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On Lipschitz Extension of Interval Maps

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Let $[a, b]$ be a compact real interval and f: $[a, b] \to [a, b]$ a continuous map from $[a, b]$ into itself. We say that f is topologically mixing if for any open $U, V \subset [a, b]$ there exists an N such that $f^n(U) \cap V \neq \emptyset$ for any $n > N$. Denote by $A(f)$ the set of those from points a, b which have no preimages in (a, b) and $\text{ent}(f)$ the topological entropy of f. We show the following: If f: $[a, b] \to [a, b]$ satisfies the conditions (i) f is topologically mixing, (ii) $A(f) = \emptyset$, (iii) $\text{ent}(f) = \log \nu < \infty$, then f has a ν-Lipschitz extension, i.e. there exist a ν-Lipschitz map g: $[a, b] \to [a, b]$ and a nondecreasing surjective map h: $[a, b] \to [a, b]$ such that $f \circ h = h \circ g$.

Key Words: Interval map, topological mixing, finite topological entropy.

1. INTRODUCTION

There is a famous theorem of Parry [8] that enables to conjugate any piecewise monotone transitive interval map f to a piecewise linear one with slope $\pm \exp^{\text{ent}(f)}$, where $\text{ent}(f)$ is the topological entropy of f. We prove some consequences of this theorem for interval maps that are not piecewise monotone. For a sufficiently large class of topologically mixing maps with finite entropy we obtain - rather surprisingly - that any such map has a Lipschitz extension, where the achieved Lipschitz constant is the least possible and it corresponds to the topological entropy again. Our result shows that - in a sense - Lipschitz interval maps provide the same combinatorial complexity as general ones. As far as we know the research of these phenomena is quite at the beginning and most of questions remain open.

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Let $[a,b]$ be a compact real interval and let $f\colon [a,b] \to [a,b]$ be a continuous map from $[a,b]$ into itself. We say that f is topologically mixing if for any open $U, V \subset [a,b]$ there exists an N such that $f^n(U) \cap V \neq \emptyset$ for any $n > N$. Denote by $A(f)$ the set of those from points a, b which have no preimages in (a,b) and by $\mathrm{ent}(f)$ the topological entropy of f. As usually, a continuous map $g\colon [a,b] \to [a,b]$ is an extension of f if there is a surjective continuous map $h\colon [a,b] \to [a,b]$ such that

$$f \circ h = h \circ g \text{ on } [a,b].$$

We say that g is a ν-Lipschitz extension if g is a ν-Lipschitz map.

Our main result is the following.

THEOREM 21. *Let $f \in C([a,b])$ be topologically mixing and satisfying $\mathrm{ent}(f) = \log \nu \in \mathbf{R}$, $A(f) = \emptyset$. There exists a ν-Lipschitz extension of f.*

Remark 1. Let us briefly discuss two conditions from Theorem 21: (i) f is topologicaly mixing and (ii) $A(f) = \emptyset$. Using the results from [2] it is not difficult to show that instead of property (i) we could assume a more general case of dense set of periodic points of f. Concerning the property (ii) we do not know whether it cannot be removed. Moreover, we do not know what is the class of maps satisfying (i),(ii) for which Theorem 21 describes a Lipschitz map conjugated (not only semiconjugated) to f - by Parry's Theorem this class involves all piecewise monotone maps satisfying (i).

The paper is organized as follows:

In Section 2 we give some basic notation and definitions. Section 3 is devoted to the known needed results used throughout the paper. In Section 4 we prove several auxiliary results that will be useful when proving Theorem 21. The main results of this section are Lemmas 17-19. Section 5 is devoted to the proof of Theorem 21 and in Appendix we prove two technical lemmas from Section 4 - Lemma 14 and Lemma 19.

2. NOTATION AND DEFINITIONS

By $\mathbf{R}, \mathbf{N}, \mathbf{N_0}$ we denote the sets of real, positive and nonnegative integer numbers respectively. Let T be a compact subset of $\mathbf{R}$. We consider a space $C(T)$ of all continuous maps g, which are defined on T and mapping it into itself. For $g \in C(T)$ and a nonempty set (maybe onepoint) $J \subset T$ the set $\mathrm{orb}(J,g) = \{g^i(J) : \ i \in \mathbf{N}_0\}$ is called the orbit of J. We write $\mathrm{orb}(x,g)$ if $J = \{x\}$. As usually, the ω-limit set $\omega(x,g)$ of $x \in T$ consists of all the limit points of $\{g^i(x)\colon\ i \in \mathbf{N}\}$. A point $x \in T$ is called g-recurrent if $x \in \omega(x,g)$.

We say that a point $x \in T$ is periodic if $g^n(x) = x$ for some $n \in \mathbf{N}$. Such an n is called the period of x and the pair $(\mathrm{orb}(x, g), g)$ is called a cycle. The union of all periodic points of a map g will be denoted by $\mathrm{Per}(g)$.

For $T \subset \mathbf{R}$, we say that $g : T \to T$ is minimal, resp. transitive on T if for each $x \in T$, resp. for some $x \in T$ it holds $\omega(x, g) = T$. The point with the last property is called transitive. We say that g: $T \to T$ is topologically mixing if for any open $U, V \subset T$ there exists an N such that $g^n(U) \cap V \neq \emptyset$ for any $n > N$. A subset M of T is g-invariant if $g(M) \subset M$. By $\mathrm{conv}X$ we denote the (closed) convex hull of a set $X \subset \mathbf{R}$.

For $T \subset \mathbf{R}$ compact and $g : T \to T$ continuous we define a map $g_T \in C(\mathrm{conv}T)$ mapping the convex hull conv T into itself and such that $g_T|\, T = g$ and $g_T|\, J$ is affine for any interval $J \subset I$ such that $J \cap T = \emptyset$. For an interval $I \subset \mathbf{R}$, a map $g \in C(I)$ is piecewise monotone if there are some $l \in \mathbf{N}$ and points $\min I = c_0 < c_1 < \cdots < c_l < c_{l+1} = \max I$ such that g is (non-strictly) monotone on each $[c_i, c_{i+1}]$, $i = 0, \dots, l$.

Definition 2. We define $\mathcal{T}$ as the set of all pairs (T, g) such that $T \subset \mathbf{R}$ is compact, $g : T \to T$ continuous and g is transitive on T.

Remark 3. It is well-known that for $(T, g) \in \mathcal{T}$ exactly one of the following three possibilities is satisfied [4]: (i) T is finite and then (T, g) is a cycle; (ii) T is a Cantor set; (iii) T is a union of finitely many disjoint closed intervals. Notice that if for $g \in C(T)$ a point $x \in T$ is recurrent then $(\omega(x, g), g) \in \mathcal{T}$.

Let $\mathcal{I}$ be the set of all compact subintervals of $\mathbf{R}$. In the sequel we use the notation $C(\mathcal{I}) = \bigcup_{I \in \mathcal{I}} C(I)$. For two closed sets $K, L \subset \mathbf{R}$ we will write $K < L$ if $\max K < \min L$.

Definition 4. Assume there are sequences $\{K_i^1\}_{i \in \mathbf{N}_0}$, $\{K_i^2\}_{i \in \mathbf{N}_0}$ such that

(i) K_i^j is a point or closed interval,

(ii) either $K_{i(1)}^j \cap K_{i(2)}^j = \emptyset$ or $K_{i(1)}^j = K_{i(2)}^j$ for $i(1) \neq i(2)$.

We will say that the sequences $\{K_i^1\}_{i \in \mathbf{N}_0}, \{K_i^2\}_{i \in \mathbf{N}_0}$ have the same order if

$$K_{i(1)}^1 < K_{i(2)}^1 \iff K_{i(1)}^2 < K_{i(2)}^2,\ i(1), i(2) \in \mathbf{N}_0. \tag{1}$$

In particular, for $f_1, f_2 \in C(\mathcal{I})$ and closed (possibly degenerate) intervals J, K, the orbits $\mathrm{orb}(J, f_1), \mathrm{orb}(K, f_2)$ have the same order if (1) is satisfied for the sequences $\{K_i^1 = f_1^i(J)\}_{i \in \mathbf{N}_0}$, $\{K_i^2 = f_2^i(K)\}_{i \in \mathbf{N}_0}$.

Definition 5. For the notion of topological entropy we use the Bowen's definition [7]. A set $E \subset T$ is (n, ε)-separated (with respect to g) if, whenever $x, y \in E$, $x \neq y$ then $\max_{0 \leq i \leq n-1} |g^i(x) - g^i(y)| > \varepsilon$.

For a compact set $K \subset T$ we denote $s(n, \varepsilon, K)$ the largest cardinality of any (n, ε)-separated subset of K. Put

$$\mathrm{ent}(g, K) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, K)$$

and $\mathrm{ent}(g) = \mathrm{ent}(g, T)$. The quantity $\mathrm{ent}(g)$ is called the topological entropy of g.

3. KNOWN NEEDED RESULTS

In the first lemma we recall known properties of dynamical systems given by topologically mixing interval maps. These properties will be useful when proving our results.

Lemma 6.

(i) Let $g \in C([a, b])$ and $p \in \mathrm{Per}(g)$. Then for $T = \mathrm{orb}(p, g)$ it holds $\mathrm{ent}(g_T) \leq \mathrm{ent}(g)$.

(ii) Let (T, g) be a cycle of odd period greater than 1. Then the map $g_T \in C(\mathrm{conv} T)$ is topologically mixing.

(iii) Let $g \in C([a, b])$ be topologically mixing. The following is true.

(iii_1) The map g is transitive and the set of all transitive points is dense in $[a, b]$.

(iii_2) The set of all periodic points with odd period is dense in $[a, b]$.

(iii_3) Let $\eta \in [a, b]$ be transitive and $\{p(n)\} \subset [a, b]$ be a sequence of periodic points, denote $P(n) = \mathrm{orb}(p(n), g)$. Then we have

$$\lim_{n \to \infty} p(n) = \eta \;\Rightarrow\; \lim_{n \to \infty} \mathrm{ent}(g_{P(n)}) = \mathrm{ent}(g).$$

(iii_4) For each $n \in \mathbf{N}$, the map g^n is topologically mixing, hence there is no periodic interval of g different from $[a, b]$.

Proof. See [1]. ▮

In order to study transitive pairs we need some method that will help us to recognize that a fixed map $f \in C(I)$ has such a pair of prescribed order. The following lemma satisfies this requirement.

Lemma 7. *Let* $f \in C(I)$ *and* $(T, g) \in \mathcal{T}$. *Assume there is a sequence* $\{K_i\}_{i \in \mathbf{N}_0}$ *such that*

(i) $K_i \subset I$ *is a point or closed interval,*
(ii) either $K_{i(1)} \cap K_{i(2)} = \emptyset$ *or* $K_{i(1)} = K_{i(2)}$ *for* $i(1) \neq i(2)$,
(iii) $f^i(K_0) = K_i$ *and for some transitive point* $t \in T$ *the orbits* $\mathrm{orb}(K_0, f)$, $\mathrm{orb}(t, g)$ *have the same order.*

Then there is an f*-recurrent point* $t^* \in I$ *such that the orbits* $\mathrm{orb}(t^*, f)$, $\mathrm{orb}(t, g)$ *have the same order.*

Proof. It is literally the same as the proof of Lemma 2.2 in [6]. ∎

For $f \in C([a, b])$ let $A(f)$ be the set of those from points a, b which have no preimages in (a, b).

Lemma 8. *([5]) If* $f \in C([a, b])$ *is topologically mixing then there are the following possibilities for* $A = A(f)$*: 1)* $A = \emptyset$*; 2)* $A = \{a\}$, $f(a) = a$*; 3)* $A = \{b\}$, $f(b) = b$*; 4)* $A = \{a, b\}$, $f(a) = a$, $f(b) = b$*; 5)* $A = \{a, b\}$, $f(a) = b$, $f(b) = a$*. Moreover, if* $J \subset [a, b]$ *is a closed interval,* $J \cap A = \emptyset$*, then for any open* U *there exists* n *such that* $f^m(U) \supset J$ *for* $m > n$*. In particular, if* $A = \emptyset$ *then for any open* U *there exists* n *such that* $f^m(U) = [a, b]$ *for* $m > n$*.*

Proposition 9. *([8]) Any continuous, transitive, piecewise monotone map* $f \in C(I)$ *is topologically conjugate to a piecewise linear map* $g \in C(I)$ *which has slope* $\pm\beta$ *(*$\log \beta$ *is the topological entropy of* f*) on each linear piece.*

The following lemma is well known. We use it in Appendix.

Lemma 10. *([3]) For* $f \in C([a, b])$ *let* $\{I_k\}_{k=0}^{m-1}$ *be a sequence of intervals satisfying* $f(I_k) \supset I_{k+1 (\mathrm{mod}\, m)}$*. There is a periodic point* $p \in [a, b]$ *such that* $f^k(p) \in I_k$ *for* $k = 0, \dots, m-1$ *and* $f^m(p) = p$*.*

4. OUR LEMMAS

Definition 11. We say that $f \in C([a, b])$ has an increasing lap (α, ω) if for points $\alpha, \omega \in (a, b)$ it holds $a < \alpha < \omega < b$ and $f(\alpha) = a$, $f(\omega) = b$. In this case we sometimes shortly say that f has an increasing lap.

Remark 12. Note that by our definition a map f may not be increasing on its increasing lap.

In what follows we will need the following six pairwise disjoint classes of maps from $C([a, b])$: $C_1 = \{f\colon f(a) = a,\ f(b) = b\}$, $C_2 = \{f\colon f(a) =$

$a,\ f(b) \in (a,b)\}$, $C_3 = \{f\colon\ f(b) = b,\ f(a) \in (a,b)\}$, $C_4 = \{f\colon\ f(a) = f(b) = a\}$, $C_5 = \{f\colon\ f(a) = f(b) = b\}$, $C_6 = \{f\colon\ f(a), f(b) \in (a,b)\}$.

Clearly, the set $\bigcup_{i=1}^{6} C_i$ is a proper subset of $C([a,b])$.

LEMMA 13. *Let $f \in C([a,b])$ be topologically mixing with $A(f) = \emptyset$. The following is true.*

(i) There exists an n such that f^m has some increasing lap for each $m > n$.

(ii) There is an $m \in \mathbf{N}$ such that $f^m \in \bigcup_{i=1}^{6} C_i$ and f^m has an increasing lap.

Proof. (i) It directly follows from Lemma 8. Let us prove (ii). Using (i) of this lemma we fix $n_0 \in \mathbf{N}$ such that every f^n, $n > n_0$ has some increasing lap. If $f^n(b) = b$, resp. $f^n(a) = a$ for some $n \in \mathbf{N}$ then we can put $m = 2nn_0$. Really, f^m belongs to $C_1 \cup C_3 \cup C_5$, resp. $C_1 \cup C_2 \cup C_4$ and since $m = 2nn_0 > n_0$ by the previous f^m has an increasing lap.

So, we have to check the case when $\{a,b\} \cap \mathrm{Per}(f) = \emptyset$. Then

$$\#\{n \in \mathbf{N}\colon\ f^n(a) = b\} \le 1 \ \&\ \#\{n \in \mathbf{N}\colon\ f^n(b) = a\} \le 1,$$

hence $f^m \in C_6$ for sufficiently large $m > n_0$. ▮

Let $g \in C([a,b])$. In the sequel we use the notation

$$U(a,x) = \{y \in [a,x]\colon\ g(y) \ge y\},\ \ D(x,b) = \{y \in [x,b]\colon\ g(y) \le y\}.$$

LEMMA 14. *Let $g \in C([a,b])$ be topologically mixing with $A(g) = \emptyset$. The following is true.*

i) If $a \in \mathrm{Fix}(g)$ then for every positive ε and $x \in (a,b]$ there is an interval $J \subset [a, a+\varepsilon]$ and an $n \in \mathbf{N}$ such that $g^n(J)$ is a neighbourhood of x and $\bigcup_{i=0}^{n-1} g^i(J) \subset U(a,x)$.

(ii) If $b \in \mathrm{Fix}(g)$ then for every positive ε and $x \in [a,b)$ there is an interval $J \subset [b-\varepsilon, b]$ and an $n \in \mathbf{N}$ such that $g^n(J)$ is a neighbourhood of x and $\bigcup_{i=0}^{n-1} g^i(J) \subset D(x,b)$.

Proof. See Appendix. ▮

Construction. Let $f \in C([a,b])$ be topologically mixing and $\eta \in [a,b]$ be a transitive point. By Lemma 6 there is a sequence $\{p(n)\}_{n\in\mathbf{N}}$ of periodic points of odd period such that for each n

$$p(n) \in (\eta - 1/n, \eta + 1/n). \tag{2}$$

Put $P(n) = \mathrm{orb}(p(n), f)$. We know from Lemma 6(ii) that each map $f_n = f_{P(n)} \in C(\mathrm{conv}P(n))$ is transitive. Using Proposition 9 we can consider an increasing surjective map $h_n \colon \mathrm{conv}P(n) \to [a, b]$ such that

$$g_n = h_n \circ f_n \circ h_n^{-1}, \tag{3}$$

where $g_n \in C([a, b])$ is a piecewise linear map with slope $\pm\nu_n$ ($\log \nu_n$ is the topological entropy of f_n) on each linear piece.

Let us suppose that the topological entropy ent $(f) = \log \nu$. It is known - see Lemma 6(i) - that for each n it holds ent $(g_n) = \log \nu_n \le \log \nu$. If $\log \nu \in \mathbf{R}$ we obtain as a consequence that the sequence $\{g_n\}_{n=1}^{\infty}$ given by (3) is equicontinuous and equibounded hence by Ascoli-Arzela Theorem also relatively compact in the space $C([a, b])$ (equipped with the supremum metric). We use the notation

$$\mathcal{L}\ (f) = \bigcap_{m \ge 1} \overline{\bigcup_{n > m} \{g_n\}}.$$

By its definition, the set $\mathcal{L}\ (f)$ is a closed subset of $C([a, b])$. If we denote by $C_\nu([a, b])$ the set of all ν-Lipschitz maps from $C([a, b])$, then $\mathcal{L}\ (f) \subset C_\nu([a, b])$.

In the next lemma we use the notation from the above construction.

Lemma 15. *Let $f \in C([a, b])$ be topologically mixing, put $h_n(p(n)) = q(n)$ for $n \in \mathbf{N}$, fix $g \in \mathcal{L}\ (f)$. The following is true.*

(i) There is a sequence $\{n(i)\}_{i \in \mathbf{N}}$ and a $\theta \in [a, b]$ such that for each $j \in \mathbf{N}$

$$g_{n(i)}^j \xrightarrow{i} g^j \quad \& \quad q(n(i)) \xrightarrow{i} \theta. \tag{4}$$

(ii) If for $\theta \in [a, b]$ from (i) it holds $\theta \notin \mathrm{Fix}(g)$ then the orbits $\mathrm{orb}(\eta, f)$, $\mathrm{orb}(\theta, g)$ have the same order.

Proof. The property (i) follows directly from our Construction. Let us prove (ii). Supposing $\theta \notin \mathrm{Fix}(g)$ we show that $\mathrm{orb}(\theta, g)$ is not finite. In order to show this fact let to the contrary $\#\mathrm{orb}(\theta, g) \in \mathbf{N}$. Then there are $r \in \mathbf{N}_0$ and $s \in \mathbf{N}$ such that $g^r(\theta) \in \mathrm{Per}(g)$ and has a period s. Let r, s be the least values with this property. We can write

$$\mathrm{orb}(g^r(\theta), g) = \{p_0 < \cdots < p_{s-1}\}.$$

Choose positive integers k, l greater than r such that $f^k(\eta) < \eta$ and $f^l(\eta) > \eta$. It is possible since $\eta \in (a, b)$ is a transitive point of f. Then for each sufficiently large n we obtain from (2) and (3)

$$f_n^k(p(n)) < p(n),\ f_n^l(p(n)) > p(n)\ \&\ g_n^k(q(n)) < q(n),\ g_n^l(q(n)) > q(n),$$

hence by (4) we get

$$g^k(\theta) \leq \theta\ \&\ g^l(\theta) \geq \theta.$$

The last inequalities together with $\theta \notin \mathrm{Fix}(g)$ show that $s > 1$. Let $(\sigma(0), \dots, \sigma(s-1))$ be the unique permutation of numbers $(0, \dots, s-1)$ satisfying

$$g^{r+\sigma(i)+sj}(\theta) = p_i,\ (i,j) \in \{0, \dots, s-1\} \times \mathbf{N}_0.$$

Then by definition of g we obtain that for each $j(0), \dots, j(s-1) \in \mathbf{N}_0$ it holds

$$f^{r+\sigma(0)+sj(0)}(\eta) < \cdots < f^{r+\sigma(s-1)+sj(s-1)}(\eta)$$

and the sets

$$J_i = \overline{\{f^{r+\sigma(i)+sj}(\eta)\colon\ j \in \mathbf{N}_0\}},\ i \in \{0, \dots, s-1\}$$

are periodic intervals (of period $s > 1$) of topologically mixing f - a contradiction with Lemma 6(iii$_4$). Thus, the set $\mathrm{orb}(\theta, g)$ is not finite.

Now, it is again an easy consequence of our Construction that the orbits $\mathrm{orb}(\eta, f)$ and $\mathrm{orb}(\theta, g)$ have the same order. ▌

Let p be a periodic point of a map $f \in C([a,b])$, put $P = \mathrm{orb}(p, f)$. We denote by p_+, resp. p_- uniquely determined points from P defined by

$$f(p_-) = \min P\ \&\ f(p_+) = \max P.$$

Definition 16. Let $f \in C([a,b])$ be topologically mixing. We say that f is bordered by a triple $(\beta, \gamma, \eta) \in (a,b)^3$ if $\beta < \gamma$ and η is a transitive point such that

$$\forall\ n\ \exists\ p(n) \in \mathrm{Per}(f) \cap (\eta - 1/n, \eta + 1/n)\colon\ p_-(n), p_+(n) \in (\beta, \gamma). \quad (5)$$

If it is not important to emphasize the values $\beta, \gamma \in (a,b)$ we will sometimes write $(\cdot, \cdot, \eta)$ instead of (β, γ, η).

Lemma 17. *Let $f \in C([a,b])$ with $\mathrm{ent}\ (f) = \log \nu \in \mathbf{R}$ be bordered by a triple (β, γ, η), let $\{p(n)\}_{n \in \mathbf{N}} \subset \mathrm{Per}(f)$ be a sequence given by (5), suppose that f_n, h_n, g_n are the same as in Construction, put $h_n(p(n)) = q(n)$ for $n \in \mathbf{N}$. Then*

$$\exists\ k, l \in \mathbf{N}\ \exists\ n_0 \in \mathbf{N}\ \forall\ n > n_0\colon\ |\ g_n^k(q(n)) - g_n^l(q(n))\ | \geq \frac{b-a}{\nu}. \quad (6)$$

Proof. Since η is transitive there exist positive integers k, l such that $f^k(\eta) < \beta$ and $f^l(\eta) > \gamma$. From (5) follows that an analogous statement is true for f_n and $p(n)$ for n sufficiently large, i.e. for k, l and β, γ as above we have

$$\exists\, n_0\ \forall\, n > n_0\colon\ f_n^k(p(n)) < \beta\ \&\ f_n^l(p(n)) > \gamma.$$

Then using (5) again we can also write for each $n > n_0$

$$f_n^k(p(n)) < \min\{p_+(n), p_-(n)\}\ \&\ f_n^l(p(n)) > \max\{p_+(n), p_-(n)\}. \quad (7)$$

The relation (3) in particular gives $h_n(p_+(n)) = q_+(n)$ and $h_n(p_-(n)) = q_-(n)$. By virtue of (7) and (3) we get for each $n > n_0$

$$g_n^k(q(n)) < \min\{q_+(n), q_-(n)\}\ \&\ g_n^l(q(n)) > \max\{q_+(n), q_-(n)\}. \quad (8)$$

But by the previous $g_n(q_+(n)) = b$ and $g_n(q_-(n)) = a$. Since g_n is a ν_n-Lipschitz map we can see that

$$\mid q_+(n) - q_-(n) \mid \geq \frac{b-a}{\nu_n} \geq \frac{b-a}{\nu}. \quad (9)$$

Summarizing, from (8),(9) we obtain (6). This proves the lemma. ∎

LEMMA 18. *Let $f \in C([a,b])$ be topologically mixing and* ent $(f) \in \mathbf{R}$. *Suppose that for some $m \in \mathbf{N}$ the map f^m is bordered by a triple $(\cdot, \cdot, \eta)$. The following is true.*

(i) The point $\eta \in (a,b)$ is transitive for f.

(ii) Using η in Construction, there is a map $g \in \mathcal{L}\ (f)$ and a g-recurrent point $\theta \in [a,b]$ such that the orbits $\mathrm{orb}(\eta, f)$, $\mathrm{orb}(\theta, g)$ *have the same order.*

Proof. The property (i) is clear.

We show (ii). Let $\tilde{f} = f^m$ be bordered by (β, γ, η). Since the entropy of f is finite we can write $\mathrm{ent}(\tilde{f}) = \log \nu \in \mathbf{R}$. Let $\{p(n)\}_{n\in\mathbf{N}} \subset \mathrm{Per}(\tilde{f}) = \mathrm{Per}(f)$ be a sequence given for $\tilde{f}$ by (5). Apply Construction for f, η, $\{p(n)\}$, resp. for $\tilde{f}$, η, $\{p(n)\}$ and denote maps from this construction f_n, h_n, g_n, resp. $\tilde{f}_n, \tilde{h}_n, \tilde{g}_n$. Using Proposition 9 we get $h_n = \tilde{h}_n$ and $g_n^m = \tilde{g}_n$ for each n and we can put $h_n(p(n)) = \tilde{h}_n(p(n)) = q(n)$, $n \in \mathbf{N}$.

Fix $\tilde{g} \in\ \mathcal{L}\ (\tilde{f})$. Obviously there are an increasing sequence $\{n(i)\}_{i\in\mathbf{N}}$ and a $\theta \in [a,b]$ such that for each $j \in \mathbf{N}$

$$\tilde{g}^j_{n(i)} \xrightarrow{\ i\ } \tilde{g}^j \quad \& \quad q(n(i)) \xrightarrow{\ i\ } \theta; \quad (10)$$

without loss of generality we can assume (taking a subsequence of $\{n(i)\}$ if necessary) that also $g^j_{n(i)} \xrightarrow{i} g^j$ for some $g \in \mathcal{L}(f)$ (obviously $g^m = \tilde{g}$) and each $j \in \mathbf{N}$.

We show that $\theta \notin \mathrm{Fix}(\tilde{g})$. Suppose to the contrary the relation $\theta \in \mathrm{Fix}(\tilde{g})$ and fix an ε positive for which $4\varepsilon < \frac{b-a}{\nu}$. For the numbers k, l given in (6), by (10) there is a positive integer i_1 such that for each $i > i_1$ it holds ($\tilde{g}^k(\theta) = \theta$, $\tilde{g}^l(\theta) = \theta$)

$$\begin{aligned} |\tilde{g}^k_{n(i)}(q(n(i))) - \tilde{g}^l_{n(i)}(q(n(i)))\,| \;&\le\; |\,\tilde{g}^k_{n(i)}(q(n(i))) - \tilde{g}^k(q(n(i)))\,| \;+ \\ +\; |\,\tilde{g}^k(q(n(i))) - \tilde{g}^k(\theta)\,| \;+\; |\,\tilde{g}^l(\theta) - \tilde{g}^l(q(n(i)))\,| \;+ \\ +\; |\,\tilde{g}^l(q(n(i))) - \tilde{g}^l_{n(i)}(q(n(i)))\,| \;&\le\; 4\varepsilon \;<\; \frac{b-a}{\nu}, \end{aligned}$$

what is impossible by (6). We have shown that $\theta \notin \mathrm{Fix}(\tilde{g})$, hence we get $\theta \notin \mathrm{Fix}(g)$. Now, it follows from Lemma 15 that the orbits $\mathrm{orb}(\eta, f)$, $\mathrm{orb}(\theta, g)$ have the same order. Using Lemma 7 we can suppose that the point θ is g-recurrent. This proves the lemma. ▌

Lemma 19. *If $f \in C([a,b])$ is topologically mixing and satisfying $A(f) = \emptyset$ then there is an $m \in \mathbf{N}$ such that f^m is bordered by some triple (β, γ, η).*

Proof. See Appendix. ▌

5. MAIN RESULTS

Our goal in this section is to use lemmas developed in the previous section to prove the main results.

Let J, K be two compact subintervals of $\mathbf{R}$; we denote by $\mathcal{H}(J,K)$, resp. $\mathcal{H}(J)$ the set of all continuous, non-decreasing maps mapping J onto K, resp J.

Definition 20. A map $g \in C(J)$ is called the (interval) extension of $f \in C(K)$ if there is a map $h \in \mathcal{H}(J,K)$ such that

$$f \circ h = h \circ g \text{ on } J. \tag{11}$$

We say that g satisfying (11) is a ν-Lipschitz extension if $g \in C_\nu(J)$.

For $h \in \mathcal{H}(J,K)$ we put

$$\mathrm{supp}(h) = \{x \in J : \ h(L) \text{ is not a point for any open interval } L \subset J \text{ with } x \in L\}.$$

THEOREM 21. *Let $f \in C([a,b])$ be topologically mixing and satisfying $\mathrm{ent}(f) = \log \nu \in \mathbf{R}$, $A(f) = \emptyset$. There exists a ν-Lipschitz extension of f.*

Proof. By Lemma 19 there is an $m \in \mathbf{N}$ such that f^m is bordered by some triple (β, γ, η) and we can use Lemma 18. By that lemma there is a map $\tilde{g} \in \mathcal{L}(f)$ and a $\tilde{g}$-recurrent point $\theta \in [a,b]$ such that the orbits $\mathrm{orb}(\eta, f)$, $\mathrm{orb}(\theta, \tilde{g})$ have the same order.

Put $T = \overline{\mathrm{orb}(\theta, \tilde{g})} = \omega(\theta, \tilde{g})$ and denote $g = \tilde{g}_T \in C(\mathrm{conv} T)$. By the previous we have $g \in C_\nu(\mathrm{conv} T)$.

Notice that since the orbits $\mathrm{orb}(\eta, f)$ and $\mathrm{orb}(\theta, g)$ have the same order and $T = \omega(\theta, g)$, resp. $[a,b] = \omega(\eta, f)$, we can consider the map $h \in \mathcal{H}(\mathrm{conv}\ T, [a,b])$ fulfilling the conditions $\mathrm{supp}(h) = T$, $h(\theta) = \eta$ and (see Remark 3)

$$f^m(\eta) = f^m(h(\theta)) = h(g^m(\theta)) \text{ for each } m \in \mathbf{N}_0.$$

Extending the last equality to the whole interval conv T, we obtain $f \circ h = h \circ g$ on $\mathrm{conv} T$. This proves the theorem. ▮

6. APENDIX - THE PROOFS OF TECHNICAL RESULTS

DEFINITION 22. Let $g \in C([a,b])$. We say that a point $x \in (a,b)$ is regular if for every neighbourhood $U(x)$ of x the set $g(U(x))$ is a neighbourhood of $g(x)$.

For $g \in C([a,b])$ we use notation

$$U(a,x) = \{y \in [a,x]\colon\ g(y) \geq y\},\ D(x,b) = \{y \in [x,b]\colon\ g(y) \leq y\}.$$

Proof of Lemma 14. We show the property (i). The proof of (ii) is analogous.

Fix $\varepsilon > 0$ and first take $x \in (a, a+\varepsilon]$. Since g is topologically mixing the set $g([a,x])$ is not a subset of $[a,x]$. From $g(a) = a$ we can see that there has to be a regular point $z \in (a,x)$ for which $g(z) = x$. Clearly, if a closed interval J is a sufficiently small neighbourhood of z, J and $n = 1$ satisfy our conclusion for x. Thus, if we put

$$A = \{x \in (a,b]\colon \text{ conclusion does not hold for } x\},$$

then $A \subset (a+\varepsilon, b]$. Suppose that $A \neq \emptyset$. Obviously the value $x_1 = \inf A$ is well defined and $a + \varepsilon \leq x_1$.

Similarly as above, the interval $[a,x_1]$ is not g-invariant. Together with the equality $g(a)=a$ we obtain that for some regular point $z_1\in(a,x_1)$ it holds $g(z_1)=x_1$. But $z_1<x_1$ hence for some closed interval $J\subset[a,a+\varepsilon]$ and a $k\in\mathbf{N}$ the set $g^k(J)$ is a neighbourhood of regular z_1 and $\bigcup_{i=0}^{k-1} g^i(J)\subset U(a,z_1)$. Then for sufficiently small $J_1\subset J$ for which the set $g^k(J_1)$ is a neighbourhood of z_1 and $g^k(J_1)\subset U(a,x_1)$ we obtain that J_1 and $n=k+1$ have the required property for x_1, hence also for all points from some right neighbourhood of x_1 - a contradiciton. This proves the lemma.

Proof of Lemma 19. By virtue of Lemma 6(iii$_4$) and Lemma 13 we obtain that there is an $m\in\mathbf{N}$ such that $g=f^m\in\bigcup_{i=1}^6 C_i$, g is topologically mixing and it has an increasing lap (α,ω). Put

$$\beta=\begin{cases}\alpha & ,\ g(a)=a\\ \min\mathrm{Fix}(g) & ,\ g(a)>a\end{cases}\qquad \gamma=\begin{cases}\omega & ,\ g(b)=b\\ \max\mathrm{Fix}(g) & ,\ g(b)<b\end{cases}\tag{12}$$

Clearly, $a<\beta<\gamma<b$. Denote $p(\alpha)=\min\mathrm{Fix}\ (g)\cap(\alpha,b]$ and fix a transitive point $\eta\in(\alpha,p(\alpha))$. It is possible since by Lemma 6(iii$_1$) the set of all transitive points is dense in $[a,b]$.

Choose a neighbourhood $U(\eta)\subset(\alpha,p(\alpha))$ of η arbitrarily. We need to show that there is a periodic point $p\in U(\eta)$ such that $p_-,p_+\in(\beta,\gamma)$. We will do it for $g\in C_1\cup C_6$. The cases when $g\in C_2\cup C_3\cup C_4\cup C_5$ are analogous and we leave them to the reader. Since g is topologically mixing and $A(g)=\emptyset$, from Lemma 8 we know that for every interval $J\subset[a,b]$ it holds $g^k(J)=[a,b]$ for some $k\in\mathbf{N}$. We use this fact liberally throughout the proof.

I. $g\in C_6$. Choose an $\varepsilon>0$ to satisfy $a+\varepsilon<\beta$, $\gamma<b-\varepsilon$ and put $K_1=[a,a+\varepsilon]$ and $K_2=[b-\varepsilon,b]$. There exist positive integers $n(0),n(1),n(2)$ and a closed interval $K_0\subset U(\eta)$ such that $g^{n(0)}(K_0)\subset K_1$, $g^{n(0)+n(1)}(K_0)\subset K_2$ and $g^{n(0)+n(1)+n(2)}(K_0)\supset K_0$. By Lemma 10 there is a periodic point $p\in K_0$ such that $g^i(p)\in g^i(K_0)$, $i\in\{0,1,\dots,n(0)+n(1)+n(2)-1\}$. Since for $g\in C_6$ from (12) follows $\min\{g(x)\colon\ x\in[a,\beta]\}\ge\beta$ and $\max\{g(x)\colon\ x\in[\gamma,b]\}\le\gamma$ we get $p_-,p_+\in(\beta,\gamma)$. The reader can see that the point p can be chosen to have an odd period.

II. $g\in C_1$. From (12) we see that $\beta=\alpha$ and $\gamma=\omega$.

(A) There is a sufficiently small $\varepsilon>0$, a closed interval $K_0\subset U(\eta)$ and an $n(0)\in\mathbf{N}$ satisfying

(B) $\min\{g(x)\colon\ x\in[p(\alpha)-\varepsilon,p(\alpha)]\}>a$, $g^{n(0)}(K_0)=[p(\alpha)-\varepsilon,p(\alpha)]$. Since for $g\in C_1$ it holds $a,b\in\mathrm{Fix}\ (g)$ the properties (B) imply that

$$a<m=\min\{g(x)\colon x\in\bigcup_{i=0}^{n(0)} g^i(K_0)\}<M=\max\{g(x)\colon x\in\bigcup_{i=0}^{n(0)} g^i(K_0)\}<b.$$

Thus, we can consider a positive δ for which $a+\delta < m$ and $M < b-\delta$. From (12) we have $g(\beta) = a$ and $g(\gamma) = b$.

Obviously **(C)** there is a closed interval $K_1 \subset [p(\alpha)-\varepsilon, p(\alpha)]$ and an $n(1) \in \mathbf{N}$ such that

$$g^{n(1)}(K_1) = [a, a+\delta] \ \& \ \bigcup_{i=0}^{n(1)-1} g^i(K_1) \subset [\beta, p(\alpha)].$$

Now, using Lemma 14 ($g(a) = a$)

(D) we can consider a closed interval $K_2 \subset [a, a+\delta]$ and an $n(2) \in \mathbf{N}$ such that $g^{n(2)-1}(K_2)$ is a left neighbourhood of γ, $g^{n(2)}(K_2) \subset [b-\delta, b]$ is a neighbourhood of b and

$$\bigcup_{i=0}^{n(2)-2} g^i(K_2) \subset U(a, \gamma).$$

Using Lemma 14 again ($g(b) = b$)

(E) we can consider a closed interval $K_3 \subset g^{n(2)}(K_2)$ and an $n(3) \in \mathbf{N}$ such that $g^{n(3)}(K_3)$ is a neighbourhood of $p(\alpha)$ and

$$\bigcup_{i=0}^{n(3)-1} g^i(K_3) \subset D(p(\alpha), b).$$

Finally,

(F) for some closed interval $K_4 \subset g^{n(3)}(K_3)$ and an $n(4) \in \mathbf{N}$ we obtain $g^{n(4)}(K_4) = K_0$.

Summarizing **(A-F)**, for a closed interval $K \subset K_0$ it holds $g^{n(0)+\cdots+n(4)}(K) = K_0$, hence by Lemma 10 there is a periodic point $p \in K \subset U(\eta)$. By our construction, $p_- \in g^{n(0)+n(1)-1}(K) \subset [\beta, p(\alpha)]$ (see **(C)**) and $p_+ \in g^{n(0)+n(1)+n(2)-1}(K) \subset [p(\alpha), \gamma]$ (see **(D)**). Similarly as above the point p can be chosen to have an odd period. This proves this part of the lemma.

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