QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **4**, 115–124 (2003)
ARTICLE NO. 56

Fundamental Groups of some Ergodic Equivalence Relations of Type II_∞

Sergey L. Gefter*

Department of Mechanics and Mathematics, Kharkov National University, Freedom Sq. 4, 61077 Kharkov, Ukraine

B.Verkin Istitute for Low Temperature Physics and Engineering, Lenin Ave 47, 61103 Kharkov, Ukraine
E-mail: gefter@univer.kharkov.ua

Automorphisms of equivalence relations generated by translations of dense arithmetic subgroups and groups of **Q**-rational points on semisimple non-compact Lie groups (real and p-adic) are studied. These automorphisms are proved to preserve a Haar measure.

Key Words: ergodic equivalence relations, fundamental groups, dense subgroups.

1. INTRODUCTION

Recently a plenty of remarkable results have been obtained in the orbit theory of non-amenable groups. A. Furman [9] essentially improved Zimmer's rigidity theorem for ergodic actions of lattices in higher rank semisimple Lie groups. Using G. Levitt's invariant cost and the notion of treeable equivalence relation studied by S. Adams, D. Gaboriau proved in [10], in particular, that measure preserving free ergodic actions of free groups of different ranks on probability spaces cannot be orbit equivalent. The reader's attention is attracted to the fact that the results mentioned above refer to ergodic actions with finite invariant measure. The orbit theory of non-amenable actions with infinite invariant measure (i.e. type II_∞ actions) is still not a subject of such an extensive research. This paper deals with the specific form of ergodic group actions as follows.

* The author is grateful to Professor A. Boutet de Monvel for the hospitality at University Paris VII and to the Ministry of Foreign Affairs of France for the financial support granted for the period of work on this paper.

Let G be a continuous locally compact second countable group with a left Haar measure μ and Γ a countable dense subgroup of G. Then (G,μ) is a standard measure space and Γ acts on (G,μ) by left translations: $g \overset{\gamma}{\longmapsto} \gamma g$, $\gamma \in \Gamma$, $g \in G$. The action of Γ on (G,μ) is ergodic because Γ is dense in G. Such actions forms a simple but important and interesting class of ergodic actions (see, for example [16, 22, 23]). Denote the Γ-orbital equivalence relation by R_Γ. If G is amenable as a discrete group (for example, if G is Abelian or solvable), the equivalence relation R_Γ is amenable too (see [5]). Therefore this case is particularly interesting from the orbital point of view. The opposite case is partially described by the following important Zimmer's theorem (see [23]).

Theorem 1 (R. Zimmer, 1987). *If G is a connected non-solvable Lie group, then R_Γ is non-amenable.*

Pairs (G,Γ), where G is a connected Lie group and $\Gamma \subset G$ a dense finitely generated subgroup, and such that the action of Γ on G (by translations) is stably orbit equivalent to an action of some semisimple Lie group, are studied in Zimmer's paper [24]

A lot of interesting questions on the equivalence relation R_Γ arise in the non-amenable case. In this work we consider only one of those, specifically that of computing the fundamental group of R_Γ (see Definition 2) in some particular cases when Γ acts by translations on a non-compact group G. Section 1 contains a preliminary information on automorphism groups and fundamental groups of equivalence relations. In Section 2 we consider the case when Γ is a projection of an irreducible lattice in a direct product of some simple Lie groups (real or p-adic); we demonstrate that all the automorphisms of R_Γ are measure preserving, i.e. $F(R_\Gamma) = \{1\}$ (see Theorem 5). To prove this fact, we use both Zimmer's rigidity theorem for ergodic actions of semisimple groups [21] and the notion of fundamental group for ergodic actions of continuous locally compact groups (the latter notion was introduced in the work [13]; see also [12]). Section 3 is dedicated to the actions by translations of groups of $\mathbf{Q}$-rational points for semisimple algebraic groups. In this case, to prove the triviality of the fundamental group (Theorem 10) we use a method which was introduced in A. Connes' work [4] and was developed in [14], [12] and [11] (see the proof of Proposition 12). This method was applied in [12] to computation the fundamental groups for actions on non-compact groups of the form $K \times \mathbf{Z}$, where K is a connected compact group with a finite center.

2. PRELIMINARIES

Let Γ be a countable discrete group and (X, μ) a standard free ergodic Γ-space with infinite σ-finite invariant measure. Denote by R_Γ the Γ-orbital equivalence relation. Let $\theta : X \to X$ be a non-singular automorphism of the measure space (X, μ). θ is called an automorphism of the equivalence relation R_Γ if $\theta(R_\Gamma(x)) = R_\Gamma(\theta(x))$ for a.e. $x \in X$, where $R_\Gamma(x) = \{\gamma x \mid \gamma \in \Gamma\}$ is the Γ-orbit of x. Denote the automorphism group of R_Γ by $\mathrm{Aut}(R_\Gamma)$. It is well known that if $\theta \in \mathrm{Aut}(R_\Gamma)$ then there exists $\lambda = \mathrm{mod}\ \theta > 0$, such that $\mu \circ \theta = \lambda\mu$. We refer the reader to [8], [7] and [18] for a detailed exposition of the theory of ergodic equivalence relations and their automorphisms.

Definition 2. The fundamental group of the equivalence relation R_Γ is the following subgroup in $\mathbf{R}^*_+$:

$$F(R_\Gamma) = \{\mathrm{mod}\ \theta \mid \theta \in \mathrm{Aut}(R_\Gamma)\}.$$

It is easy to see that the fundamental group is invariant of stable orbit equivalence (see [7]). If the equivalence relation R_Γ is amenable, then $F(R_\Gamma) = \mathbf{R}^*_+$ (see [5] and [8]).

Remark 3. If (X, μ) is a free ergodic Γ-space with *finite* invariant measure and R_Γ is a Γ-orbital equivalence relation, then

$$F(R_\Gamma) \overset{def}{=} F(R_\Gamma \times I_\infty),$$

where I_∞ is the transitive equivalence relation on $\mathbf{Z}$. The properties of fundamental groups of type II_1 equivalence relations are exposed in [12] and [9, §2].

Recall the definition of the fundamental group for the equivalence relation associated to an ergodic action of a continuous locally compact group [13, 12].

Let T be a continuous locally compact second countable unimodular group and (Y, ν) a standard free properly ergodic T-space with finite invariant measure. Consider the ergodic equivalence relation R_T. Let $S \subset Y$ be a complete countable type II_∞ section for the action of T. Then $R_T \cong \widetilde{R} \times I$, where $\widetilde{R}$ is the equivalence relation on S with countable orbits (discrete reduction of type II_∞) and I the transitive equivalence relation generated by the translation of the circle (see Th. 6.4 in [7]). Let θ be an automorphism of R_T. According to Theorem 2.4 [15], there exist $\widetilde{\theta} \in \mathrm{Aut}(\widetilde{R})$ and an inner automorphism w of R_T, such that $\theta = (\widetilde{\theta} \times id)w$. We set $\mathrm{mod}\ \theta \overset{def}{=} \mathrm{mod}\ \widetilde{\theta}$ (see Definition 2.9 and Remark 2.10 in [15]). Clearly, mod θ does not depend on the choice of a discrete reduction $\widetilde{R}$ and on the representation of θ as a product $(\widetilde{\theta} \times id)w$.

Definition 4. The group $F(R_T) = \{\bmod \theta \mid \theta \in \mathrm{Aut}(R_T)\}$ is called the fundamental group of the equivalence relation R_T.

We will use the standard notation and terminology concerning the theory of algebraic groups(see Ch.I in [18] and Ch.3 in [21]).

3. ACTIONS OF IRREDUCIBLE LATTICES

For each prime $p \in \mathbf{N}$ denote by $\mathbf{Q}_p$ the field of p-adic numbers, and set $\mathbf{Q}_\infty = \mathbf{R}$. Let $V = \{primes\ in\ \mathbf{N}\} \cup \{\infty\}$, and $p_1, p_2, \ldots, p_m \in V$. Suppose that for each p_i, H_{p_i} is a connected almost $\mathbf{Q}$-simple linear algebraic $\mathbf{Q}$-group such that the group $H_{p_i}(\mathbf{Q}_{p_i})$ is not compact. Let $B = \prod_{i=1}^{m} H_{p_i}(\mathbf{Q}_{p_i})$, so that B is a locally compact non-compact group. Suppose that Λ is an irreducible lattice in B (see 2.2.4 and p.188 in [21]) and fix some non-trivial subset $I_0 \subset \{1, 2, \ldots, m\}$. Consider $G = \prod_{i \in I_0} H_{p_i}(\mathbf{Q}_{p_i})$ and $\Gamma = \pi_{I_0}(\Lambda)$, where $\pi_{I_0} : B \to G$ is a projection onto G. Then G is a locally compact non-compact group and Γ is a dense subgroup in G. Consider the action of Γ on G by left translations and denote by R_Γ the corresponding equivalence relation.

Theorem 5. *Suppose* $\sum_{i \notin I_0} \mathbf{Q}_{p_i}\text{–rank}(H_{p_i}) \geq 2$ *(see Section 5 of Ch. VII in [17]). Then all the automorphisms of* R_Γ *preserve a Haar measure on* G, *i.e.* $F(R_\Gamma) = \{1\}$.

We need the following

Lemma 6. *Let* G *and* T *be locally compact second countable groups,* Λ *be a closed subgroup of* $G \times T$ *and* $\Gamma = \pi_G(\Lambda)$, *where* $\pi_G : G \times T \to G$ *is a projection. Then the action of* Γ *on* G *by left translations is stably orbit equivalent to the* T*-action on the homogeneous space* $X = \Lambda \setminus (G \times T)$.

Proof. Let R_Γ and R_T be Γ-orbital and T-orbital equivalence relations, respectively, $x_1, x_2 \in X$, $x_1 = \Lambda(g_1, t_1)$ and $x_2 = \Lambda(g_2, t_2)$. We first prove that $x_1 \underset{R_T}{\sim} x_2 \Leftrightarrow g_1 \underset{R_\Gamma}{\sim} g_2$. Indeed, $x_1 \underset{R_T}{\sim} x_2 \Leftrightarrow \Lambda(g_2, t_2) = \Lambda(g_1, t_1 t^{-1})$ for some $t \in T$, i.e., $(g_2, t_2) = (\lambda_1 g_1, \lambda_2 t_1 t^{-1})$, where $(\lambda_1, \lambda_2) \in \Lambda$. It follows that $g_2 = \lambda_1 g_1$, i.e., $g_1 \underset{R_\Gamma}{\sim} g_2$. Conversely, suppose that $g_1 \underset{R_\Gamma}{\sim} g_2$. Then there exists $\gamma = \pi_G(\gamma, t)$ such that $g_2 = \gamma g_1$ and $(\gamma, t) \in \Lambda$. Hence $\Lambda(g_2, t_2) = \Lambda(\gamma g_1, t_2) = \Lambda(g_1, t^{-1} t_2) = \Lambda\left(g_1, t_1 \left(t_2{}^{-1} t t_1\right)^{-1}\right)$, i.e.,

$x_1 \underset{R_T}{\sim} x_2$. We now consider a Borel section $s : X \to G \times T$ of the natural projection and define a Borel map $F : X \times \Lambda \to G \times T$ by: $F(x, \lambda) = \lambda s(x)$. Then F is a nonsingular Borel isomorphism. We claim to show that F is also an isomorphism of the equivalence relations $R_T \times I_\Lambda$ and $R_\Gamma \times I_T$, where I_Λ and I_T are transitive equivalence relations on Λ and T, respectively. Indeed, let $s(x_i) = (g'_i, t'_i), \quad i = 1, 2$. Then $\Lambda(g_i, t_i) = \Lambda(g'_i, t'_i), \quad i = 1, 2$. Now we have $F(x_1, \lambda_1) \underset{R_\Gamma \times I_T}{\sim} F(x_2, \lambda_2) \Leftrightarrow \lambda_1 s(x_1) \underset{R_\Gamma \times I_T}{\sim} \lambda_2 s(x_2) \Leftrightarrow \lambda_1(g'_1, t'_1) \underset{R_\Gamma \times I_T}{\sim} \lambda_2(g'_2, t'_2) \Leftrightarrow g'_1 \underset{R_\Gamma}{\sim} g'_2 \Leftrightarrow g_1 \underset{R_\Gamma}{\sim} g_2 \Leftrightarrow x_1 \underset{R_T}{\sim} x_2 \Leftrightarrow (x_1, \lambda_1) \underset{R_T \times I_\Lambda}{\sim} (x_2, \lambda_2)$. ▮

Proof (of the Theorem 5). We set $T = \prod_{i \notin I_0} H_{p_i}(\mathbf{Q}_{p_i})$. Then $B = G \times T$ and $\operatorname{rank}(T) = \sum_{i \notin I_0} \mathbf{Q}_{p_i} - \operatorname{rank}(H_{p_i}) \geq 2$. Since Λ is an irreducible lattice, the action of T on $X = \Lambda \backslash B$ is irreducible too (see 2.2.11, 2.2.12 and p.118 in [21]). Next, consider the T-orbital equivalence relation R_T. In view of Lemma 6, $R_\Gamma \times I$ is isomorphic to R_T, where I is the transitive equivalence relation generated by translations of the circle. Hence $F(R_\Gamma) = F(R_T)$. Now our statement follows from general Zimmer's rigidity theorem [21, Th.10.1.8] and from the arguments used to prove Th.1 in [13] (see also Th.B.2 in [12]). ▮

Example 7. We denote by $\mathbf{Z}[\sqrt{2}]$ the ring of integers of the field $\mathbf{Q}(\sqrt{2})$. For $a + b\sqrt{2} \in \mathbf{Z}[\sqrt{2}]$, we set $\sigma(a + b\sqrt{2}) = a - b\sqrt{2}$. Let $\Gamma = \mathrm{SL}_n(\mathbf{Z}[\sqrt{2}])$, so that for $\gamma \in \Gamma$, $\sigma(\gamma)$ has an obvious meaning. Consider now $\Lambda = \{(\gamma, \sigma(\gamma)) \mid \gamma \in \Gamma\}$. Then Λ is an irreducible lattice in $\mathrm{SL}_n(\mathbf{R}) \times \mathrm{SL}_n(\mathbf{R})$ and $\Gamma \cong \Lambda$ (see Example (v), p.296 in [17]). Consider the equivalence relation R_Γ generated by left translations of Γ on $G = \mathrm{SL}_n(\mathbf{R})$. If $n \geq 3$, then $\mathbf{R}-\operatorname{rank}(G) = n - 1 \geq 2$ [21, 5.1]. Hence all automorphisms of R_Γ preserve a Haar measure on G.

Example 8. For each finite subset $S = \{p_1, p_2, \ldots, p_m\}$ of primes, let $\mathbf{Z}[S^{-1}]$ be the ring of rationals whose denominators (in reduced form) have prime factors in S, and let $\Gamma = \mathrm{SL}_n(\mathbf{Z}[S^{-1}])$, $n \geq 3$. Identify Γ with its image under the diagonal embedding into $B = \mathrm{SL}_n(\mathbf{R}) \times \mathrm{SL}_n(\mathbf{Q}_{p_1}) \times \ldots \times \mathrm{SL}_n(\mathbf{Q}_{p_m})$. Then Γ is an irreducible lattice in B (see Example (iii), p.295 in [17]). Consider the equivalence relation R_Γ generated by translations of Γ on $G = \mathrm{SL}_n(\mathbf{R})$. Then $F(R_\Gamma) = \{1\}$.

Example 9. With S, Γ and B as above, we let now $G = \mathrm{SL}_n(\mathbf{Q}_{p_1}) \times \ldots \times \mathrm{SL}_n(\mathbf{Q}_{p_m})$. Identify Γ with its image under the diagonal embedding into B. Then Γ is a dense subgroup of G. It follows from Theorem 5 that all the automorphisms of equivalence relation R_Γ preserve a Haar measure on G.

4. ACTIONS OF THE GROUPS Q-RATIONAL POINTS

Actions by translations of the group $\mathbf{Q}$-rational points of semisimple linear algebraic groups are considered in this section. The proof of Theorem 10 is based on the methods of the works [4], [14] and [12].

Let H be a connected semisimple linear algebraic $\mathbf{Q}$-group, p a prime in $\mathbf{N}$, $G = H(\mathbf{Q}_p)$, and $\Gamma = H(\mathbf{Q})$. Suppose that H is a simply connected, almost $\mathbf{Q}$-simple, and assume $H(\mathbf{Q}_p)$ is non-compact and $H(\mathbf{R})$ has no compact factors. Then G is a locally compact non-compact group and Γ is a dense subgroup in G (see II.6.8 in [17]).

Theorem 10. *Suppose that $H(\mathbf{R})$ has Kazhdan's property (T) (see Ch. III in [17]), and that the center of $H(\mathbf{R})$ is trivial. Consider the equivalence relation R_Γ generated by left translations of Γ on G. Then all the automorphisms of R_Γ preserve a Haar measure on G.*

Proof. Set $\Lambda = H(\mathbf{Z})$, and let K denote the closure of Λ in G. It is well known that Λ is a lattice in $H(\mathbf{R})$ and K is open and compact in G (see [1] and [19]). Moreover, since $H(\mathbf{R})$ has Kazhdan's property (T), Λ has also Kazhdan's property (T))(see Ch.III, 2.12 in [17]). We set $\Gamma_1 = \Gamma \cap K$. Since H is connected simply connected semisimple $\mathbf{R}$-group, $H(\mathbf{R})$is connected (see [17, pp. 52-53]). Therefore Γ_1 is an ICC-group with respect to Λ, i.e. for each $\gamma \in \Gamma_1$, $\gamma \neq e$ the set $\{\lambda\gamma\lambda^{-1} \mid \lambda \in \Lambda\}$ is infinite (see Pr.1.6 in [12]).

We need the following simple lemma and two general propositions.

Lemma 11. *Let G be a locally compact second countable group, Γ be a countable dense subgroup of G, L an open subgroup in G and $\Gamma_1 = \Gamma \cap L$. Set*

$$A = \begin{cases} \mathbf{Z}\,, & if \quad [G:L] = \infty \\ \mathbf{Z}/m\mathbf{Z}\,, & if \quad [G:L] = m \end{cases}$$

Consider a Borel section $s : G/L \to G$ such that $G = \bigcup_{n\in A} \gamma_n L$, where $\{\gamma_n \mid n \in A\} = s(G/L) \subset \Gamma$. Define the Borel map $\varphi : L \times A \to G$ by setting $\varphi(k,n) = \gamma_n k$. Then φ is an isomorphism of the equivalence relations $R_{\Gamma_1} \times I_A$ and R_Γ, where I_A is a transitive equivalence relation on A. Moreover $\varphi(r_t^L \times id)\varphi^{-1} = r_t$, $t \in L$, with r_t being the right translation, i.e. $r_t(g) = gt^{-1}$, $g \in G$, and r_t^L a restriction of r_t on L.

Proposition 12. *Let G, Γ, r_t be as in 11. Suppose L is compact, but G is non-compact. Consider the action of Γ on G by left translations and denote by R_Γ the corresponding equivalence relation. Let $\theta \in \mathrm{Aut}(R_\Gamma)$ and ε*

be the natural projection from $\mathrm{Aut}(R_\Gamma)$ *onto* $\mathrm{Out}(R_\Gamma) = \mathrm{Aut}(R_\Gamma)/\mathrm{Int}(R_\Gamma)$. *If* $\varepsilon(\theta)$ *belongs to the centralizer of* $\varepsilon(\{r_t \mid t \in L\})$ *in* $\mathrm{Out}(R_\Gamma)$, *then* θ *preserves a Haar measure on* G.

Proof. Let Γ_1 and φ be as in Lemma 11. Since $[G : L] = \infty$, φ is an isomorphism of the equivalence relations $R_\Gamma \times I_\infty$ and R_Γ. Moreover, $\varphi^{-1} r_t \varphi = r_t^L \times id, \quad t \in L$. Next, since $\varepsilon(\theta)$ belongs to the centralizer of $\varepsilon(\{r_t \mid t \in L\})$, we have $\theta r_t = \delta_t r_t \theta, \quad t \in L$, where $\delta_t \in \mathrm{Int}(R_\Gamma)$. Hence we obtain $(\varphi^{-1}\theta\varphi)(r_t^L \times id) = (\varphi^{-1}\delta_t\varphi)(r_t^L \times id)(\varphi^{-1}\theta\varphi), \quad t \in L$. By Proposition 5.3 [12], $\varphi^{-1}\theta\varphi$ preserves the measure, thus so does θ. ∎

PROPOSITION 13. *Let* G *be a locally compact non-compact second countable unimodular group,* Γ *a countable dense subgroup in* G, K *an open compact subgroup in* G, *and* $\Gamma_1 = \Gamma \cap K$. *Let* R_Γ *be an equivalence relation generated by left translations of* Γ *on* G. *Suppose that there exists a subgroup* Λ *in* Γ_1 *such that* Λ *has property (T),* Λ *is dense in* K, *and* Γ_1 *is an ICC-group with respect to* Λ. *Moreover, assume the following conditions are satisfied:*

(i) for each open subgroup $K_1 \subset K$ *and its continuous injective homomorphism* $f : K_1 \to G$ *there exist an open subgroup* $K_2 \subset K_1$ *and a continuous automorphism* $\sigma \in \mathrm{Aut}(G)$ *such that* $f(k) = \sigma(k)$, $k \in K_2$;

(ii) for each $g \in G$, $g \neq e$, *the right translation* r_g *is an outer automorphism of* R_Γ;

(iii) $\mathrm{Out}(G)$ *is torsion.*

Then all the automorphisms of R_Γ *preserve a Haar measure on* G.

Proof. By Lemma 11, there is an isomorphism of equivalence relations R_Γ and $R_{\Gamma_1} \times I_\infty$ which intertwines the right translations from $\mathrm{Aut}(R_{\Gamma_1} \times I_\infty)$ with those from $\mathrm{Aut}(R_\Gamma)$. It follows from the condition *(ii)*, the proof of Lemma 5.1 [12], and Proposition 5.2 [12] that $\varepsilon(\{r_g \mid g \in G\})$ is an open subgroup in $\mathrm{Out}(R_\Gamma)$ which is topologically isomorphic to G (for the definition and basic properties of the topology on $\mathrm{Aut}(R_\Gamma)$, we refer the reader to [6] §3 and [12] §§2, 5). Therefore the space $X = \mathrm{Out}(R_\Gamma)/G$ is discrete (we identify $\varepsilon(\{r_g \mid g \in G\})$ with G here). Let us consider the action of K on X by left translations: $x = qG \overset{k}{\longmapsto} kx = kqG$, where $q \in \mathrm{Out}(R_\Gamma)$ and $k \in K$. Since K is compact, the orbit of any $x \in X$ is finite. Let $\theta \in \mathrm{Aut}(R_\Gamma)$ and K_1 be the stabilizer of $\varepsilon(\theta)G$, i.e. $K_1 = \{k \in K \mid k\varepsilon(\theta)G = \varepsilon(\theta)G\}$. Then K_1 is open in K and $\varepsilon(\theta)^{-1}K_1\varepsilon(\theta) \subset G$. Define the homomorphism $f : K_1 \to G$ by setting $f(k) = \varepsilon(\theta)^{-1}k\varepsilon(\theta), \quad k \in K_1$. It follows from *(i)* that there exist an open subgroup $K_2 \subset K_1$ and $\sigma \in \mathrm{Aut}(G)$ such that $f(k) = \varepsilon(\theta)^{-1}k\varepsilon(\theta) = \sigma(k), \quad k \in K_2$. Now let $\sigma^n \in \mathrm{Int}(G)$, i.e. there exists $g_0 \in G$ such that $\sigma^n(g) = g_0 g g_0^{-1}, \quad g \in G$. Set up $L = \bigcap_{j=0}^{n-1} \sigma^{-j}(K_2)$. Then L is open in K_2 and $\sigma^j(L) \subset K_2$, $j = 0, 1, \ldots, n-1$. Therefore we obtain $\varepsilon(\theta)^{-n} t \varepsilon(\theta)^n = \sigma^n(t) = g_0 t g_0^{-1}$ for all $t \in L$. This implies that $\varepsilon(\theta)^n g_0 t = t \varepsilon(\theta)^n g_0$, $t \in L$, i.e. $\varepsilon(\theta^n r_{g_0}) \varepsilon(r_t) = \varepsilon(r_t) \varepsilon(\theta^n r_{g_0})$, $t \in L$. Now an application of Lemma 11 and Proposition 12 shows that the automorphism $\theta^n r_{g_0}$ preserves a Haar measure. Hence θ also has this property. ▮

We turn back to the proof of Theorem 10. Now it suffices only to verify only that the conditions (i), (ii) and (iii) of Proposition 13 are satisfied. Let $f : K_1 \to G$ be a continuous injective homomorphism, for K_1 an open subgroup in K. Since K_1 is open in G, K_1 is a p-adic Lie subgroup of $G = H(\mathbf{Q}_p)$. Thus, by Cartan's theorem [2, Ch.III, §8, Th.1], f is an analytic homomorphism of p-adic Lie groups. Therefore f induces a homomorphism $L(f)$ of the p-adic Lie algebras $L(K_1)$ and $L(G)$. Since K_1 is open in G, $L(K_1) = L(G)$. Moreover, $L(f)$ is injective (see Ch.III, §3, Pr.8 in [2]). Thus, $L(f)$ is an automorphism of $L(G)$. We may also view $L(f)$ as a $\mathbf{Q}_p$-automorphism of the Lie algebra $L(H)$. Since H is semisimple and simply connected, there exists an automorphism $\sigma \in \mathrm{Aut}(H)$ whose differential is $L(f)$, Furthermore, σ is defined over $\mathbf{Q}_p$ (see Pr.1.4.13, Ch.I in [17]). Hence $\sigma(G)$ is a finite index subgroup in G (see p.34 in [21]). It follows from 2.3.2 and 2.3.6, Ch.I [17] that $\sigma(G) = G$, i.e. $\sigma \in \mathrm{Aut}(G)$. According to Theorem 3 of §7, Ch.III in [2] there exists an open subgroup K_2 in K_1 such that $f(k) = \sigma(k)$ for $k \in K_2$. Next, consider $r_g \in \mathrm{Int}(R_\Gamma)$. The subgroup $N = \{r_h \mid h \in G\} \cap \mathrm{Int}(R_\Gamma)$ is normal in $\{r_h \mid h \in G\} \cong G$. We shall show that $N \neq G$. Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be generators of the group Λ (see Ch.III in [17]). If $r_{\gamma_i} \in \mathrm{Int}(R_\Gamma)$, then there exist open subgroups $F_i \subset G$, $i = 1, \ldots, m$ such that γ_i lies in the centralizer of F_i (see Remark 2.8 in [11]). Set up $F = \bigcap_{i=1}^{m} F_i$. Then F is an open subgroup in G and F is contained in the centralizer of Λ. It follows from Borel's density theorem that $F \subset Z(G)$ (see Ch. V in [20]). We obtain the contradiction. Therefore $r_{\gamma_i} \in \mathrm{Int}(R_\Gamma)$, i.e. $N \neq G$. It follows from 2.3.2 and 2.3.6, Ch.I [17] that $N \subset Z(G)$. Hence $g \in Z(G)$. Since $r_g \in \mathrm{Int}(R_\Gamma)$, we obtain $g \in \Gamma$, i.e. $g \in Z(\Gamma) \subset Z(H(\mathbf{R})) = \{e\}$. Finally, since the group of outer automorphisms $\mathrm{Aut}(G)/_{\mathrm{Int(G)}}$ is finite (this follows from the results of §5 (see sections 3, 4 and Exercise 11), Ch.VIII [3]), we have that the condition (iii) is satisfied.

Thus Theorem 10 is proved completely. ▮

Example 14. Let $G = SL_n(\mathbf{Q}_p)$ and $\Gamma = SL_n(\mathbf{Q})$, $n \geq 3$. Then $F(R_\Gamma) = \{1\}$. In fact, if n is odd, then $SL_n(\mathbf{R})$ has the trivial center. Hence the result follows from the theorem. Suppose now n is even. Then $Z(SL_n(\mathbf{R})) = \{\pm 1\}$ and the center of $SL_n(\mathbf{R})/_{\{\pm 1\}} = PSL_n(\mathbf{R})$

is trivial. We consider the action of $\widetilde{\Gamma} = \Gamma/\{\pm 1\}$ on $\widetilde{G} = G/\{\pm 1\}$. It is easy to see that equivalence relations R_Γ and $R_{\widetilde{\Gamma}}$ are isomorphic. Hence $F(R_\Gamma) = F(R_{\widetilde{\Gamma}})$. Next, if θ is an automorphism of the Lie algebra $L(\widetilde{G}) = sl_n(\mathbf{Q}_p)$, then either $\theta(x) = axa^{-1}$ or $\theta(x) = -a\ {}^t xa^{-1}$, where $a \in GL_n(\mathbf{Q}_p)$ (see §13, Ch. VIII [3]). Therefore, for every $\theta \in \mathrm{Aut}(L(\widetilde{G}))$ there exists an automorphism $\sigma \in \mathrm{Aut}(\widetilde{G})$ whose differential is θ. Our result now follows from Proposition 13 and from the proof of Theorem 10.

To conclude, we find it worthwhile to mention some interesting open problems related to the fundamental group of R_Γ.

PROBLEM 15. *Compute $F(R_\Gamma)$ in the following cases:*

(i) $G = SL_n(\mathbf{R})$, $\Gamma = SL_n(\mathbf{Q})$, $n = 2$ and $n \geq 3$;
(ii) $G = SL_2(\mathbf{Q}_p)$, $\Gamma = SL_2(\mathbf{Q})$.

PROBLEM 16. *Do the groups like G and Γ with $F(R_\Gamma) \neq \mathbf{R}_+^*$ and $F(R_\Gamma) \neq \{1\}$ exist?*

REFERENCES

1. A. BOREL AND HARISH-CHANDRA, Arithmetic subgroups of algebraic groups, *Ann. of Math.*, **75** (1962), 485–535.
2. N. BOURBAKI, *Groupes et Algébres de Lie*, Ch. I–III, Hermann, Paris, 1971.
3. N. BOURBAKI, *Groupes et Algébres de Lie*, Ch. VII, VIII, Hermann, Paris, 1975.
4. A. CONNES, A factor of type II_1 with countable fundamental group, *J. Operator Theory*, **4** (1980), 151–153.
5. A. CONNES, J. FELDMAN AND B. WEISS, An amenable equivalence relation is generated by a single transformation, *Ergod. Theory* & *Dynam. Sys.*, **1** (1981), 431–450.
6. A.I. DANILENKO, The topological structure of Polish groups and groupoids of measure space transformations, *Publ. RIMS, Kyoto Univ.*, **31** (1995), 913–940.
7. J. FELDMAN, P. HAHN AND C.C. MOORE, Orbit structure and countable sections for actions of continuos groups, *Adv. in Math.*, **28** (1978), 186–230.
8. J. FELDMAN AND C.C. MOORE, Ergodic equivalence relations, cohomology, and von Neumann algebras I, *Trans. Amer. Math. Soc.*, **234** (1977), 289–324.
9. A. FURMAN, Orbit equivalence rigidity, *Ann. of Math*, **150** (1999), 1083–1108.
10. D. GABORIAU, Coût des relations d'équivalence et des groupes, *Invent. math.* **139** (2000), 41–98.
11. S.L. GEFTER, Outer automorphism group of the ergodic equivalence relation generated by translations of dense subgroup of compact group on its homogeneous space, *Publ. RIMS, Kyoto Univ.*, **32** (1996), 517–538.
12. S.L. GEFTER AND V.YA. GOLODETS, Fundamental groups for ergodic actions and actions with unit fundamental groups, *Publ. RIMS, Kyoto Univ.*, **24** (1988), 821–847.

13. S.L. Gefter, V.Ya. Golodets and S.D. Sinel'shchikov, The fundamental group for ergodic actions of semisimple Lie groups and their lattices, (in Russian), *Dokl. UkrSSR Acad. Sci.*, **8** (1987), 6–9.

14. V.Ya. Golodets and N.I. Nessonov, T-Property and nonisomorphic full factors of types II and III, *J. Funct. Anal.*, **70** (1987), 80–89.

15. V.Ya. Golodets and S.D. Sinel'shchikov, Outer conjugacy for actions of continuous amenable groups, *Publ. RIMS, Kyoto Univ.*, **23** (1987), 737–769.

16. G.W. Mackey, Ergodic transformation groups with a pure point spectrum, *Ill. J. Math.*, **8** (1964), 593-600.

17. G.A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Springer, Berlin, 1991.

18. C.C. Moore, Ergodic theory and von Neumann algebras, *Proc. Symp. Pure Math*, **38** (1982), part 2, 179–226.

19. V.P. Platonov, Arithmetical theory of algebraic groups, (in Russian), *Uspekhi Mat. Nauk*, **37** (1982), 3–54.

20. M.S. Raghunathan, *Disrete subgroups of Lie groups*, Springer-Verlag, Berlin–Heidelberg–New-York, 1972.

21. R.J. Zimmer, *Ergodic theory and Semisimple Groups*, Birkhäuser, Boston, 1985.

22. R.J. Zimmer, Normal ergodic actions, *J. Funct. Anal.*, **25** (1977), 286–305.

23. R.J. Zimmer, Amenable actions and dense subgroups of Lie groups, *J. Funct. Anal.*, **72** (1987), 58–64.

24. R.J. Zimmer, Arithmeticity of holonomy groups of Lie foliations, *J. of the Amer. Math. Soc.*, **1** (1988), 35–58.