QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **4**, 31–37 (2003)
ARTICLE NO. 50

On Dense Embeddings of Discrete Groups into Locally Compact Groups

Maxim S. Boyko

Institute for Low Temperature Physics and Engineering, Lenin Ave 47, 61103 Kharkov, Ukraine
E-mail: mboyko@land.ru

Sergey L. Gefter

Institute for Low Temperature Physics and Engineering, Lenin Ave 47, 61103 Kharkov, Ukraine
E-mail: gefter@univer.kharkov.ua

Konstantin M. Kulagin

Institute for Low Temperature Physics and Engineering, Lenin Ave 47, 61103 Kharkov, Ukraine
E-mail: kkulagin@land.ru

We consider a class of discrete groups which have no ergodic actions by translations on continuous non-compact locally compact groups. We also study dense embeddings of $\mathbf{Z}^n$ ($n > 1$) into non-compact locally compact groups. Moreover, we study some discrete groups which admit no embeddings into almost connected locally compact groups. In particular, we prove that a lattice in a simple Lie group with property (T) cannot be embedded densely into a connected non-compact locally compact group.

Key Words: locally compact groups, discrete groups, dense subgroups.

1. INTRODUCTION

Let G be a non-discrete locally compact group with a left Haar measure μ, and Γ a countable dense subgroup of G. Then Γ acts on the measure space (G, μ) by left translations. Such actions form a simple but important class of ergodic actions (see [10] and [16]). A lot of interesting questions arise on actions mentioned above (see, for example, [8], [9], [11], [17] and [4]). Dense embeddings of group $\mathbf{Z}^n$ ($n > 1$) into non-compact locally compact groups are considered in the present paper (see Section 2). Moreover, we study some specific discrete groups which admit no embeddings into almost connected locally compact groups (see Section 3).

The paper contains the extended text of the two lectures given at the Conferens on "Dynamical Systems and Ergodic Theory" Katsiveli (Crimea, Ukraine), August 21-30, 2000.

2. DENSE EMBEDDINGS OF DISCRETE ABELIAN GROUPS

A topological group H is called monothetic if there exists a cyclic subgroup Λ which is dense in H. It is well known that a locally compact monothetic group is either compact or discrete (see, for example, Theorem 19 in [13]). To rephrase, the group of integers $\mathbf{Z}$ cannot be embedded densely into a non-discrete non-compact locally compact group.

Definition 1. We say that a discrete group Γ has Z-property if Γ cannot be embedded densely into a continuous locally compact non-compact group.

Example 2. For a prime p we set $\mathbf{C}_{p^\infty} = \{z \in \mathbf{T} \mid z^{p^n} = 1 \, for\ some\ n\}$. Let us show that the quasicyclic group $\mathbf{C}_{p^\infty}$ has Z-property. Let $\varphi : \mathbf{C}_{p^\infty} \to G$ be a dense embedding of $\mathbf{C}_{p^\infty}$ into a locally compact group G. Then G is Abelian. Hence there exists an open subgroup $G_1 \subset G$ which is topologically isomorphic to $\mathbf{R}^m \times K$, where K is a compact group (see Th. 24.30 in [6]). Let $\Gamma_1 = \varphi^{-1}(G_1)$. Then $\varphi(\Gamma_1)$ is dense in G_1. If Γ_1 is finite, then G_1 is finite and open. Therefore G is discrete. Assume now that Γ_1 is infinite. Then $\Gamma_1 = \mathbf{C}_{p^\infty}$. Since $\overline{\varphi(\mathbf{C}_{p^\infty})} = G$ and G_1 is open, $G_1 = G$. Hence φ is a dense embedding of $\mathbf{C}_{p^\infty}$ into $\mathbf{R}^m \times K$. Since the elements of $\mathbf{C}_{p^\infty}$ have finite order $m = 0$, i.e. G is compact.

Problem 3. *Prove that $\mathbf{C}_{p^\infty}$ has property (Z) without using the structural theory of locally compact Abelian groups.*

The following theorem provides a complete description of Abelian groups with Z-property.

Theorem 4 ([5, Theorem 1.1]). *Let Γ be an infinite discrete Abelian group with Z-property. Then either $\Gamma \cong \mathbf{Z} \times F$ or $\Gamma \cong \mathbf{C}_{p^\infty} \times F$, where F is a finite Abelian group.*

Remark 5. There exists discrete Abelian groups which admit dense embeddings only into non-compact locally-compact groups of the form $K \times D$, where K is a compact group and D is a discrete group. Let us say that these groups have weak property (Z). The infinite direct sum of cyclic groups $\mathbf{Z}(p)$ is an example of such a group. This follows from J.Braconnier's theorem (see [6, Section 25.29]).

Remark 6. It is interesting to observe that an infinite periodic Abelian group cannot be embedded densely into a compactly generated (and, in particular, into an almost connected) locally compact non-compact group. Indeed, let G be a compactly generated locally compact group and Γ a periodic Abelian dense subgroup of G. By Theorem 9.8 [6], G is topologically isomorphic to $\mathbf{R}^m \times \mathbf{Z}^n \times K$, where K is a compact group. Since Γ is dense in G, $m = n = 0$, i.e. G is compact.

The following theorem is the main result of this Section. It contains a description of the structure for non-compact locally compact Abelian groups which have finitely generated dense subgroups.

THEOREM 7. *Let G be a non-compact locally compact Abelian group. If $\mathbf{Z}^n$ $(n > 1)$ can be densely embedded into G then G is topologically isomorphic to $K \times \mathbf{R}^l \times \mathbf{Z}^m$, where K is a compact group, $l + m + 1 \leq n$ and $l + m \geq 1$.*

Proof. We shall show that G is a compactly generated group. Let $\varphi : \mathbf{Z}^n \to G$ be a dense embedding, G_0 the component of the identity of G, and π be the natural projection from G onto $\widetilde{G} = G/G_0$. Then $\widetilde{G}$ is totally disconnected and $(\pi \circ \varphi)(\mathbf{Z}^n)$ is dense in $\widetilde{G}$. Let Q be an open compact subgroup in $\widetilde{G}$ (see Theorem 7.7 in [6]). Then the image of $\mathbf{Z}^n$ in the factor group $\widetilde{G}/Q$ is dense and $\widetilde{G}/Q$ is discrete. Therefore $\widetilde{G}/Q$ is finitely generated. By Proposition 5.39 (i) of [6], $\widetilde{G}$ is compactly generated group. Hence G also has this property (see Theorem 7.4 and Proposition 5.39 (i) in [6]). According to Theorem 9.8 of [6] G is topologically isomorphic to $K \times \mathbf{R}^l \times \mathbf{Z}^m$, where K is a compact. To prove the inequality $l+m+1 \leq n$ we generalize the arguments from Section 2 of [2]. Let H be a topologically finitely generated group. We set

$$\sigma(H) = \min\{k : \textit{ there exist } h_1, h_2, \ldots, h_k \in H \textit{ such that } H \textit{ is topologically generatedby } h_1, h_2, \ldots, h_k\}. \tag{1}$$

It may be proved that $\sigma(\mathbf{R}^l \times \mathbf{Z}^m) = l + m + 1$ (see Proposition 2.3 in [2]). Since $\mathbf{R}^l \times \mathbf{Z}^m$ is a quotient of $K \times \mathbf{R}^l \times \mathbf{Z}^m$, we obtain from Lemma 2.2 of [2]

$$\sigma(K \times R^l \times \mathbf{Z}^m) \geq \sigma(\mathbf{R}^l \times \mathbf{Z}^m) = l + m + 1.$$

Hence $n \geq \sigma(K \times \mathbf{R}^l \times \mathbf{Z}^m) = l + m + 1.$ ∎

The following theorem generalizes Proposition 2.4 of [2].

Theorem 8. *Let K be a compact monothetic group. If l and m are non-negative integers, then there exists a dense embedding of $\mathbf{Z}^{l+m+1}$ into $K \times \mathbf{R}^l \times \mathbf{Z}^m$.*

Proof. We show that $\sigma(K \times \mathbf{R}^l \times \mathbf{Z}^m) = l + m + 1$, where σ is defined in (0). Consider the exact sequence

$$0 \to \mathbf{Z}^l \times \mathbf{Z}^m \to K \times \mathbf{R}^l \times \mathbf{Z}^m \to K \times \mathbf{T}^l \to 0.$$

Since $K \times \mathbf{T}^l$ is monothetic (see Theorem 25.17 in [6]), $\sigma(K \times \mathbf{T}^l) = 1$. By Lemma 2.1 of [2], we have

$$\sigma(K \times \mathbf{R}^l \times \mathbf{Z}^m) \leq \sigma(\mathbf{Z}^{l+m}) + \sigma(K \times T^l) = l + m + 1.$$

By Lemma 2.2 and Proposition 2.3 of [2], we obtain

$$\sigma(K \times \mathbf{R}^l \times \mathbf{Z}^m) \geq \sigma(\mathbf{R}^l \times \mathbf{Z}^m) = l + m + 1.$$

Hence $\sigma(K \times \mathbf{R}^l \times \mathbf{Z}^m) = l + m + 1$. ▌

3. DENSE EMBEDDINGS OF DISCRETE GROUPS INTO ALMOST CONNECTED GROUPS

Let G be a topological group. Let us denote by G_0 the connected component of the identity in G. We say that G is almost connected if the quotient group G/G_0 is compact.

Theorem 9. *Let Γ be a simple discrete group; assume it to be nonlinear over the field $\mathbf{C}$ (i.e. Γ cannot be embedded into any $\mathrm{GL}(n, \mathbf{C})$). Then Γ cannot be embedded into an almost connected locally compact group.*

Proof. The proof consists of several steps.

(1) Show that Γ cannot be embedded into a compact group. Let G be a compact group and $\varphi : \Gamma \to G$ be an embedding. According to the Peter - Weyl theorem, for a $\gamma \in \Gamma$, $\gamma \neq e$, there exists a homomorphism $\psi : G \to U(n)$ such that $\psi(\varphi(\gamma)) \neq I$ (see §22 in [6]). We may assume that $\psi : G \to \mathrm{GL}(n, \mathbf{C})$. Consider the homomorphism $\vartheta : \Gamma \to \mathrm{GL}(n, \mathbf{C})$, $\vartheta = \psi \circ \varphi$. Then $\vartheta(\gamma) \neq I$. Hence $\mathrm{Ker}\, \vartheta \neq \Gamma$. On the other hand $\mathrm{Ker}\, \vartheta$ is a normal subgroup of Γ and Γ is a simple group. Therefore, $\mathrm{Ker}\, \vartheta = \{e\}$, i.e., $\vartheta : \Gamma \to \mathrm{GL}(n, \mathbf{C})$ is an injective homomorphism. This contradicts the assumption of nonlinearity of Γ.

(2) Check that Γ cannot be embedded into a connected Lie group. Suppose $\varphi : \Gamma \to G$ is such an embedding. Let $Ad_G : G \to \mathrm{GL}\,(g)$ be the adjoint representation of G, where g is the Lie algebra of G ($dim\, g < \infty$).

It is well-known that $\operatorname{Ker} Ad_G = Z$, where Z is the center of G. We may assume that $Ad_G : G \to \mathrm{GL}(n, \mathbf{C})$. Consider the homomorphism $\vartheta : \Gamma \to \mathrm{GL}(n, \mathbf{C})$, $\vartheta = Ad_G \circ \varphi$. Since Γ is simple, $\operatorname{Ker} \vartheta = \{e\}$ or $\operatorname{Ker} \vartheta = \Gamma$. But Γ is nonlinear over $\mathbf{C}$. Hence $\operatorname{Ker} \vartheta = \Gamma$, i.e., $\varphi(\Gamma) \subset \operatorname{Ker} Ad_G = Z$. Since the homomorphism φ is injective, it follows from the above inclusion that Γ is Abelian. This contradicts our assumption on Γ.

(3) Now prove that Γ cannot be embedded into a connected locally compact group. Suppose $\varphi : \Gamma \to G$ is an embedding and G is a connected locally compact group. Let U be a compact neighborhood of the identity in G. It is well-known from the structure theory of locally compact groups that there exists a closed normal subgroup $N \subset U$ such that ${}^{G}/_{\mathrm{N}}$ is a connected Lie group (see [14]). Since U is compact, N is compact too. Let $\pi : G \to {}^{G}/_{\mathrm{N}}$ be the natural projection. Consider the homomorphism $\vartheta : \Gamma \to {}^{G}/_{\mathrm{N}}$, $\vartheta = \pi \circ \varphi$. As in the previous part of the proof, we deduce that $\operatorname{Ker} \vartheta = \{e\}$ or $\operatorname{Ker} \vartheta = \Gamma$. If $\operatorname{Ker} \vartheta = \{e\}$, then $\vartheta : \Gamma \to {}^{G}/_{\mathrm{N}}$ is an embedding of Γ into a connected Lie group, which contradicts the part (2). If $\operatorname{Ker} \vartheta = \Gamma$, then $\varphi(\Gamma) \subset N$. Thus $\varphi : \Gamma \to N$ is an embedding of Γ into a compact group, which contradicts part (1).

(4) Finally, let G be an almost connected group and $\varphi : \Gamma \to G$ an embedding. Consider the natural map $\pi : G \to {}^{G}/_{G_0}$. Set $\vartheta = \pi \circ \varphi$. Then $\vartheta : \Gamma \to {}^{G}/_{G_0}$. If $\operatorname{Ker} \vartheta = \{e\}$, then ϑ is an embedding of Γ into a compact group, which contradicts part (1). If $\operatorname{Ker} \vartheta = \Gamma$, then $\varphi(\Gamma) \subset G_0$. Therefore, $\varphi : \Gamma \to G_0$ is an embedding of Γ into a connected locally compact group, which contradicts part (3).

This completes the proof. ∎

Corollary 10. *Let S_∞ be the group of finite permutations on a countable set. Then S_∞ cannot be embedded into an almost connected (and, in particular, into a connected) locally compact group.*

Proof. It suffices to show our statement for the subgroup A_∞ of even permutations. It follows from the classical Jordan theorem about finite subgroups of $\mathrm{GL}(n, \mathbf{C})$ (see [12, p.404]) that A_∞ is nonlinear (see also Theorem 8.6 in [7]). Since A_∞ is simple, the result now follows from the theorem. ∎

Problem 11. *Find a straightforward proof of this statement which does not use the theory of unitary representations and the structure theory of locally compact groups.*

Corollary 12. *Let Γ be an infinite finitely generated group without nontrivial finite quotient groups. Then Γ cannot be embedded into an almost connected locally compact group.*

Proof. By Mal'cev's theorem finitely generated linear groups are residually finite. ∎

EXAMPLE 13. Let Γ be G.Higman's group; that is, Γ is defined by generators x_1, x_2, x_3, x_4 and the relations

$$x_2 x_1 x_2^{-1} = x_1^2,$$

$$x_3 x_2 x_3^{-1} = x_2^2,$$

$$x_4 x_3 x_4^{-1} = x_3^2,$$

$$x_1 x_4 x_1^{-1} = x_4^2.$$

Then Γ has no nontrivial finite quotient groups (see, [15, pp. 9-10]). Thus, Γ cannot be embedded into an almost connected (in particular, into a compact and into a connected) locally compact group.

For the proof of the following theorem we shall use Margulis' superridigity theorem for lattices in semisimple Lie groups (see [11], [18] and [3]).

THEOREM 14. *Let H be a connected non-compact simple Lie group with finite center and Γ a lattice in H. Suppose that H has Kazhdan's property (T) (see Ch.III in [11]). Then Γ cannot be embedded densely into a connected non-compact locally compact group.*

Proof. Let G be a connected locally compact group and $\varphi : \Gamma \to G$ be a homomorphism with a dense image. We shall show that G is compact.

We first consider the case when G is a connected Lie group. Let R be a solvable normal subgroup in G such that the quotient group $G_1 = {}^{G}/_{R}$ is a semi-simple Lie group. We may take G_1 to be the adjoint group (see §8, Ch. III in [1]). If $\pi : G \to G_1$ be the natural projection, then $\rho = \pi \circ \varphi$ is a homomorphism from Γ into G_1 with a topologically dense image. In particular $\rho(\Gamma)$ is Zariski dense and by superrigidity G_1 is a compact group, because the other alternative $\rho(\Gamma)$ being a lattice in G_1 contradicts the density assumption (see §5, Ch.III and §5, Ch.VII in [11], and [3]). Next, since Γ has property (T), G and R also have this property (see Ch.III, Corollary 2.13 in [11]). We obtain from amenability of R, that R is compact (see Corollary 7.1.9 in [18]). Therefore, G is a compact group.

Now let G be an arbitrary connected locally compact group. Then there exists a compact normal subgroup $K \subset G$ such that ${}^{G}/_{K}$ is a connected Lie group (see [14]). If $\theta : G \to {}^{G}/_{K}$ be the natural projection and $\psi = \theta \circ \varphi$, then we obtain a homomorphism $\psi : \Gamma \to {}^{G}/_{K}$ with $\psi(\Gamma)$ dense in ${}^{G}/_{K}$. Hence ${}^{G}/_{K}$ is a compact Lie group and G is compact too.

This complete the proof. ▮

Remark 15. If H has no property (T), then the conclusion of Theorem 14 may not be fulfilled. For example the lattice $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ in $H =$

$SL_2(\mathbf{R})$ is virtually a free group, hence Γ can be densely embedded into a connected non-compact locally compact group. The referee of present paper conjectured that such an embedding may be built for a lattice in any rank-one group without property (T), i.e. in $H = SO(n, 1)$ or $H = SU(n, 1)$.

ACKNOWLEDGMENT

The work is partially supported by CRDF - grant UM1-2092.

The authors are grateful to the referee for pointing out a gap in the last part of the proof of Theorem 14 in a previous version of this article, and for suggesting a simpler proof of this theorem. We would also like to thank him for interesting comments to Remark 15.

REFERENCES

1. N. Bourbaki, *Groupes et Algébres de Lie*, Ch. I-III, Hermann, Paris, 1971.
2. J. Cleary and S.A.Morris, Generators for locally compact groups, *Proc. of the Edinburgh Math. Soc.*, **10**(1993), 463–467.
3. K. Corlette, Archimedean superrigidity and hyperboliques geometry, *Ann. of Math.*, **135**(1992), 165–182.
4. S.L. Gefter and V.Ya. Golodets, Fundamental groups for ergodic actions and actions with unit fundamental groups, *Publ. RIMS, Kyoto Univ.*, **24**(1988), 821–847.
5. S.L. Gefter and K.M.Kulagin, On dense embeddings of discrete Abelian groups into locally compact groups, *Bull. Belg. Math. Soc.*, **9**(2002), 161–165.
6. E. Hewitt and K.A. Ross, *Abstract Harmonic analysis*, vol. 1, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
7. I. Kaplansky, *An introduction to differential algebra*, Herman, Paris, 1957.
8. A. Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Verlag, Basel-Boston-Berlin, 1994.
9. A. Lubotzky and B.Weiss, Groups and expanders, *DIMACS Ser. in Discrete Math. and Th. Computer Science*, vol. 10, 1993, 95–109.
10. G.W. Mackey, Ergodic transformation groups with a pure point spectrum, *Ill. J. Math.*, **8** (1964), 593-600.
11. G.A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer, Berlin, 1991.
12. Ju.I. Merzljakov, *Rational groups*, (in Russian), Nauka, Moscow, 1980.
13. S.A. Morris, *Pontryagin duality and the structure of locally compact Abelian groups*, Cambridge University Press, Cambridge, 1977.
14. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, New York-London, 1965.
15. J-P. Serre, *Trees*, Springer-Verlag, New York, 1980.
16. R.J. Zimmer, Normal ergodic actions, *J. Funct. Anal.*, **25** (1977), 286–305.
17. R.J. Zimmer, Amenable actions and dense subgroups of Lie groups, *J. Funct. Anal.*, **72** (1987), 58–64.
18. R.J. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, Boston, 1985.