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Versions of the Closing Lemma for Certain Dynamical Systems on Tori

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Let $\mathbf{T}^n$ be the n-torus. We show that strengthened versions of the C^r-closing lemma ($r \geq 1$) take place for several classes of dynamical systems on tori; namely, 1) for Herman actions of the group $\mathbf{Z}^k$ on $\mathbf{T}^1$; 2) for foliations without compact leaves on $\mathbf{T}^3$; 3) for diffeomorphisms of $\mathbf{T}^1$ with wandering chain recurrent points; 4) for flows on $\mathbf{T}^2$ with wandering chain recurrent trajectories and without fixed points. We also prove a version of the C^r-closing lemma for generalized interval exchange transformations on $\mathbf{T}^1$ under the assumption that a nontrivially recurrent point has symbolic expansions sufficiently large, and as a corollary we get a version of the C^r Closing lemma under similar assumption in terms of symbolic coding for C^r vector fields with finitely many singularities of saddle type on an orientable surface of genus≥ 2.

Key Words: Closing lemma, flows, interval exchange transformations, recurrent points.

1. INTRODUCTION

Let M be a manifold and $\chi^r(M)$ the space of C^r-smooth vector fields on M with the C^r-topology ($r \geq 1$). Suppose that $f \in \chi^r(M)$ has a nontrivially recurrent point $x \in M$. The following assertion is called the *C^r-closing lemma* for f: there exists $g \in \chi^r(M)$ arbitrarily close to f in the C^r-topology such that x is a periodic point for g. If one requires every point of M to be periodic point of g one usually calls the corresponding assertion the *strengthened C^r-closing lemma*. It is easy to give similar definitions for discrete dynamical systems and for foliations on M as well.

In 1939 A. G. Maier [15] proved the C^r-closing lemma ($r \geq 1$) for orientation preserving diffeomorphisms of the circle $\mathbf{T}^1$ (this result was obtained independently in [21]; see also a modern exposition in [19], ch.1). M. V. Jakobson [10] proved C^1-closing lemma for C^1-endomorphisms of the circle and also of the interval. In [27], Lai-Sang Young proved the C^r-closing lemma ($2 \leq r \leq \infty$) for C^r maps of the interval. The C^1-closing lemma and a general density theorem for diffeomorphisms of compact manifolds was proved by C. Pugh [22], [23] (see also [24]).

It is often of use (in particular, in the perturbation theory) to have versions of the closing lemma under weaker conditions on the point $x \in M$ than the condition of nontrivial recurrency. For example, C.Pugh [23] proved the C^1-closing lemma for nonwandering points on compact manifolds; M.Peixoto [21] proved C^r-closing lemma ($r \geq 1$) for prolongationally recurrent points of vector fields on the plane with singularities which are either semihyperbolic or satisfy the shadowing property.

In the present paper we prove the strengthened C^r-closing lemma ($r \geq 1$) for wandering chain recurrent points of diffeomorphisms of the circle $\mathbf{T}^1$ (denoted also by S^1) and for wandering chain recurrent points of flows without fixed points on $\mathbf{T}^2$, see Section 2. In Section 3, we consider the so called Herman actions (or H-actions) of $\mathbf{Z}^k$ on S^1 and prove the strengthened C^∞-closing lemma for them. This result is then used for foliations without compact leaves on $\mathbf{T}^3$.

In Section 4, we give sufficient conditions for the Closing lemma for piecewise diffeomorphisms of the circle (i.e., for generalized interval exchange transformations) in terms of the so called p-expansions, see precise definition in Section 4. This result is closely related to the result of C. Gutierrez [8], which states that the C^r-closing lemma ($r \geq 1$) holds true for torus flows with finitely many fixed points provided that Poincaré rotation number of the flow is of nonconstant type (the latter means that if $rot(f^t) = [a_0, a_1, \ldots, a_n, \ldots]$ is the continued fraction of the Poincaré rotation number, then $\underline{\overline{lim}}_{n\to\infty} a_n = +\infty$). By using symbolic representations for orbits of piecewise diffeomorphism $f : S^1 \to S^1$, we define for any nontrivially recurrent point of f, the so called left and right p-expansions, which characterize a kind of "periodicity" of the orbit; these p-expansions form two sequences of natural numbers which are constructed in accordance with successive repetitions of blocks in the symbolic representation of x. In a sense, p-expansions can be regarded as a generalization of the continued fractions of Poincaré rotation numbers. Theorem 5 states that if the p-expansions of a given recurrent point are sufficiently large (more precisely, if they contain infinitely many numbers bigger than 2), then the C^r-closing lemma ($r \geq 1$) holds for this point. As applications, we get sufficient conditions for the C^r-closing lemma for vector fields with finitely many singularities of saddle type on an orientable surface of genus≥ 2 (see therems 6-8 below). These sufficient conditions are in terms of the Koebe-Morse coding and the Bowen-Series expansions of geodesics corresponding to nontrivially recurrent trajectories (recall that every nontrivially recurrent trajectory corresponds to a unique geodesic with the same asymptotic directions). These results can be regarded as an extension of Gutierrez's theorem [8] to surfaces of genus≥ 2 (for further extensions see [3]).

2. THE STRENGTHENED CLOSING LEMMA FOR WANDERING CHAIN RECURRENT POINTS AND TRAJECTORIES

Let $f : M \to M$ be a homeomorphism of the manifold M with metric d. A point $x \in M$ is called *chain recurrent* if for any $\epsilon > 0$ there exist points $x = x_0, x_1, \ldots, x_n = x$ such that $d(f(x_i), x_{i+1}) < \epsilon$, $i = 0, \ldots, n-1$. A point $x \in M$ is a *wandering point* of f if there is a neighborhood U of x such that $f^n(U) \cap U = \emptyset$ for all $n \neq 0$.

Let $Diff^r(S^1)$ be the space of C^r-diffeomorphisms of the circle endowed with the standard metric d_r which induces the C^r-topology. In the following theorem we denote by R_α the rigid rotation of the circle with rotation number α.

THEOREM 1. *Suppose $f \in Diff^r(S^1), r \geq 1$, has a wandering chain recurrent point. Then for any $\epsilon > 0$ there is $g \in Diff^r(S^1)$ such that $d_r(f, g) < \epsilon$ and all orbits of g are compact (and thus periodic).*

Proof. Since any orientation reversing homeomorphism of the circle has no wandering chain recurrent points, it follows that f is orientation preserving. Thus for f, the rotation number $rot(f)$ is well defined. If $rot(f)$ is irrational and $r > 1$ then due to the well known Denjoy's Theorem [6],

f cannot have wandering points. Therefore we need to consider two cases as follows:

1. $rot(f)$ is irrational and $r = 1$, and
2. $rot(f)$ is rational and $r \geq 1$

In the case 1) it follows from the Herman Theorem on instability of irrational rotation numbers (see proposition 4.1.1 of [11]) that there exists an analytical diffeomorphism $\phi : S^1 \to S^1$ such that $d_r(f, \phi) < \epsilon/2$, $rot(\phi) \in A$, where $A \subset \mathbf{R}$ is the set of full Lebesgue measure (the numbers $\alpha \in A$ satisfy the Diophantine condition on approximation by rational numbers:

$$\left| \alpha - \frac{p}{q} \right| > \frac{K}{q^{2+\delta}}$$

for all rationals $\frac{p}{q}$, where K and δ are some positive constants) and moreover, there exists an analitical diffeomorphism h which conjugates ϕ to $R_{rot(\phi)}$, i.e., $\phi = h^{-1}R_\alpha h$, where $\alpha = rot(\phi)$. Take a number μ such that $d_r(h^{-1}R_\alpha h, h^{-1}R_\mu h) < \epsilon/2$. So we have $d_r(f, g) < \epsilon$. If we take the number μ to be rational, then all orbits of g are compact. So the case 1) is completed.

In the case 2) we have $rot(f) = p/q$ for some integers p, q. Therefore f^q has fixed points. Consider the lift $\overline{f}^q$ for f^q such that $\overline{f}^q \in [0; 1)$. Since f has a wandering chain recurrent point, it follows that either $\overline{f}^q(x) \geq x$ or $\overline{f}^q(x) \leq x$. We assume for definiteness that $\overline{f}^q(x) \geq x$ for all $x \in \mathbf{R}$. Consider the family $\overline{f}_\lambda(x) = \overline{f}(x) + \lambda$ with $\lambda > 0$. One has $\overline{f}^q_\lambda(x) \geq \overline{f}^q(x) + \lambda \geq x + \lambda$. Therefore $rot(\overline{f}_\lambda) > rot(f)$. Because of continuity in λ of the rotation numbers, there exists $\lambda_0 > 0$ such that $rot(f_{\lambda_0}) \in A$ and $d_r(f, f_{\lambda_0}) < \epsilon/3$. To finish the proof it remains now to use the same arguments as in the end of the proof of the case 1). ▮

We now consider an analog of the previous result for flows on the torus $\mathbf{T}^2$. Let f^t be a flow on the manifold M. A point $x \in M$ is a *chain recurrent* point of f^t if for any $\epsilon > 0$ and $T > 0$ there are finite sequences of points $x = x_0, x_1, \ldots, x_n = x$ and numbers $t_0, \ldots, t_{n-1}$ such that $t_i > T$ and $d(f^{t_i}(x_i), x_{i+1}) < \epsilon$ for all $i = 0, \ldots, n-1$. The set of chain recurrent points of f^t is a closed invariant set. Let $W(f^t)$ be the set of wandering points of f^t, and $\chi^r(\mathbf{T}^2)$, $r \geq 1$, be the space of C^r-flows on the torus $\mathbf{T}^2$ with the C^r-metric d_r.

THEOREM 2. *Let f^t be a C^r flow without fixed points on $\mathbf{T}^2$ and assume f^t has a chain recurrent wandering point. Then for any $\epsilon > 0$ there is a C^r flow g^t on $\mathbf{T}^2$ such that $d_r(f^t, g^t) < \epsilon$ and all trajectories of g^t are closed.*

Proof. Using the well known result (which follows from the Denjoy Theorem) that C^r-flows on $\mathbf{T}^2$ without fixed points have no wandering points provided $r > 1$ and $rot(f^t)$ irrational, it remains to consider two cases:

1. f^t has no closed trajectories and $r = 1$,
2. f^t has a closed trajectory and $r \geq 1$

In the case 1) the flow f^t has a closed transversal, say C, and f^t induces the forward Poincaré map $p : C \to C$, which is C^r-smooth. Since f^t has no closed trajectories, the map p has irrational rotation number. Due to Theorem 1, the diffeomorphism p can be approximated by a C^r-diffeomorphism with all orbits closed. This implies the result in the case 1).

Now we divide the case 2) into two subcases: 2a) f^t has a global transversal circle C; and 2b) f^t has no global transversal circles.

In subcase 2a), the forward Poincaré map $p : C \in C$ has rational rotation number and p satisfies the conditions of Theorem 1. Therefore this subcase is treated similarly to the case 1).

In subcase 2b), consider the set $Per(f^t)$, the union of all periodic trajectories of f^t. Since f^t has no fixed points, $Per(f^t)$ is compact. Hence there is a finite collection $\Sigma_1, \ldots, \Sigma_k$ of transversal segments such that every closed trajectory intersects exactly one transversal segment Σ_i and there are forward Poincaré maps $f_i : \Sigma_i \to \Sigma_i$, $(i = 1, \ldots, k)$. Choose a parametrization $x_i : \mathbf{R} \to \Sigma_i$ for each $1 \leq i \leq k$. Since f^t has wandering chain recurrent trajectories, the functions $f_i(x_i) - x_i$ do not reverse the sign. So there exist disjoint closed annuli $U_1, \ldots, U_k$ such that the following properties hold for each $i = 1, \ldots, k$:

1. the boundary ∂U_i is the union of two transversal circles;
2. $Per(f^t) \subset \bigcup_{i=1}^k U_i$;
3. the vector field of the flow f^t is directed inwards (resp. outwards) with respect to U_i on one of the two transversal (resp. on the other transversal boundary circle) .

Since the functions $f_i(x_i) - x_i$ do not change the sign, there is an arbitrarily small (in the C^r-topology) perturbation of the vector field of f^t in the domain $U := \bigcup_{i=1}^k (intU_i)$ (without any change outside U) such that the resulting flow g^t has no fixed points nor closed trajectories homotopic to the closed trajectories of f^t. Hence the flow g^t has a global transversal circle (any boundary circle of ∂U_i can be chosen for this). So subcase 2b) reduces to subcase 2a). ∎

3. THE STRENGTHENED CLOSING LEMMA FOR SOME SMOOTH ACTIONS AND FOLIATIONS

Recall that a group homomorphism $\rho : \mathbf{Z}^k \to Diff(S^1)$ is a C^r-*action* of $\mathbf{Z}^k$ on S^1 if the evaluation map $(\gamma, x) := \rho(\gamma)(x), \quad x \in S^1$, is C^r-

smooth ($r \geq 0$) for each $\gamma \in \mathbf{Z}^k$. The space $G^r(\mathbf{Z}^k, S^1)$ of such actions is equipped with the standard C^r-metric for finite r: a sequence ρ_n of C^r-actions converge to ρ_0 in $G^r(\mathbf{Z}^k, S^1)$ if the sequence $\rho_n(\gamma)$ converge to $\rho_0(\gamma)$ in $Diff^r(S^1)$ for any $\gamma \in \mathbf{Z}^k$.

A diffeomorphism $f \in Diff^\infty(S^1)$ is said to be an *H-diffeomorphism* if it is C^∞-conjugate to rigid rotation with irrational rotation number. If the rotation number of a C^∞-diffeomorphism f satisfy the Diophantine condition as in section 2, than f is an H-diffeomorphism (see for example, Theorem 1.2 [26]). The action $\rho \in G^\infty(\mathbf{Z}^k, S^1)$ is called an *H-action* if there is $\gamma \in \mathbf{Z}^k$ such that $\rho(\gamma)$ is an H-diffeomorphism.

Theorem 3. *Let $\rho \in G^\infty(\mathbf{Z}^k, S^1)$ be an H-action. Then for any $\epsilon > 0$ and any finite $r \in \mathbf{N}$ there is $\rho_c \in G^\infty(\mathbf{Z}^k, S^1)$ ϵ-close to ρ (in the space $G^r(\mathbf{Z}^k, S^1)$) such that all orbits of ρ_c are compact.*

Proof. By the definition of H-action there is a diffeomorphism $\rho(\gamma_0)$, $\gamma_0 \in \mathbf{Z}^k$, with irrational rotation number. Therefore $\rho(\gamma_0)$ is conjugate to a transitive rotation, i.e., there is a homeomorphism $h : S^1 \to S^1$ such that $h^{-1} \circ \rho(\gamma_0) \circ h$ is a rigid rotation. It is well known that the centralizer of transitive rigid rotations consists of rotations. Since all diffeomorphisms $\rho(\gamma)$,$\gamma \in \mathbf{Z}^k$, are mutually commutative it follows that every $\rho(\gamma)$ which is not the identity homeomorphism has no fixed points. Due to [9] (Theorem 2.1, see also section II.1 in [11]), $h^{-1} \circ \rho(\gamma) \circ h$ is a rigid rotation for every $\gamma \in \mathbf{Z}^k$. By the definition of H-action we may assume that h is a C^∞-diffeomorphism.

Since $\mathbf{Z}^k$ is finitely generated and the action $\rho^h : \gamma \to h^{-1} \circ \rho(\gamma) \circ h$, $\gamma \in \mathbf{Z}^k$, consists of rigid rotations, there is an action $\rho' \in G^\infty(\mathbf{Z}^k, S^1)$ arbitrarily close to ρ^h in the space $G^r(\mathbf{Z}^k, S^1)$ such that all orbits of ρ' are compact. Because ρ' is arbitrarily close to ρ^h and h is a C^∞-diffeomorphism, the action $\rho_c = h \circ \rho' \circ h^{-1}$ is arbitrarily close to ρ in the C^r-topology. It is easy to see that all orbits of ρ_c are compact. ∎

We now apply the previous result for foliations on $\mathbf{T}^3$. For codimension one foliations without holonomy on $\mathbf{T}^3$, a topological invariant, the so called *rotation functional*, was introduced by H.Rosenberg and R.Roussarie. The rotation functional, which is denoted as a pair of real numbers (λ, μ), describes the foliation as follows. If the numbers λ, μ are rational then all leaves of the foliation are compact (being 2-dimensional tori); if at least one of λ, μ is irrational and λ, μ are dependent over the field of rational numbers $\mathbf{Q}$ then all leaves are annuli; finally, if λ, μ are independent over $\mathbf{Q}$, both λ and μ being irrational, then all leaves are 2-planes.

It is well known that if a foliation on $\mathbf{T}^3$ has no compact leaves then it is a foliation without holonomy. As a consequence one gets that a foliation on $\mathbf{T}^3$ without compact leaves has the rotation functional (λ, μ) with at least one of λ, μ irrational.

THEOREM 4. *Let F be a C^∞-foliation without compact leaves on $\mathbf{T}^3$. Assume that at least one of the two numbers of the rotation functional of F satisfies the Diophantine condition on approximation by rationals. Then for any $r \in \mathbf{N}$ there exists a codimension one foliation F_c arbitrarily close to F in the C^r-topology such that all leaves of F_c are compact (and hence being 2-dimensional tori).*

Proof. Since the foliation F has a noncompact leaf, there is a closed simple transversal T_0 of F. As F is a foliation without holonomy, the transversal T_0 is nonhomotopic to zero. Let $\pi : \mathbf{R}^3 \to \mathbf{T}^3$ be a universal covering. Since T_0 is nonhomotopic to zero then the full preimage $\pi^{-1}(T_0)$ is a family of pairwise disjoint nonclosed curves $\overline{T_i}$, $i \in \mathbf{N}$.

Let $\overline{F}$ be a covering foliation for F. Then every $\overline{T_i}$ is a transversal curve for the foliation $\overline{F}$. Moreover, since F is the foliation without holonomy it follows that any leaf of $\overline{F}$ intersects every curve $\overline{T_i}$, i.e., $\overline{T_i}$, $i \in \mathbf{N}$, is a global section of the foliation $\overline{F}$ (see [9]). As a consequence we have that the foliation F induces the C^∞-action $\rho \in G^\infty(\mathbf{Z}^2, S^1)$. Since at least one of the two numbers of the rotation functional satisfies the Diophantine condition, it follows that ρ is an H-action. Therefore there is a C^∞-diffeomorphism $h : S^1 \to S^1$ which conjugates ρ to the action consisting of rigid rotations of S^1. The diffeomorphism h can be extended to the C^∞-diffeomorphism $\phi : \mathbf{T}^3 \to \mathbf{T}^3$ which maps F to the linear foliation F_1. It is easy to see that there exists a codimension one C^∞-foliation F_2 arbitrarily close to F_1 in the C^r-topology such that all leaves of F_2 are 2-dimensional tori. Then $F_c = \phi^{-1}(F_2)$ is the desired foliation. ▮

Remark. Theorem 4 can be generalized for codimension one foliations which are defined by the Pfaff 1-forms $dx_{n+1} = \sum_{i=1}^{n} P_i(x_1, \dots x_{n+1})dx_i$ on $\mathbf{T}^{n+1}$, $n \geq 3$.

4. CLOSING LEMMA FOR PIECEWISE DIFFEOMORPHISMS OF THE CIRCLE

In this section we prove a version of the C^r-closing lemma for piecewise diffeomorphisms of the circle (generalized interval exchange transformations) under the assumption that symbolic expansions of recurrent points are sufficiently large. Then we apply this result to C^r vector fields with finitely many singularities of saddle type on surfaces. To state the result we need to give some preliminaries.

Let $\pi : \mathbf{R} \to S^1 = \mathbf{R}/\mathbf{Z}$ be the natural projection. Fix an integer $k \geq 2$ and let $\{a_i\}_{i=1}^k$, $\{b_i\}_{i=1}^k$ be two sets of points on S^1, where the points of each set are cyclically denoted. A one-to-one map $f : S^1 - \{a_i\}_{i=1}^k \to S^1 - \{b_i\}_{i=1}^k$ will be called a C^r *piecewise diffeomorphism* if the restriction of f to each interval $I_i = (a_i, a_{i+1})$ is a C^r-diffeomorphism to its image, $r \geq 0$, $i = 1, \ldots, k$ (where $a_{k+1} = a_1$). Throughout this section we assume that $\{a_i\}_{i=1}^k$ are the points of discontinuity of f. The set of all C^r piecewise diffeomorphisms with k points of discontinuity will be denoted by $\mathcal{M}^r(k)$.

Consider a map $f \in \mathcal{M}^r(k)$. Each restriction $f|I_i$ $(i = 1, \ldots, k)$ can be easily extended by continuity to the semiclosed interval $[a_i, a_{i+1})$. Denote by $f_l : S^1 \to S^1$ the resulting map, the *left extension* of f. Obviously, if f is increasing for all I_i (or decreasing for all I_i) then f_l is also one-to-one. (In particular, if $f|I_i$ is linear with slope $+1$ for all I_i, then f_l becomes the usual interval exchange transformation.) In similar way, one can define the map f_r (the *right extension* of f). If both f_l and f_r are C^r maps on the disjoint union of the corresponding semiclosed intervals, then we will write $f \in \mathcal{M}^{r+0}(k)$.

Let us define the C^r-topology on $\mathcal{M}^{r+0}(k)$. For $f \in \mathcal{M}^{r+0}(k)$ with points of discontinuity $\{a_1, \ldots, a_k\}$, denote by $\overline{I}_i(f)$ the closed interval $[a_i, a_{i+1}]$, $i = 1, \ldots, k$. Given $\varepsilon > 0$, we define the ε-neighborhood $U_\varepsilon(f)$ of f with respect to the C^r-topology on $\mathcal{M}^{r+0}(k)$ as follows: a map $g \in \mathcal{M}^{r+0}(k)$ belongs to $U_\varepsilon(f)$ of f if there is an orientation preserving C^r-diffeomorphism $h : S^1 \to S^1$ such that h is ε-close to the identity in the C^r-topology, $h(\overline{I}_i(f)) = \overline{I}_i(g)$, $i = 1, \ldots, k$, and $g \circ h$ is ε-close to f in the C^r-topology on each $\overline{I}_i(f)$.

We define now the symbolic model for maps in $\mathcal{M}^r(k)$. Let $f \in \mathcal{M}^r(k)$ with the points a_i, b_i and the intervals I_i as before, and let $\mathcal{J}$ be the set consisting of k symbols $J_1, \ldots, J_k$. We put $A = \{a_1, \ldots, a_k\}$, $B = \{b_1, \ldots, b_k\}$ and $A^\infty = \bigcup_{i=0}^\infty f^{-i}(A)$. Then for any $x \notin A^\infty$, its forward f-orbit is well defined and one associates the *itinerary* of x as follows: $i_f(x) = (i_0(x), \ldots, i_n(x), \ldots)$, where $i_n(x) = J_i$ if $f^n(x) \in I_i$. Further we put $B^\infty = \bigcup_{i=0}^\infty f^i(B \setminus A^\infty)$. Then for any $x \notin B^\infty$, its backward f-orbit is well defined. So for any $x \notin A^\infty \bigcup B^\infty \overset{\text{def}}{=} D^\infty$, the full f-orbit $O(x) = \bigcup_{-\infty}^{+\infty} f^n(x)$ of the point x is well defined. (It is easy to see that D is at most countable).

A point $x \notin A^\infty$ (resp. B^∞) is called ω-*reccurent* (resp. α-*recurrent*) if it is contained in its ω-limit set (resp. α-limit set). A point is called *recurrent* if it is both ω- and α-recurrent. A recurrent point is called *nontrivial* if it is neither fixed nor periodic point. It is clear that if x is recurrent then every point of $O(x)$ is recurrent as well. Therefore we may speak of recurrent orbits.

Suppose $x \in I_\nu = (a_\nu, a_{\nu+1})$ is a nontrivial recurrent point. Let $q_1(r)$ (the letter r being for "right") be the minimal positive integer for which $f^{q_1(r)}(x) \in (x, a_{\nu+1})$, or else $q_1(r) = \infty$ (i.e., $q_1(r) = \infty$ when $f^i(x) \notin$

$(x, a_{\nu+1})$ for all $i > 0$). Now we define by induction: if $q_{n-1} \neq \infty$ then $q_n(r)$ is the minimal positive integer for which $f^{q_n(r)}(x) \in (x, f^{q_{n-1}(r)}(x))$, or else $q_n = \infty$. In the same way, to describe approaching of the orbit of x from the left, one can define the integers $q_n(l)$, starting with the number $q_1(l)$, which is the minimal positive integer with $f^{q_1(l)}(x) \in (a_\nu, x)$.

Given a positive integer n consider the finite block

$$B_n^r = < i_0(x), \ldots, i_{q_n(r)-1}(x) >$$

of the itinerary $i_f(x)$. Let $r_n \geq 0$ be the maximal number of successive repetitions of B_n^r in $i_f(x)$ starting with $i_{q_n(r)}(x)$ (we suppose that $r_n < \infty$, otherwise the situation is trivial). Formally, this means that $i_k(x) = i_{k+jq_n(r)}(x)$ for $0 \leq j \leq r_n$, $0 \leq k \leq q_n(r)-1$ and r_n is the maximal number for which these equalities hold. The sequence $R(x) = \{r_1(x), \ldots, r_n(x), \ldots\}$ is called the *right p-expansion of the point* x. If one replace $q_n(r)$ by $q_n(l)$ then one gets $L(x)$, the *left p-expansion* of x.

We now are in position to state the main result of this section.

THEOREM 5. *Suppose $f \in \mathcal{M}^{r+0}(k)$, $r \geq 1$, is a piecewise diffeomorphism of the circle S^1 and f is increasing on all monotonicity intervals. Let $x \in S^1$ be a nontrivially recurrent point and $L(x) = \{l_i\}_1^\infty$, $R(x) = \{r_i\}_1^\infty$ the left and right p-expansions of x respectively. If*

$$\limsup_{n\to\infty} l_n \geq 3 \qquad \text{and} \qquad \limsup_{n\to\infty} r_n \geq 3,$$

then for any neighborhood $U(f)$ of f in the C^r-topology there exists $g \in \mathcal{M}^{r+0}(k) \cap U(f)$ such that x is a periodic point for g.

Proof. Let $x \in I_\nu = (a_\nu, a_{\nu+1})$. Since x is nontrivially recurrent, we see that at least one of the intervals (a_ν, x), $(x, a_{\nu+1})$ contains infinitely many points of $O(x)$ arbitrarily close to x. To be definite, assume that it is the interval $(x, a_{\nu+1})$. First of all, remark that if there is an interval $(x, c) \subset (x, a_{\nu+1})$ such that the restriction $f^n|(x, c)$ is a homeomorphism for all $n \in \mathbb{N}$, then the Theorem holds true without any assumptions on p-expansions of x . Indeed, in this case the proof is similar to that for diffeomorphisms (see [21], [19]). So we may assume that for any interval $(x, c) \subset (x, a_{\nu+1})$ there is $n_0 \in \mathbb{N}$ such that $f^{n_0}|(x, c)$ is not a homeomorphism (i.e., $f^{n_0-1}(x, c)$ contains at least one point of discontinuity). By our assumption, the itinerary $i_f(x)$ contains infinitely many blocks of the form $\underbrace{B_n^r \ldots B_n^r}_{r_n \text{ times}}$, where

$$B_n^r = < i_0(x), \ldots, i_{q_n(r)-1}(x) >, \quad i_0(x) = i_{q_n(r)}(x) = J_\nu.$$

Hence $f^{jq_n}(x) \in (x, a_{\nu+1})$ for $0 \leq j \leq r_n$. By passing to a subsequence if necessary, one may assume that $r_n \geq 3$.

Later on, essential parts of the proof we indicate as steps.

Step 1. The restriction of f^{q_n} on each interval

$$[f^{(j-1)q_n}(x), f^{jq_n}(x)], \quad 1 \leq j \leq r_n - 1,$$

is a homeomorphism

$$[f^{(j-1)q_n}(x), f^{jq_n}(x)] \to [f^{jq_n}(x), f^{(j+1)q_n}(x)].$$

Moreover, the restriction of f to each interval

$$[f^{m+(j-1)q_n}(x), f^{m+jq_n}(x)], \quad 0 \leq m \leq q_n$$

is a homeomorphism for $1 \leq j \leq r_n - 2$.

Proof of step 1. By assumption, the points $f^m(x)$, $f^{m+q_n}(x)$ belong to the same interval of continuity of f for each $0 \leq m \leq q_n$. Hence every interval $[f^m(x), f^{m+q_n}(x)]$ contains no points of discontinuity and the restrictions

$$f\Big|\ [f^m(x), f^{m+q_n}(x)] : [f^m(x), f^{m+q_n}(x)] \to [f^{m+1}(x), f^{m+1+q_n}(x)],$$

$0 \leq m \leq q_n - 1$, are homeomorphisms. As a consequence, we have that the restriction

$$f^{q_n}\Big|\ [x, f^{q_n}(x)] : [x, f^{q_n}(x)] \to [f^{q_n}(x), f^{2q_n}(x)]$$

is a homeomorphism. By the same arguments, one can prove that the restriction

$$f^{q_n}\Big|\ [f^{m+(j-1)q_n}(x), f^{m+jq_n}(x)] :$$

$$[f^{m+(j-1)q_n}(x), f^{m+jq_n}(x)] \to [f^{m+jq_n}(x), f^{m+(j+1)q_n}(x)]$$

is a homeomorphism for every $1 \leq j \leq r_n - 1$. This completes the proof of step 1.

Step 2. The restriction of f^{q_n} to the interval $[x, f^{(r_n-1)q_n}(x)]$ is a homeomorphism

$$[x, f^{(r_n-1)q_n}(x)] \to [f^{q_n}(x), f^{r_n q_n}(x)].$$

Proof of step 2. Since f is increasing on each monotonicity interval, we have that $[f^{(j-1)q_n}(x), f^{jq_n}(x)]$ is adjoint to the interval

$$[f^{jq_n}(x), f^{(j+1)q_n}(x)], \quad 1 \leq j \leq r_n - 1.$$

Combining this fact and step 1 we conclude the proof of step 2.

Step 3. Either $f^{(r_n-1)q_n}(x) \to x$ or $f^{r_n q_n}(x) \to x$ as $n \to \infty$.

Proof of step 3. Suppose the contrary. Then there are subsequences $f^{(r_{i_n}-1)q_{i_n}}(x)$ and $f^{r_{i_n} q_{i_n}}(x)$ converging to some points $y \in (x, a_{\nu+1}]$ and $z \in (x, a_{\nu+1}]$ respectively (it may happen that $z = y$). Let us take $\delta > 0$ such that $|y-x| > 2\delta$. Then the interval $I_\delta = (x+\delta, y-\delta)$ is a proper subinterval of (x, y). We may assume that $I_\delta \cap O(x) \neq \emptyset$ for sufficiently small δ because the interval $(x, a_{\nu+1})$ contains points of $O(x)$ arbitrarily close to x. By assumption, there is $m_0 \in \mathbb{N}$ such that the restriction $f^{m_0}|I_\delta$ is not a homeomorphism. On the other hand, $f^{q_n}(x) \to x$ as $n \to \infty$, due to the definition of the numbers q_n. Hence, $f^{q_{i_n}}(x) \in (x, x+\delta)$, $f^{(r_{i_n}-1)q_{i_n}}(x) \in (y-\delta, y+\delta)$ for all n sufficiently large. According to step 1, the restriction of $f^{q_{i_n}}$ on the interval $[x, f^{(r_{i_n}-1)q_{i_n}}(x)]$ is a homeomorphism. For $q_{i_n} \geq m_0$ we get the contradiction because the interval I_δ belongs to $[f^{q_{i_n}}(x), f^{r_{i_n} q_{i_n}}(x)]$ for n sufficiently large. This contradiction concludes the proof of step 3.

Obviously, if $f^{r_n q_n}(x) \to x$, then $f^{(r_n-1)q_n}(x) \to x$ as $n \to \infty$. Thus the convergence $f^{(r_n-1)q_n}(x) \to x$ takes place. As a consequence of step 3, we get the following step.

Step 4. The lengths of the intervals

$$[x, f^{q_n}(x)], \ldots, [f^{(r_n-2)q_n}(x), f^{(r_n-1)q_n}(x)]$$

tend uniformly to zero as $n \to \infty$.

As in Pugh's proof of the C^1-closing lemma, it is enough to prove that given any neighborhood $U(f)$ (in the space $\mathcal{M}^{r+0}(k)$) and a neighborhood $V(x)$ of x on S^1 there is $g \in U(f)$ with periodic orbit through $V(x)$ (see [22], [24], [8]). Let $U(f) \subset \mathcal{M}^{r+0}(k)$ be any neighborhood and $I_\gamma = (x - \gamma, x + \gamma) \subset I_\nu$ any interval. Without loss of generality we may assume that $f(x) \notin I_\gamma$. Let $h : S^1 \to S^1$ be a C^∞ diffeomorphism such that h is equal to the identity outside of I_γ and $h|I_\gamma$ has no fixed points. To be definite, we assume that $h(z) < z$, $z \in I_\gamma$. Take h so small (in the usual C^r-topology) that $f \circ h \in U(f)$. Moreover, we may assume that there is a C^∞-isotopy $h_t : S^1 \to S^1$ such that the following holds:

1. $h_0 = id$, $h_1 = h$, and all the maps h_t $(0 \leq t \leq 1)$ are equal to the identity outside of I_γ and have no fixed points inside I_γ.
2. $h_t(z) < z$, $z \in I_\gamma$, and $f \circ h_t \in U(f)$ for $0 \leq t \leq 1$.
3. $|h_{t_1}(z) - z| \leq |h_{t_2}(z) - z|$ for all $z \in I_\gamma$ provided that $t_1 < t_2$.

Let us take an arbitrary interval $I_\alpha = (x - \alpha, x + \alpha)$, $\alpha < \gamma$. By the property 1) above, there is

$$\beta = \min_{z \in I_\alpha} |h(z) - z| > 0.$$

According to step 4, there is n_0 such that the lengths of intervals

$$[x, f^{q_n}(x)], \ldots, [f^{(r_n-2)q_n}(x), f^{(r_n-1)q_n}(x)]$$

are less than β as $n \geq n_0$. According to step 3, $f^{2q_n}(x) \in I_\alpha$ (hence, $f^{q_n}(x) \in I_\alpha$) for sufficiently large n. Fix such an n and put $q_n = q$, $f^{q_n}(x) = x_q$, $f^{2q_n}(x) = x_{2q}$. Our aim is to prove the existence of $t^* \in (0,1)$ such that $(f \circ h_{t^*})^q(x_{2q}) = x_{2q}$.

Let $f^{j_1}[x, x_{2q}]$, $\ldots$, $f^{j_s}[x, x_{2q}]$, $0 < j_1 < \ldots < j_s < q$, be the intervals such that $f^{j_k}(x_{2q}) \in I_\gamma$, $1 \leq k \leq s$, and there are no other intervals $f^j[x, x_{2q}]$, $0 < j < q$, with the above inclusion. First, let us consider the following case

$$h(x_{2q}) > x, h \circ f^{j_1} \circ h(x_{2q}) > f^{j_1}(x), \quad h \circ f^{j_2-j_1} \circ h \circ f^{j_1} \circ h(x_{2q}) > f^{j_2}(x), \ldots,$$

$$h \circ f^{j_s-j_{s-1}} \circ h \circ \cdots \circ h \circ f^{j_2-j_1} \circ h \circ f^{j_1} \circ h(x_{2q}) > f^{j_s}(x).$$

Since f preserves the orientation of intervals of continuity and due to properties 2) and 3) for h_t, the points

$$h_t(x_{2q}), h_t \circ f^{j_1} \circ h_t(x_{2q}), h_t \circ f^{j_2-j_1} \circ h_t \circ f^{j_1} \circ h_t(x_{2q}),$$

$$h_t \circ f^{j_s-j_{s-1}} \circ h_t \circ \cdots \circ h_t \circ f^{j_2-j_1} \circ h_t \circ f^{j_1} \circ h_t(x_{2q})$$

belong to the interval $[x, x_{2q}]$ for all $t \in [0,1]$. So using the fact that by our assumption h_t is equal to the identity outside of I_γ, we have

$$(f \circ h_t)^q(x_{2q}) = f^{q-j_s} \circ h_t \circ f^{j_s-j_{s-1}} \circ h_t \circ \cdots \circ h_t \circ f^{j_2-j_1} \circ h_t \circ f^{j_1} \circ h_t(x_{2q}).$$

Since $x_{2q} \in I_\alpha$, it follows that $h(x_{2q}) - x_{2q} \geq \beta$. As a consequence, $h(x_{2q}) \leq x_q$. Therefore,

$$(f \circ h)^q(x_{2q}) \leq f^{q-j_s} \circ f^{j_s-j_{s-1}} \circ h \circ \cdots \circ h \circ f^{j_1}(x_q) \leq f^q(x_q) = x_{2q}.$$

On the other hand, $(f \circ h_0)^q(x_{2q}) = f^q(x_{2q}) > x_{2q}$. Hence there is t^* such that $(f \circ h_{t^*})^q(x_{2q}) = x_{2q}$. Suppose now that $h(x_{2q}) < x$. Since $h_t(x_{2q})$ increases continuously provided that t decreases and $h_0(x_{2q}) = x_{2q} > x_q > x$, we see that there is $0 < t_1 < 1$ such that $x < h_{t_1}(x_{2q}) < x_q$. As a consequence,

$$(f \circ h_{t_1})^{j_1}(x_{2q}) = f^{j_1} \circ h_{t_1}(x_{2q}) \in [f^{j_1}(x), f^{j_1+q}(x)].$$

Note that due to the definition of the number j_1, $(f \circ h_t)^{j_1}(x_{2q}) = f^{j_1} \circ h_t(x_{2q})$ as $0 < t \leq t_1$. If $h_{t_1}[(f \circ h_{t_1}(x_{2q})] < f^{j_1}(x)$, then there is $0 < t_2 < t_1$ such that

$$f^{j_1}(x) < h_{t_2}[(f \circ h_{t_2})^{j_1}(x_{2q})] < f^{j_1+q}(x)$$

because $h_t \circ (f \circ h_t)^{j_1}(x_{2q}) = h_t \circ f^{j_1} \circ h_t(x_{2q})$ increases continuously provided that t decreases and

$$h_0 \circ (f \circ h_0)^{j_1}(x_{2q}) = f^{j_1}(x_{2q}) > f^{j_1+q}(x).$$

Hence,

$$(f \circ h_{t_2})^{j_2}(x_{2q}) = (f \circ h_{t_2})^{j_2-j_1}[f^{j_1} \circ h_{t_2}(x_{2q})] =$$

$$= f^{j_2-j_1} \circ h_{t_2}[f^{j_1} \circ h_{t_2}(x_{2q}) \in [f^{j_2}(x), f^{j_2+q}(x)].$$

Proceeding in a similar way, one gets $0 < t_s < t_{s-1} < \cdots t_1 < 1$ such that

$$(f \circ h_{t_s})^{j_s}(x_{2q}) \in [f^{j_s}(x), f^{j_s+q}(x)]$$

and moreover, $(f \circ h_{t_s})^{j_s}(x_{2q}) = f^{j_s+q}(x) = f^{j_s}(x_q)$. Then

$$(f \circ h_{t_s})^q(x_{2q}) = (f \circ h_{t_s})^{q-j_s} \circ (f \circ h_{t_s})^{j_s}(x_{2q}) = f^{q-j_s}[f^{j_s}(x_q)] = x_{2q}.$$

This completes the proof. ▮

4.1. Some applications

This subsection is concerned with applications of Theorem 5 for flows on orientable two-manifolds M_h^2 of genus $h \geq 2$. To state the results we need some notations. Let Δ be the hyperbolic (or Lobachevsky) plane regarded as the unit disk $|z| < 1$ of the complex z-plane endowed with the metric $ds = 2|dz|/(1 - |z|^2)$. Any closed orientable surface $M = M_h^2$ of genus $h \geq 2$ can be thought of as the quotient space Δ/Γ, where Γ is a finitely generated Fuchsian group of the first kind acting freely in the unit disc Δ by isometries. Let D be a geodesic polygon which is a fundamental region of Γ; it has an even number of sides which are identified in pairs according with generators $\Gamma_D \subset \Gamma$. Let N be the net of images of ∂D under Γ.

Due to [14] and [17], we label the sides of D by elements of Γ_D as follows (for more details see [25], [12]) : if the side s is identified in D with the side $\gamma_j(s)$, $\gamma_j \in \Gamma_D$, we label the side s by γ_j. Each side of N is labeled by the same generator as the corresponding side of D. Then each oriented geodesic $\overline{g} \in \Delta$ can be coded by a two-sided sequence of generators of Γ_D in accordance with crossing the successive sides of N by this geodesic. Following [25] we call such a coding the *Koebe-Morse coding*. For simplicity, we wil write $[\overline{g}] = \ldots i_{-n} \ldots i_0 \ldots i_n \ldots$ omitting the symbol γ in the Koebe-Morse coding. Given an integer n and a natural number m, we denote by $B(i_n, m)$ the finite block in $[\overline{g}]$ starting with i_n and ending at i_{n+m}.

Suppose $\overline{g}$ is a lift of a nontrivially recurrent geodesic $g \subset M$. Then any symbol of $[\overline{g}]$ occurs in $[\overline{g}]$ infinitely many times. A finite block in $[\overline{g}]$ with the same first and last symbol is called a *circle block*. Given an integer n, we denote by x_n a unique point of the intersection of $\overline{g}$ with N which corresponds to i_n in $[\overline{g}]$. For $i_0 \in [\overline{g}]$ let us define an increasing sequence of natural numbers $q_n(r)$ as follows. First, fix an orientation on all geodesics of N assuming that congruent geodesics have consistent orientation. So $x_0 \in s_0 = [a, b]$, where s_0 is some side in N with vertices a, b. Let $q_1(r)$ be the first natural number such that $x_{q_1(r)}$ is congruent to some point, say x'_{q_1}, on (x_0, b). Suppose by induction that $q_n(r)$ is the first natural number such that x_{q_n} is congruent to some point, say x'_{q_n}, on $(x_0, x'_{q_{n-1}})$. In the same way, one can define the integers $q_n(l)$ starting with the number $q_1(l)$ being the first natural such that $x_{q_1(l)}$ is congruent to some point on (a, x_0).

Take a circle block $B(i_0, q_n(r)) \subset [g(l)]$. Let $r_n(i_0) \in \mathbf{N}$ be the maximal number of successive repetitions of the block $B(i_0, q_n(r) - 1)$ in $[g(l)]$ starting with i_0. The sequence $R(i_0) := \{r_n(i_0)\}_{n=1}^{\infty}$ is called the *right p-expansion* with respect to the symbol i_0. If one replaces $q_n(r)$ by $q_n(l)$ one gets the *left p-expansion* denoted by $L(i_0)$. We say that $[\overline{g}]$ has *p-expansions of unrestricted type* if there is a symbol $i_0 \in [g]$ such that both sequences $R(i_0)$ and $L(i_0)$ are unbounded. Notice that congruent geodesics have the same Koebe-Morse coding. Hence the above definitions are well defined for geodesics on M.

Following [5], to each point of the circle at infinity $S_\infty = \{z \in \mathbf{C} : |\mathbf{z}| = \mathbf{1}\}$ one associates the so called *f-expansion*. It allows one to represent a geodesic $\overline{g} \subset \Delta$ as a two sided sequence of symbols $\{j_n\}_{-\infty}^{+\infty}$ by juxtaposing the f-expansions of their ideal endpoints. We call the sequence $\{j_n\}_{-\infty}^{+\infty}$ the *Bowen-Series expansion* of $\overline{g}$. As above, one can assign the both right and left p-expansions $R_n(j_0)$, $L_n(j_0)$ to each symbol j_0 in such a representation.

As a consequence of Theorem 5, we have the following result.

THEOREM 6. *Let $X \in \chi^r(M_h^2)$, $1 \le r \le \infty$, $h \ge 2$, be a vector field and $E(X)$ the number of its singularities of saddle type, $E(X) < \infty$. Let λ be a nontrivially recurrent trajectory of X through a point $m \in \lambda$, $g = g(\lambda)$ be the geodesic corresponding to λ and $[g]$ its Koebe-Morse coding. If there is $i_0 \in [g]$ with*

$$\limsup_{n\to\infty} r_n(i_0) \ge 3E(X) + 1, \quad \limsup_{n\to\infty} l_n(i_0) \ge 3E(X) + 1, \tag{1}$$

then there exists $Y \in \chi^r(M_h^2)$ arbitrarily close to X in the C^r-topology such that Y has a periodic trajectory through m.

Proof. It follows from the existence of the nontrivially recurrent trajectory that there is a closed transversal C. Since all singularities of X are of saddle type, the Poincaré forward map f induced on C by X is a C^r piecewise diffeomorphism. Due to inequality (1), f satisfies the conditions of Theorem 5. Hence there exists a functional rotation of X along C which yields a closing. ▮

To state further consequences of Theorem 5 we need the term of *corresponding geodesic*. Let λ be a nontrivially recurrent trajectory. A geodesic g is called the corresponding geodesic for λ and denoted $g = g(\lambda)$ if $g(\lambda)$ and λ have the same asymptotic directions. Note that using closed transversals for a flow, one can introduce a coding for λ which corresponds in principle to the coding of $g(\lambda)$ (see [3] for details).

Now Theorem 6 immediately implies

THEOREM 7. *Let $X \in \chi^r(M_h^2)$, $1 \le r \le \infty$, $h \ge 2$, be a vector field with finitely many singularities of saddle type, and let λ be a nontrivially recurrent trajectory of X through a point $m \in \lambda$. Then there exists $Y \in \chi^r(M_h^2)$ arbitrarily close to X in the C^r-topology such that Y has a periodic trajectory through m provided that the Koebe-Morse coding of $g(\lambda)$ has unbounded p-expansions, where $g(\lambda)$ is the corresponding geodesic of λ.*

In terms of Bowen-Series expansions, one also has

THEOREM 8. *Let $X \in \chi^r(M_h^2)$, $1 \le r \le \infty$, $h \ge 2$, be a vector field with finitely many, say $E = E(X)$, singularities of saddle type and let λ be a nontrivially recurrent trajectory of X through a point m. Suppose that the Bowen-Series expansion of the geodesic g corresponding to λ has a symbol s_0 with the following properties:*

$$\limsup_{n\to\infty} R_n(s_0) \ge 3E+1, \quad \limsup_{n\to\infty} L_n(s_0) \ge 3E+1. \tag{2}$$

Then there exists $Y \in \chi^r(M_h^2)$ arbitrarily close to X in the C^r-topology such that Y has a periodic trajectory through m.

Proof. Due to Theorem I [25], this result reduces to Theorem 6 because (2) implies (1). ▮

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