gh^{-1}h=g \quad \text{for any } g\in G,$$

$$\varphi_1(\gamma(\alpha_1)(x,h),(x,h))=\varphi_1((\gamma x,\alpha_1(\gamma x,x)h),(x,h))=e \quad \text{for any } \gamma\in\Gamma$$

On the other hand, if $(x',h')=\gamma(\alpha_1)V(g)(x,h)$, then

$$\varphi_2\circ(\Phi\times\Phi)((x',h'),(x,h))=\varphi_2(\Phi(\gamma(\alpha_1)V(g)(x,h)),\Phi(x,h))=$$

$$=\varphi_2(\widetilde{\gamma}(\alpha_2)\tau(g)V(g)\Phi(x,h),(x,h))=g$$

where $\widetilde{\gamma}\in\Gamma$, $\tau(g)\in\mathrm{Int}\mathcal{E}_2$. This implies $\varphi_2\circ(\Phi\times\Phi)=\varphi_1$.

Denote by $\pi_1: X\times G\to X$ and $\pi_2: X\times G\to G$ the projections. Set $\Phi_1=\pi_1\circ\Phi$, $\omega(x)=\Phi_1(x,e)$. Then $\omega: X\to X$ is continuous and $\omega^{-1}(x)$ is no more than countable for any $x\in X$. Let U be a proper clopen subset of X with $Y=\omega^{-1}(U)$ being a proper subset of X. Lets consider a Borel equivalence relation on $Y: \mathcal{R}_\omega=\{(y_1,y_2)\in Y\times Y:\omega(y_1)=\omega(y_2)\}$. One sees $\mathcal{R}_\omega \subset \mathcal{R}_\Gamma|_{Y\times Y}$, so $\mathcal{R}_\omega$ is a generic equivalence relation. Besides $\mathcal{R}_\omega$ is smooth, and hence it is of discrete type. Thus one may assume that there exists a clopen $\mathcal{R}_\omega$-transversal B with $\omega : B \to U$ to be a homeomorphism.

Use [24, lemma 1.6] to find homeomorphisms δ, $\delta' \in \mathrm{Int}\mathcal{R}_\Gamma$ with $\delta = \delta^{-1}$, $\delta' = {\delta'}^{-1}$, δ interchanges B and $X \setminus B$, δ' interchanges $\omega(B)$ and $X \setminus \omega(B)$. Define a homeomorphism $\Theta : X \to X$ by setting:

$$\Theta x = \begin{cases} \omega x, & \text{if } x \in B \\ (\delta' \circ \omega \circ \delta)x, & \text{if } x \in X \setminus B \end{cases}$$

It is easy to see that $\Theta \in \mathrm{Aut}\mathcal{R}_\Gamma$. Note also that $(\Theta \times \mathrm{id})\Phi^{-1} \in \mathrm{Int}\,\mathcal{R}_{2,G}$. Therefore $\Phi = (\Theta \times \mathrm{id})\tau$, where $\tau \in \mathrm{Int}\,\mathcal{R}_{2,G}$. Now, if $x_1\, \mathcal{R}_\Gamma\, x_2$, then $(x_1, e)\, \mathcal{R}_{1,G}\, (x_2, e)$ and

$$\varphi_2(\Phi(x_1, e), \Phi(x_2, e)) = \varphi_1((x_1, e)(x_2, e)) = \alpha_1(x_1, x_2)$$

On the other hand,

$$\varphi_2(\Phi(x_1, e), \Phi(x_2, e)) = \varphi_2((\Theta \times \mathrm{id})\tau(x_1, e), (\Theta \times \mathrm{id})\tau(x_2, e)) =$$

$$= \varphi_2((\Theta \times \mathrm{id})(\pi_2 \circ \tau)(x_1, e)), (\Theta \times \mathrm{id})(\pi_2 \circ \tau)(x_2, e) =$$

$$= ((\pi_2 \circ \tau)(x_1, e))^{-1}\alpha_2(\Theta x_1, \Theta x_2)(\pi_2\circ\tau)(x_2, e) = f(x_1)^{-1}\alpha_2(\Theta x_1, \Theta x_2)f(x_2)$$

where $f(x) = (\pi_2 \circ \tau)(x, e) : X \to G$ is a Borel function. Thus $f(x_1)^{-1}\cdot \alpha_1(x_1, x_2)f(x_2) = \alpha_2(\Theta x_1, \Theta x_2)$ for all $(x_1, x_2) \in \mathcal{R}_\Gamma$. ▌

Remark. Given a free ergodic action $W(G)$ of a countable group G on a perfect Polish space X define a cocycle $\varrho \in Z^1(\mathcal{R}_{W(G)}, G)$ by: $\varrho(gx, x) = g$ (the *return* cocycle). Then the generic Mackey action $W_\varrho(G)$ is isomorphic to $W(G)$, so any transient cocycle with a given generic Mackey action $W(G)$ is weakly equivalent to the return cocycle ϱ.

REFERENCES

1. W. Ambrose, S. Kakutani, *Structure and continuity of measurable flows*, Duke Math. J., **9** (1942), 25 – 42.

2. S. Bezuglyi, *Groups of automorphisms of measure space and weak equivalence of cocycles*, London Math. Soc. Lecture Note Series, **277** (2000), Cambridge Univ. Press, Cambridge. 59 – 86.

3. S. Bezuglyi and V. Golodets, *Weak equivalence and the structure of cocycles of an ergodic automorphism*, Publ. RIMS Kyoto Univ., **27** (1991), 577 – 625.

4. S. Bezuglyi and V. Golodets, *Weak equivalence of cocycles of an ergodic automorphism and the Connes-Krieger theorem for actions of discrete non-principal amenable groupoids*, Preprint, Heidelberg University, 1991.
5. H.A. Dye, *On groups of measure-preserving transformations I*, Amer. J. Math., **81** (1959), 119 – 159.
6. E. Effros, *Transformation groups and $C^\star$-algebras.*, Ann. of Math., **81** (1965), 36–55.
7. Feldman J., Moore C.C., *Ergodic equivalence relations, cohomology, and von Neumann algebras I*, Trans. Amer. Math. Soc., **234** (1977), 289 – 359.
8. V. Golodets, V. Kulagin, and S. Sinel'shchikov, *Orbit properties of pseudo-homeomorphism groups and their cocycles*, London Math. Soc. Lecture Note Series, **277** (2000), Cambridge Univ. Press, Cambridge, 211 – 229.
9. V. Golodets and S. Sinel'shchikov, *Existence and uniqueness of cocycles of an ergodic automorphism with dense ranges in amenable groupa*, Preprint, Inst. Low Temp. Physics and Engin., Kharkov, 1983.
10. V. Golodets and S. Sinel'shchikov, *Classification and structure of cocycles of amenable ergodic equivalence relations*, J. Funct. Anal., **121** (1994), 455 – 485.
11. V. Golodets and S. Sinel'shchikov, *Amenable ergodic actions of groups and images of cocycles*, Soviet Math. Dokl., **41** (1990), 523 – 526.
12. L. Harrington, A. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc., **3** No 4 (1990), 903 – 928.
13. A. Kechris *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, **156**, Springer-Verlag, New-York, (1995).
14. A. Kechris, *Descriptive Dynamics., London Math. Soc. Lecture Note Series*, **277** (2000). Cambridge Univ. Press, Cambridge, 231 – 258.
15. W. Krieger, *On ergodic flows and isomorphism of factors*, Math. Ann., **223** (1976), 19 – 70.
16. K. Kuratowski *Topology*, Academic Press, New York, (1966).
17. G. Mackey, *Borel structure in groups and their duals*, Trans. Amer. Math. Soc., **85** (1957), 134 – 165.
18. G. Mackey, *Ergodic theory and virtual groups*, Math. Ann., **166** (1966), 187 – 207.
19. D.E. Miller, *On the measurability of orbits in Borel actions*, Proc. Amer. Math. Soc., **63** No 1 (1977), 165 – 170.
20. J. Oxtoby *Measure and Category*, Springer, (1980).
21. A. Ramsay, *The Mackey-Glimm dichotomy for foliations and other Polish groupoids*, J. Funct. Analysis., **94** (1990), 358 – 374.
22. K. Schmidt, *Cocycles of Ergodic Transformation Groups*, Lecture Notes in Math. 1, MacMillan of India, (1977).
23. S.M. Srivastava, *Selection theorems for G_δ-valued multifunctions*, Trans. Amer. Math. Soc., **254** (1979), 283 – 293.
24. D. Sullivan, B. Weiss, and J. D. M. Wright, *Generic dynamics and monotone complete C^*-algebras*, Trans. Amer. Math. Soc., **295** (1986), 795 – 809.
25. B. Weiss, *A survey of generic dynamics.*, London Math. Soc. Lecture Note Series, **277** (2000), Cambridge Univ. Press, Cambridge, 273 – 291.
26. R.J. Zimmer, *Amenable ergodic group actions and an application to Poisson boundaries of random walks*, J. Funct. Anal., **27** (1978), 350 – 372.
27. R.J. Zimmer *Ergodic Theory and Semisimple Groups*, Boston, Birkhauser, (1984).