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On the Distribution and Number of Limit Cycles for Quadratic Systems with two Foci

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In this paper, we study the distribution and number of limit cycles for quadratic systems with two foci. It is proved that a quadratic system with two foci has at most one limit cycles around one of the two foci, and hence the limit cycles of the quadratic system with two foci must be $(0,1)$–distribution or $(1,i)$-distribution $(i = 0, 1, 2, \ldots)$

Key Words: Limit cycles, quadratic systems.

1. INTRODUCTION

The main goal of this paper is to prove the following result.

THEOREM 1. *A quadratic system having two foci has at most one limit cycle surrounding one of its two foci.*

This theorem has been proved by the author in several papers published in Chinese. Here, by first time, we present its complete proof in English.

2. PRELIMINARY RESULTS

This paper discusses quadratic systems with two foci, without loss of generality, we may suppose that $O(0,0)$ and $N(0,1)$ are both foci, then by [1], the quadratic system can be written in the form

$$\begin{aligned} \frac{dx}{dt} &= -y + \delta x + lx^2 + mxy + y^2 = p(x,y), \\ \frac{dy}{dt} &= x(1 + ax + by) = Q(x,y), \quad 1 + b < 0, \end{aligned} \tag{1}$$

without loss of generality, we can suppose $a \geq 0$.

Since $O(0,0)$ and $N(0,1)$ are both foci, $|\delta| < 2$ and $(m+\delta)^2+4(b+l) < 0$. The coordinates (x,y) of the other finite singular points of (1) are determined by the following equations.

$$\begin{cases} (lb^2+a(a-mb))x^2+(a(b+2)+b(b\delta-m))x+(1+b)=0, \\ y=-(1+ax)/b. \end{cases} \tag{2}$$

The equation which determines the singular points at infinity of system (1) is

$$\phi(\lambda)=a\lambda^3+(b-l)\lambda^2-m\lambda-1=0. \tag{3}$$

Since we study the number of limit cycles around one of the two foci for system (1), first we consider the problem of concentric distribution of the limit cycles for a quadratic system (1).

The following results, which we state as lemmas, are proved in [4]. Since the proofs are easy, here the proofs are omitted.

LEMMA 2. *If $ma(b+2l) \geq 0$, then the limit cycles of system (1) must be concentric; if $ma(b+2l) \geq 0$ and $m \neq 0$, $-\delta/m \geq -1/b$ ($\leq$), then system (1) has no limit cycles around the singular point $O(0,0)$ ($N(0,1)$).*

LEMMA 3. *If $a-mb \leq 0$, then the limit cycles of system (1) must be concentric; if $a-mb \leq 0$ and $m+\delta > 0$ ($\delta < 0$) system (1) has no limit cycles around the singular point $O(0,0)$ ($N(0,1)$).*

LEMMA 4. *If system (1) satisfies one of the following conditions:*

(i) $(b+2l)\lambda+m < 0(> 0)$, $m < 0(> 0)$,
(ii) $b+2l+m\lambda(l-a\lambda) < 0(> 0)$, $m < 0(> 0)$,

then the limit cycles of system (1) must be concentric; and if system (1) satisfies also the condition $-\delta/m \geq \max(y_0,-1/b)(\leq)$, then system (1) has no limit cycle around the singular point $O(N)$, where λ is maximum positive real root of equation (3) and

$$y_0=-(\lambda^2-\delta\lambda+1)/\lambda(2l\lambda-2a\lambda^2-b\lambda+m).$$

If $lb^2+a(a-mb) > 0$, then system (1) has other two finite singular points in addition to the two foci O an N. Let the points $M_1(x_l,y_1)$ and $M_2(x_2,y_2)$ ($x_1 < 0 < x_2$) denote these two singular points whose coordinates satisfy system (2).

LEMMA 5. *If $lb^2+a(a-mb) > 0$, and $div(P,Q)_{|M_1} \cdot div(P,Q)_{|M_2} > 0$, then the limit cycle of system (1) must be concentric.*

Next we introduce the following lemmas:

LEMMA 6. *Suppose that the system*

$$\begin{cases} \dfrac{dx}{dt} = y - F(x), \\ \dfrac{dy}{dt} = -g(x), \quad F(x) = \displaystyle\int_0^x f(s)ds \end{cases} \tag{4}$$

satisfies the following conditions:

(i) $g(x) \in \mathcal{C}^1$ *and* $f(x) \in \mathcal{C}^1$ *for* $x \in (x_{02}, x_{01})$, $xg(x) > 0$ *for* $x \in (x_{02}, x_{01})$ *and* $x \neq 0$, *where* $x_{02} < 0 < x_{01}$*;*
(ii) The system

$$\begin{cases} F(x_1) &= F(x_2), \\ \dfrac{f(x_1)}{g(x_1)} &= \dfrac{f(x_2)}{g(x_2)}, \end{cases} \tag{5}$$

has no solution in the region $D_1 = \{(x_1, x_2) \mid x_{02} < x_2 < 0,\ 0 < x_1 < x_{01}\}$.

Then, system (4) has no limit cycles in the strip $x_2 < x < x_{01}$.

LEMMA 7. *Suppose that system (4) satisfies condition (i) of Lemma 6, and additionally:*

(ii) There exists an $x_0 \in (x_{02}, 0)$, *such that* $f(x) < 0 (> 0)$ *for* $x \in (x_{02}, x_0)$ *and* $f(x) > 0 (< 0)$ *for* $x \in (0, x_{01})$.

(iii) $\left\{\dfrac{f(x)}{g(x)}\right\}' < 0\,(> 0)$ *for* $x \in (x_{02}, x_0)$ *and* $x \in (0, x_{01})$.

Then, system (1) has at most one limit cycle in the strip $x_{02} < x < x_{01}$.

Lemmas 6 and 7 follow from Theorem 5.4 and Theorem 6.4 of [3], respectively.

LEMMA 8. *Suppose that system (4) satisfies condition* (i) *of Lemma 7, and additionally:*

(ii) There exists an x_0 *with* $x_{02} < x_0 < 0$, *such that* $(x - x_0)f(x) > 0$ *for* $x \in (x_{02}, x_{01})$ *and* $x \neq x_0$.
(iii) The system

$$\begin{cases} \dfrac{1}{k} F(x_1) &= F(x_2), \\ \dfrac{1}{k} \dfrac{f(x_1)}{g(x_1)} &= \dfrac{f(x_2)}{g(x_2)} \quad k \geq 1, \end{cases} \tag{6}$$

has at most one solution in the region $D_2 = \{(x_1, x_2) \mid 0 < x_1 < x_{01}, x_{02} < x_2 < \Delta_0\}$ *(where* Δ_0 *with* $\Delta_0 < x_0 < 0$ *is unique zero of* $F(x)$*). Then, system (4) has at most one limit cycle in the strip* $x_{02} < x < x_{01}$.

Proof. We make Filippov's transformation as follows.

Let $z = G(x) = \int_0^x g(s)ds$, $x_{02} < x < x_{01}$, $z_{0i} = G(x_{0i})$, $(i = 1, 2)$ and denote the inverse function of $z = G(x)$ by $x_i(z)$, for $(-1)^{i+1}x \geq 0$.

System (4) is equivalent to the following two equations for $x \geq 0$ and $x \leq 0$, respectively:

$$\frac{dz}{dt} = F_i(z) - y, \quad 0 \leq z \leq z_{01},$$

where $F_i(z) = F(x_i(z))$, $(i = 1, 2)$.

In order to prove this lemma, by Theorem 4.10 of [4] it is sufficient to show that

$$H_k'(z) < F_2'(u) \text{ when } H_k(z) = F_2(u),\ G(\Delta_0) < u \leq z, \tag{7}$$

where $H_k(z) = (1/k)F_1(k^2 z)$.

A calculation shows that (7) is equivalent to

$$\frac{1}{k}\frac{f(x_1)}{g(x_1)} < \frac{f(x_2)}{g(x_2)} \text{ when } \frac{1}{k}F(x_1) = F(x_2),\ G(\Delta_0) < u = G(x_2) < z = G(x_1).$$

It is easy to see by condition (*iii*) of Lemma 8 that (7) holds. So, the proof is complete. ∎

For the equivalent equations of (4)

$$\begin{cases} \dfrac{dx}{dt} &= u, \\ \dfrac{du}{dt} &= -g(x) - f(x)u, \end{cases} \tag{8}$$

we have the following lemmas.

Lemma 9. *Suppose that system (8) satisfies the following conditions:*

(i) $g(x) \in \mathcal{C}^1$ *and* $f(x) \in \mathcal{C}^1$ *for* $x \in (x_{02}, x_{01})$, $xg(x) > 0$ *for* $x \in (x_{02}, x_{01})$ *and* $x \neq 0$, *where* $x_{02} < x < x_{01}$.

(ii) There exists a $x_0 \in (0, x_{01})$ *(or* $x \in (x_{02}, 0)$*) such that* $(x - x_0)f(x) > 0$ *for* $x \in (x_{02}, x_{01})$ *and* $x \neq x_0$.

(iii) The simultaneous equations

$$\int_0^{x_1} f(s)ds = \int_0^{x_2} f(s)ds \tag{9}$$

and

$$\frac{f(x_1)}{g(x_1)} = \frac{f(x_2)}{g(x_2)} \tag{10}$$

have at most one solution in the region $D = \{(x_1, x_2) \mid x_{02} < x_2 < 0, x_0 < x_1 < x_{01}\}$ *(or* $D' = \{(x_1, x_2) \mid x_{02} < x_2 < x_0, 0 < x_1 < x_{01}\}$*) (We denote this solution by* $x_2 = x_{20}$*,* $x_1 = x_{10}$ *with* $x_{02} < x_{20} < 0$ *and* $0 < x_{10} < x_{01}$*).*

(iv) The function

$$\int_0^x f(s)ds \frac{f(x)}{g(x)}$$

is monotone increasing in the interval (x_{10}, x_{01}) *(for the case* $x_0 \in (x_{02}, 0)$*,* $F(x_{02}+0) \le F(x_{01}-0)$ *is also satisfied), or the function*

$$\int_0^x f(s)ds \frac{f(x)}{g(x)}$$

is monotone decreasing in the interval (x_{02}, x_{20}) *(for the case* $x_0 \in (0, x_{01})$*,* $F(x_{01}-0) \le F(x_{02}+0)$ *is also satisfied) where* $F(x) = \int_0^x f(s)ds$*.*

Then system (8) has at most one limit cycle in the strip $x_{02} < x < x_{01}$*.*

Proof. Let $x = x$, $y = u + \int_0^x f(s)ds$, then (4) is changed into (8). We prove only the case $x_0 \in (0, x_{01})$ (the proof for the case $x_0 \in (x_{02}, 0)$ is similar).

Suppose that the system has the periodic orbit L, then the subarcs of L in $x > x_{10}$, $x < x_{20}$, $x_{20} < x < 0$, $0 < x < \Delta_0$ and $\Delta_0 < x < x_{10}$ are denoted by Lx_{10}^+, Lx_{20}^-, $Lx_{20}0$, $L0\Delta_0$ and $L\Delta_0 x_{10}$, respectively (where Δ_0 is the unique real zero of $\int_0^x f(s)ds$). By Theorem 4.11 of [4], (5,15) and (5,16) of [4], we have

$$\int Lx_{10}^+ \cup Lx_{20}^- f(x)dt > 0.$$

From Lemma 4.1 of [4], it follows that

$$\int L0\Delta_0 f(x)dt > 0.$$

By Lemma 4.3 of [4], (E4) (taking $k = l$) in page 285 of [4] and the conditions of this lemma (see (4.62) of [4])), we have

$$\int Lx_{20}0 \cup L\Delta_0 x_{10} f(x)dt > 0.$$

Therefore $\oint_L f(x)dt > 0$. The proof is complete. ∎

In what follows we shall need the following two lemmas. Lemma 10 is equivalent to Lemma 9, and Lemma 11 follows from Lemma 6.

LEMMA 10. *If system (8) satisfies the following conditions:*

(i) $g(x) \in \mathcal{C}^1$ and $f(x) \in \mathcal{C}^1$ for $x \in (\bar{x}_{02}, \bar{x}_{01})$, there exists an $\bar{a} \in (\bar{x}_{02}, \bar{x}_{01})$, such that $(x-\bar{a})g(x) > 0$ for $x \in (\bar{x}_{02}, \bar{x}_{01})$, and $x \neq \bar{a}$.

(ii) There exists a $\bar{x}_0 \in (\bar{a}, \bar{x}_{01})$ (or $\bar{x}_0 \in (\bar{x}_{02}, \bar{a})$) such that $(\bar{x}-\bar{x}_0)f(x) > 0$ for $x \in (\bar{x}_{02}, \bar{x}_{01})$ and $x \neq \bar{x}_0$.

(iii) The simultaneous equations

$$\int_{\bar{a}}^{x_1} f(s)ds = \int_{\bar{a}}^{x_2} f(s)ds \tag{11}$$

and (10) have at most one solution in the region $\bar{D} = \{(x_1, x_2) \mid \bar{x}_{02} < x_2 < \bar{a}, \bar{x}_0 < x_1 < \bar{x}_{01}\}$ (or $\bar{D}' = \{(x_1, x_2) \mid \bar{x}_{02} < x_2 < \bar{x}_0, \bar{a} < x_1 < \bar{x}_{01}\}$). (We denote this solution by $x_2 = x_{20}$, $x_1 = x_{10}$ with $\bar{x}_{02} < \bar{x}_{20} < \bar{a}$ and $\bar{a} < \bar{x}_{10} < x_{01}$).

(iv) The function

$$\int_{\bar{a}}^{x} f(s)ds \frac{f(x)}{g(x)}$$

is monotone increasing in the interval $(\bar{x}_{10}, \bar{x}_{01})$ (for the case $\bar{x}_0 \in (\bar{x}_{02}, \bar{a})$, $F(x_{02}+0) \le F(x_{02}-0)$ is also satisfied); or the function

$$\int_{\bar{a}}^{x} f(s)ds \frac{f(x)}{g(x)}$$

is monotone decreasing in the interval $(\bar{x}_{02}, \bar{x}_{20})$ (for the case $\bar{x}_0 \in (\bar{a}, \bar{x}_{01})$, $F(x_{01}-0) \le F(x_{02}+0)$ is also satisfied), where $F(x) = \int_{\bar{a}}^{x} f(s)ds$.

Then, system (8) has at most one limit cycle in the strip $\bar{x}_{02} < x < \bar{x}_{01}$.

LEMMA 11. *Suppose that system (8) (satisfies conditions (i) and (ii) ($\bar{a} = -\lambda$ or 0) of Lemma 10, and additionally:*

(iii) The simultaneous equations (11) and (10) have no solution in the region $\bar{D}$ (or $\bar{D}'$).

Then, system (8) has no limit cycles in the strip $\bar{x}_{02} < x < \bar{x}_{01}$.

Finally, for system (1), doing the series of regular transformations

$$\begin{aligned} &\bar{x} = x - \lambda y,\ \bar{y} = y; \\ &\xi = -(-\delta+\lambda)\bar{x} + (l - a\lambda)\bar{x}^2 + h(\bar{x})\bar{y},\ \bar{x} = x; \\ &u = \frac{\xi}{|h(x)|^r} = \bar{x},\ x = \bar{x},\ d\tau = |h(\bar{x})|^r dt; \end{aligned} \tag{12}$$

if τ denotes t again, then the system is reduced to

$$\begin{cases} \dfrac{dx}{dt} &= u, \\ \dfrac{dy}{dt} &= -g(x) - f(x)u. \end{cases} \tag{13}$$

Let $y = u + F(x)$, $x = x$, then system (13) becomes

$$\begin{cases} \dfrac{dx}{dt} &= y - F(x), \\ \dfrac{dy}{dt} &= -g(x), \end{cases} \tag{14}$$

Here λ is a positive root of (3), and

$$\begin{aligned}
r &= \frac{2l\lambda - a\lambda^2 + m}{2l\lambda - 2a\lambda^2 - b\lambda + m}, \\
g(x) &= x(x+\lambda)\bar{g}(x)/\lambda h(x) \cdot |h(x)|^{2r}, \\
F(x) &= \textstyle\int_0^x f(\xi)d\xi, \\
f(x) &= \bar{f}(x)/\lambda h(x) \cdot |h(x)|^r, \\
\bar{g}(x) &= -(\lambda^2 - \delta\lambda + 1) + (2l\lambda - 2a\lambda^2 - a + m - \delta b)x \\
&\quad -(b(l - a\lambda) + \tfrac{a}{\lambda})x^2 \\
&= -\frac{a\lambda + b}{\lambda^2}x(x+\lambda) - \frac{b}{\lambda}(x - \frac{\lambda}{b})h(x), \\
\bar{f}(x) &= \lambda\delta(\lambda^2 - \delta\lambda + 1) + ((b + 2l)\lambda + 2m\delta\lambda + 2\delta - b\lambda^2\delta - m\lambda^2)x \\
&\quad +((b + 2l) + m\lambda(l - a\lambda))x^2 \\
&= ((b + 2l + m\lambda(l - a\lambda))x + \delta(\lambda^2 - \delta\lambda + 1))(x + \lambda) \\
&\quad -(m + \delta)((b + 1)\lambda - (m + \delta)\lambda - 1)x \\
&= \frac{(b + 2l)\lambda + m}{\lambda}x(x + \lambda) + (mx - \delta\lambda)h(x), \\
h(x) &= -(\lambda^2 - \delta\lambda + 1) + (2l\lambda - 2a\lambda^2 - b\lambda + m)x.
\end{aligned} \tag{15}$$

For convince of the subsequent statement, let

$$\begin{aligned}
r_0 &= (b + 2l)\lambda + m, \quad r_1 = 2l\lambda - a\lambda^2 + m, \\
r_2 &= 2l\lambda - 2a\lambda^2 - b\lambda + m, \quad r_3 = 2l\lambda - 3a\lambda^2 - 2b\lambda + m, \\
\nu_0 &= (\lambda^2 - \delta\lambda + l)/r_2, \\
g_0 &= b(l - a\lambda) + (a/\lambda), \\
g_1 &= 2l\lambda - 2a\lambda^2 - a + m - \delta b, \\
f_1 &= b + 2l + m\lambda(l - a\lambda) = (r_0/\lambda) + mr_2, \\
f_0 &= -m(\lambda^2 + m\lambda + 1), \\
B_0 &= (b + 1)\lambda^2 - (m + \delta)\lambda - 1, \\
A_0 &= \lambda^2 - \delta\lambda + 1, \\
H &= (a\lambda^2 + b\lambda)r_1 f_1^2/g_0, \\
c_1 &= \tfrac{1}{\lambda^2}(a\lambda^3 - m\lambda(1 + m\lambda)(a\lambda^2 + b\lambda) + (m\lambda)^2(a\lambda^2 + b\lambda)^2).
\end{aligned} \tag{16}$$

In order to prove Theorem 1, we consider separately the following four cases:

(i) $lb^2 + a(a - mb) > 0$,
(ii) $lb^2 + a(a - mb) = 0$,
(iii) $lb^2 + a(a - mb) < 0$, $\delta(m + \delta) \neq 0$,
(iv) $\delta(m + \delta) = 0$.

The proofs of the theorem in the case (i) (ii), (iii) and (iv) are presented in section 2, 3, 4 and 5 respectively.

2. $\mathbf{lb^2 + a(a - mb) > 0}$

The results of this section are proved initially in [4].

THEOREM 12. *If system (1) satisfies the conditions:*

(i) $lb^2 + a(a - mb) > 0$,
(ii) $\mathrm{div}(P, Q)_{|O},\ \mathrm{div}(P, Q)_{|N} \geq 0$.

Then, system (1) has at most one limit cycle around one of its two foci.

Proof. Obviously condition (ii) of Theorem 12 is equivalent to assuming that

$$m\delta < 0,\ \delta(m + \delta) \geq 0,\ \text{or } m\delta \geq 0.$$

We prove only the case $m < 0$ (the proof for the case $m > 0$ is similar).

Since $\varphi(-b/a) = -(1/a^2)(lb^2 + a(a - mb)) < 0$, it exists a λ_1 such that $\varphi(\lambda_1) = 0$, $\varphi'(\lambda_1) > 0$ and $a\lambda_1 + b > 0$. In transformation (12), we take $\lambda = \lambda_1$(for simplicity λ_1 is rewritten as λ), and verify that system (14) satisfies the conditions of Lemma 7. From Lemmas 2 and 4, and (16) we may assume

$$a > 0,\ b + 2l > 0,\ a\lambda + b > 0,\ r_0 > 0,\ f_1 > 0,\ B_0 < 0.$$

We consider separately two cases.

Case (a): $m + \delta \geq 0$, $\delta > 0$. A calculation shows that

$$\begin{array}{llllll} r_2 & = & (B_0 - A_0)/\lambda < 0, & \nu_0 - (-\lambda) & = & B_0/r_2 > 0, \\ r_3 & = & -\varphi'(\lambda) < 0, & g_0 & = & (lb^2 + a(a - mb))/(a\lambda + b) > 0, \\ \bar{g}(-\lambda) & = & -(1 + b)B_0 < 0, & \bar{g}(0) & = & -A_0 < 0, \\ \bar{g}(\nu_0) & = & -(a\lambda + b)B_0A_0/\lambda^2 r_2^2 > 0, & \bar{f}(0) & = & \lambda\delta A_0 > 0, \\ \bar{f}(-\lambda) & = & \lambda(m + \delta)B_0 \leq 0, & \bar{f}(\nu_0) & = & r_0A_0B_0/\lambda r_2^2 < 0. \end{array} \tag{17}$$

From (17) it follows that $\bar{g}(x)$ has two real zeros which are denoted by x_1', x_2' $(x_2' < x_1')$.

We denote the real zeros of $f(x)$ by x_1^0, x_2^0, where $x_2^0 < x_1^0$. From (17) we have

$$x_2^0 < \nu_0 < x_1^0 < 0.$$

Since a limit cycle of system (1) (i.e. system (14)) surrounding O cannot surround any other singular point (see for instance [3]), we may assume $x_1' < x_1^0$ (otherwise, in the strip $x_1' < x < +\infty$ the divergence of system (14) has constant sign. Thus, system (14) has no limit cycles surrounding O. Moreover, from (17), the relationships between $-\lambda$, x_1', x_2, x_1^0, ν_0 and 0 are as follows:

$$-\lambda < x_2' < \nu_0 < x_1' < x_1^0 < 0. \tag{18}$$

Since a limit cycle of system (1) (i.e. system (14)) which surrounds O cannot surround any other singular point, it is sufficient to show that system (14) satisfies the hypothesis of Lemma 7 with $x_{02} = x_1'$, $x_{01} = +\infty$.

By (18), conditions (i) and (ii) of Lemma 7 are obviously satisfied. Next, we verify that condition (iii) (parenthesis interior) in Lemma 7

$$\left[\frac{f(x)}{g(x)}\right]' > 0,\ x \in (x_1^1, x_1^0) \cup (0, +\infty) \tag{19}$$

is satisfied.

In fact, a calculation shows that

$$\left[\frac{f(x)}{g(x)}\right]' = |h(x)|^r \frac{[M(x)\bar{g}(x)h(x) + x(x+\lambda)\bar{f}(x)W_2(x)]}{x^2(x+\lambda)\bar{g}^2(x)h(x)}, \tag{20}$$

where

$$\begin{aligned} M(x) &= (m+\delta)B_0x^2 - \delta A_0(x+\lambda)^2, \\ W_2(x) &= r_1\bar{g}(x) - h(x)\bar{g}'(x). \end{aligned}$$

Therefore, in order to prove the theorem it is sufficient to show that for $x \in (x_1^1, x_1^0) \cup (0, +\infty)$, $M(x) < 0$ and $W_2(x) < 0$.

From the conditions $m + \delta \geq 0$, $\delta \geq 0$, it is clear that $M(x) < 0$. From

$$W_2(x_1') = -h(x_1')\bar{g}'(x_1') < 0$$

and

$$W_2'(x) = (a\lambda^2 + b\lambda)\bar{g}'(x) + 2g_0h(x) < 0 \text{ for } x > x_1',$$

it follows that

$$W_2(x) < 0,\ x \geq x_1'.$$

This proves the theorem in Case (a).

Case (b): $\delta \leq 0$. Let $y - 1 = y'$, $x = x'$, then system (1) changes into:

$$\begin{cases} \dfrac{dx'}{d\tau} &= y' + (m+\delta)x' + l'x'^2 + y'^2 \\ \dfrac{dy'}{d\tau} &= x'(1 + b + ax' + by'). \end{cases}$$

Let $y' = -y$, $x' = -x/\sqrt{-(1+b)'}$ $\tau = -\sqrt{-(1+b)}t$, then system (1) becomes:

$$\begin{cases} \dfrac{dx'}{d\tau} &= -y + \delta' x + m'xy + l'x^2 + y^2 \\ \dfrac{dy}{d\tau} &= x(1 + a'x + b'y), \end{cases} \tag{21}$$

where

$$a' = \frac{1}{(-(1+b))^{3/2}},\ b' = \frac{-b}{1+b}, \qquad \delta' = \frac{-(m+\delta)}{\sqrt{-(1+b)}},$$
$$m' = \frac{m}{\sqrt{-(1+b)}}, \quad l' = \frac{l}{-(1+b)}.$$

Obviously, $\delta' > 0$, $m' = m/\sqrt{-(1+b)} < 0$, $\delta' + m' = -\delta/\sqrt{-(1+b)} \geq 0$, $1 + b' = 1/1 + b < 0$. Therefore, from the proof of Case (a), it follows that system (21) (and hence (1)) has at most one limit cycle surrounding the singular point $O(N(0,1))$. This completes the proof of the theorem. ▮

We need the following definition.

Definition 13. For a planar ordinary differential system $\frac{dX}{dt} = X(x,y)$, $\frac{dY}{dt} = Y(x,y)$, if the separatrices L_N^+ and L_N^- passing through a saddle (say N) intersect respectively a ray starting from a focus O at P_1 and P_2 (or L_N^+ and L_N^- reach the infinite singular points u_1 and u_2, respectively) such that $L_{NP_1}^+$ and $L_{P_2N}^-$ and the segment P_1P_2 (or L_N^+, L_N^- and a portion of the equator of the Poincaré sphere between u_l and u_2) forms a closed region containing only the singular point O, then the limit cycles surrounding the singular point O are said to be determined by the saddle N.

Lemma 14. *If system (1) satisfies condition* (i) *of Theorem 12, and additionally*

$$\operatorname{div}(P,Q)_{|O} \cdot \operatorname{div}(P,Q)_{|N} < 0,$$

then the limit cycles around $O(N)$ are determined by $M_1(x_1, y_1)$ $(M_2(x_2, y_2))$ at which the sign of the divergence of system (1) is opposite to that at $O(N)$.

Proof. We prove only the case $m < 0$ (the proof for the case $m > 0$ is similar). This will follow from the conditions $m + \delta < 0$, $\delta > 0$ of the lemma. Moreover, from Lemmas 2, 4 and 5 we need only to do the proof under the following conditions:

$$\begin{aligned} &b + 2l > 0,\ (b + 2l)\lambda + m > 0,\ b + 2l + m\lambda(l - a\lambda) > 0, \\ &\mathrm{div}(P, Q)_{|M_1} \cdot \mathrm{div}(P, Q)_{|M_2} < 0. \end{aligned} \tag{22}$$

Thus, the intersection point of the line $h_1 : y = -((b + 2l)/m)x - \delta/m$ and the isocline $1 + ax + by = 0$ is located between M_1 and M_2, the intersection point of the line h_1 and the y-axis is located between O and N. Hence

$$\mathrm{div}(P, Q)_{|M_2} \cdot \mathrm{div}_{|N} < 0, \quad \mathrm{div}(P, Q)_{|M_1} \cdot \mathrm{div}(P, Q)_{|O} < 0.$$

Moreover, from (14), (17) and (18), it is easy to see that the relationships between $-\lambda$, x_2', x_1', ν_0, x_2^0, x_1^0 is as follows:

$$-\lambda < x_2^0 < x_2' < \nu_0 < x_1' < x_1^0 < 0.$$

Therefore, the limit cycle of (14) surrounding $O(N')$ are determined by the saddle $M_1'(M_2')$ at which the divergence of system (14) is opposite to that at $O(N')$. This completes the proof of the lemma. ▌

THEOREM 15. *If system (1) satisfies condition* (i) *of Theorem 12, and in additionally*

$$\mathrm{div}(P, Q)_{|O} \cdot \mathrm{div}(P, Q)_{|N} < 0,$$

then system (1) has at most one limit cycle around one of its two foci.

Proof. From Lemma 14, the limit cycle surrounding the singular point $O(N)$ determined by the saddle $M_1(x_1, y_1)$ $(M_2(x_2, y_2))$ at which the sign of the divergence of system (1) is opposite to that of $O(N)$, and

$$\mathrm{div}(P, Q)_{|M_2} \cdot \mathrm{div}(P, Q)_{|O} > 0.$$

By the regular transformations, fixing the focus $O(0, 0)$ and moving the saddle $M_2(x_2, y_2)$ to $\bar{M}_2(0, 1)$, system (1) changes into (for more details see [1])

$$\begin{aligned} \frac{dx}{dt} &= -y + \delta' x + l' x^2 + m' xy + y^2 = p'(x, y), \\ \frac{dy}{dt} &= x(1 + a' x + b' y) = Q'(x, y), \quad 1 + b' > 0. \end{aligned} \tag{23}$$

Without loss of generality, suppose $a' > 0$. For simplicity δ', l', m', a' and b' are rewritten respectively as δ, l, m, a and b in the subsequent proofs. It is clear that the saddle M_1 and the focus N of system (1) become the saddle $\bar{M}_1(\bar{x}_1, \bar{y}_1)$ and the focus $\bar{N}(\bar{x}_2, \bar{y}_2)$ of system (23), respectively.

Obviously the singular points O, $\bar{M}_2$, $\bar{M}_1$ and $\bar{N}$ form still a convex quadrilateral. Moreover, by Berlinskii's Theorem [2], it is easy to see that the singular points $\bar{M}_1, \bar{N}$ are both on the right (or left) hand side of the y–axis, and hence $lb^2 + a(a - mb) > 0$ by (2).

If $\bar{N}$ and $\bar{M}_1$ are on the right hand side of the y–axis, then according to the tends of the trajectories of (23) crossing the line $O\bar{N}$ and the isocline $1 + ax + by = 0$, we can see that the singular point $\bar{N}$ is not a focus. This is contrary to the hypothesis. Therefore, under the conditions of the theorem, the focus $\bar{N}(\bar{x}_2, \bar{y}_2)$ and the saddle $\bar{M}_1(\bar{x}_1, \bar{y}_1)$ must be both on the left hand side of the y–axis and $\bar{y}_1 < \bar{y}_2$.

Moreover, from Lemma 14 and the proof of Lemma 14 it follows that the limit cycles of system (23) surrounding the point $O(\bar{N})$ are determined by the saddle $\bar{M}_1(\bar{x}_1, \bar{y}_1)(\bar{M}_2(0, 1))$ at which the sign of the divergence of system (23) is opposite to that of the point $O(\bar{N})$. Since the straight line $h_2 : y = kx - 1$ (where k is a positive real root of the equation $(1+b)(1/k)^2 - (m + \delta)(1/k) - 1 = 0$) has no contact points of system (23), it must cross the isocline $1 + ax + by = 0$ between $\bar{M}_1$ and $\bar{N}$. A calculation shows that

$$\frac{dh_2}{dt}_{|h_2=0} = k^3[a(\frac{1}{k})^3 + (b - l)(\frac{1}{k})^2 - m\frac{1}{k} - 1] > 0.$$

On the other hand, $\phi(-b/a) = -(1/a^2)(lb^2 + a(a - mb)) < 0$. Thus, equation (3) has a real root λ_0 with $-b/a < \lambda_0 < 1/k$. Hence, denoting λ by λ_0, we get that

$$(b + 1)\lambda^2 - (m + \delta)\lambda - 1 < 0, \quad a\lambda + b > 0.$$

Doing the regular transformations (12), system (23) changes into (14)' (the form of (14)' and (14) is the same), and the singular points O, $\bar{N}$, $\bar{M}_1$ and $\bar{M}_2$ of (23) become O, $\tilde{N}(\tilde{x}_2, F(\tilde{x}_2))$, $\tilde{M}_1(\tilde{x}_1, F(\tilde{x}_1))$ and $\tilde{M}_2(-\lambda, F(-\lambda))$ respectively, where $\tilde{x}_1$ and $\tilde{x}_2$ are the real roots of $\bar{g}(x) = 0$. The limit cycle of system (14)' surrounding $O(\tilde{N})$ is determined by the saddle $\tilde{M}_1(\tilde{M}_2)$ at which the sign of the divergence of system (14)' is opposite to that of the point $O(\tilde{N})$.

Since $lb^2 + a(a - mb) = (a\lambda + b)g_0 > 0$, we have $g_0 > 0$ and

$$\begin{array}{ll} r_2 = \frac{1}{\lambda}(B_0 - A_0) < 0, & r_3 = r_2 - (a\lambda^2 + b\lambda) < 0, \\ \nu_0 - (-\lambda) = B_0/r_2 > 0, & g(\nu_0) = -(a\lambda + b)A_0B_0/\lambda^2 r_2^2 > 0, \\ \bar{g}(-\lambda) = -(1 + b)B_0 > 0, & \bar{g}(0) = -A_0 < 0. \end{array}$$

Hence, the relationships between $\tilde{x}_2$, $\tilde{x}_1$, $-\lambda$, ν_0 and 0 are

$$\tilde{x}_2 < -\lambda < \nu_0 < \tilde{x}_1 < 0.$$

Therefore, it follows that

$$\bar{f}(0) \cdot \bar{f}(\tilde{x}_1) < 0, \quad \bar{f}(-\lambda) \cdot \bar{f}(\tilde{x}_2) < 0,$$

which imply that the two zeros x_2^0 and x_1^0 of $\bar{f}(x)$, where $x_2^0 < x_1^0$, are in the intervals $(\tilde{x}_2, -\lambda)$ and $(\tilde{x}_1, 0)$, respectively. Hence

$$((b+2l)\lambda + m\lambda(l-a\lambda))\delta > 0.$$

Thus, $x\bar{f}(x) > 0$ (< 0) for $x \in (\tilde{x}_1, x_1^0) \cup (0, +\infty)$ when $\delta > 0$ (< 0).

Since $\text{div}(P', Q')_{|M_2} \cdot \text{div}(P', Q')_{|O} > 0$, we need only consider the following two cases

Case (i): $m < 0$, $m + \delta > 0$; or $\delta < 0$.
Case (ii): $m > 0$, $m + \delta < 0$; or $\delta > 0$.

Obviously, conditions (i) and (ii) of Lemma 7 are satisfied by system (14)', and (20) still holds.

In a similar way to the proof of Theorem 12, we may conclude

$$\left[\frac{f(x)}{g(x)}\right]' > 0(< 0) \text{ for } x \in (\tilde{x}_1, x_1^0) \cup (0, +\infty),$$

when $\delta > 0$ (< 0). Thus, condition (iii) of Lemma 7 is also satisfied by system (14)'. Therefore, system (14)' (i.e. system (1)) has at most one limit cycle surrounding the singular point O. This completes the proof of the theorem. ▮

From Theorem 12 and 15 we obtain immediately the following result.

Theorem 16. *If $lb^2 + a(a - mb) > 0$, then system (1) has at most one limit cycle around one of its two foci.*

Since $l > 0$ and $a - mb > 0$ imply that $lb^2 + a(a - mb) > 0$, from Lemmas 2 and 3, and Theorem 16, we obtain immediately the next result.

Theorem 17. *If $m < 0$, then system (1) has at most one limit cycle around one of its two foci.*

3. $\mathbf{lb^2 + a(a - mb) = 0}$

The main result of this section is the following theorem, proved initially in [5], but here its proof is rewritten.

Theorem 18. *If $lb^2 + a(a - mb) = 0$, then system (1) has at most one limit cycle around one of its two foci.*

Proof. From Lemmas 2 and 4, Theorem 17 and (21) we can assume $a > 0$, $m > 0$, $m + \delta \geq 0$ $(\delta \neq 0)$, $b + 2l < 0$.

Since $\varphi(-b/a) = 0$, we have that $\lambda_1 = -b/a$ is the unique positive real root of (3), and $\varphi'(\lambda_1) > 0$. Now, in the transformation (12) we take $\lambda = \lambda_1$ (λ_1 is rewritten as λ). Consequently,

$$\bar{g}(x) = (ax + 1)h(x),$$

and system (14) satisfies the conditions of Lemma 7, or the conditions of Lemma 6.

Since $-1/a - (-\lambda) = ((a\lambda + b) - (b + 1))/a = -(b + 1)/a > 0$, we get that

$$-\lambda < -1/a = \bar{x}_1 < 0. \tag{24}$$

From $a\lambda^2 + b\lambda = 0$, it follows that

$$r_0 = r_1 = r_2 = r_3 = -\varphi'(\lambda_1) < 0,\ f_1 < 0,\ r = \frac{r_1}{r_2} = 1,$$

$$\bar{f}(v_0) = \frac{r_0 A_0 B_0}{\lambda r_2^2} > 0,\ \bar{f}(-\lambda) = (m + \delta)B_0\lambda < 0,\ \bar{f}(0) = \lambda\delta A_0, \tag{25}$$

Obviously $\bar{f}(x)$ has two real zeros x_1^0, x_2^0 (assume $x_2^0 < x_1^0$). From (24) and (25), by a similar argument to that used in (18) the relationships between $-\lambda_1$, $\bar{x}_1 = -1/a$, x_2^0, x_1^0 and 0 are as follows:

$$\text{as } \delta < 0,\ -\lambda \leq x_2^0 < \bar{x}_1 < x_1^0 < 0, \tag{26}$$

$$\text{as } \delta > 0,\ -\lambda \leq x_2^0 < \bar{x}_1 < 0 < x_1^0, \tag{27}$$

where the equalities hold if and only if $m + \delta = 0$. Taking $x_{02} = \max\{\bar{x}_1, \nu_0\}$ and $x_{01} = +\infty$, then system (14) satisfies condition (i) of Lemma 6, and the conditions (i) and (ii) of Lemma 7.

Next, we verify that system (14) satisfies condition (iii) of Lemma 7, or condition (ii) of Lemma 6. We consider the following three cases.

Case (a): $\delta < 0$, $m + \delta \neq 0$. We discuss the graph of the curve $y = -\bar{f}(x)/((x + \lambda)(ax + 1)x)$. Obviously the line $y = 0$ is its horizontal asymptote, and the lines $x = -\lambda$, $x = -1/a$ and $x = 0$ are its vertical asymptotes. Moreover, it follows by (26) that the curve $y = -\bar{f}(x)/((x +$

$\lambda)(ax+1)x)$ and the line $y = c$ $(c \neq 0)$ have exactly three intersection points. Thus, it is easy to see from the graph of the curve $y = -\bar{f}(x)/((x+\lambda)(ax+1)x)$ that

$$\left[\frac{f(x)}{g(x)}\right]' = \left[\frac{-\bar{f}(x)}{(x+\lambda)(ax+1)x}\right]' < 0 \quad \text{for } x \in (x_{20}, x_{01}).$$

Therefore, condition (*iii*) (parenthesis outside) of Lemma 7 is satisfied. This proves the theorem in Case (a).

Case (b): $\delta > 0$ (and consequently $m + \delta \neq 0$). Since the curve $y = -\bar{f}(x)/((x+\lambda)(ax+1)x)$ and the line $y = c$ have at most three intersection points, it is easy to see from (27) and the graph of the curve $y = -\bar{f}(x)/((x+\lambda)(ax+1)x)$ that

$$\frac{f(x_1)}{g(x_1)} = \frac{-\bar{f}(x_1)}{(x_1+\lambda)(ax_1+1)x_1} < \frac{f(x_2)}{g(x_2)} = \frac{-\bar{f}(x_2)}{(x_2+\lambda)(ax_2+1)x_2},$$

for $(x_1, x_2) \in D_1$. Therefore, condition (*ii*) of Lemma 6 is satisfied. This proves the theorem in Case (b).

Case (c): $\delta < 0$, $m + \delta = 0$. By a similar argument to that in Case (a), it is easy to prove that

$$\left[\frac{f(x)}{g(x)}\right]' = \left[-\frac{f_1 x + f_0}{(ax+1)x}\right]' < 0 \quad \text{for } x \in (x_{20}, x_{01}).$$

Therefore, system (14) satisfies condition (*iii*) (parenthesis outside) of Lemma 7. The proof is complete. ▮

4. $\mathbf{lb^2 + a(a - mb) < 0}$, $\delta(\mathbf{m} + \delta) \neq \mathbf{0}$

The main result of this section is the next theorem obtained by first time in [5, 6, 7]. We use the method of [7] to prove it. We can use the same method of this section for proving the results of Section 2.

THEOREM 19. *If $lb^2 + a(a - mb) < 0$ and $\delta(m + \delta) \neq 0$, then system (1) has at most one limit cycle around one of its two foci.*

From Lemmas 2 and 4, Theorem 17 and (21), we can assume

$$a > 0, m > 0, m + \delta > 0 (\delta \neq 0), b + 2l < 0.$$

Since $\varphi(-b/a) = -(lb^2 + a(a-mb))/a^2 > 0$, we have that (3) has a unique positive real root λ_1. Thus $a\lambda_1 + b < 0$ and $\varphi'(\lambda_1) > 0$. In transformation

(12) we take $\lambda = \lambda_1$, where for simplicity λ_1 is rewritten as λ. We verify that system (13) satisfies the conditions of Lemma 9, or the conditions of Lemma 10, or the conditions of Lemma 11.

A calculation shows that

$$g_0 = (lb^2 + a(a - mb))/(a\lambda + b),\ f_1 = \frac{r_0}{\lambda} + \lambda m r_2,\ r_3 = -\varphi'(\lambda_1) < 0,$$

$$\begin{aligned} &a\lambda^2 + b\lambda < 0,\ r_0 < r_1 < r_2 < r_3 < 0, \\ &g_0 > 0,\ f_1 < 0,\ B_0 < 0,\ \nu_0 < 0. \end{aligned} \tag{28}$$

Since $\bar{f}(\nu_0) = r_0 A_0 B_0/\lambda r_2^2 > 0$ and $\bar{f}(-\lambda) = (m + \delta)\lambda B_0 < 0$, $\bar{f}(x)$ has two real zeros x_1^0, x_2^0 ($x_2^0 < x_1^0$), obviously x_1^0, x_2^0, ν_0 and $-\lambda$ satisfy the following relationships

$$\begin{aligned} &\text{as } m + \delta > 0 \text{ and } \delta < 0,\ -\lambda < x_2^0 < \nu_0 < x_1^0 < 0, \\ &\text{as } \delta > 0 (m > 0),\ -\lambda < x_2^0 < \nu_0 < 0 < x_1^0. \end{aligned} \tag{29}$$

A calculation shows that

$$\begin{aligned} &\bar{g}(0) = -A_0 < 0,\ \bar{g}(\nu_0) = -(a\lambda + b)A_0 B_0/\lambda r_2^2 < 0, \\ &\bar{g}(-\lambda) = -(b + 1)B_0 < 0, \\ &g_1/(2g_0) - (-\lambda) = ((b + 1)(r_1 - a\lambda^2) + a - (m + \delta)b)/(2g_0) > 0, \\ &g_1 = r_1 - a\lambda^2 - a - \delta b < 0 \text{ as } \delta < 0. \end{aligned} \tag{30}$$

Therefore, if $\bar{g}(x)$ has real zeros $\bar{x}_1, \bar{x}_2 (\bar{x}_2 < \bar{x}_1)$, then there must be

$$\begin{array}{lll} & -\lambda < x_2^0 < \bar{x}_2 < \bar{x}_1 < \nu_0, & \\ \text{or} & 0 < x_1^0 < \bar{x}_2 < \bar{x}_1 & \text{as } \delta > 0 \\ \text{or} & \nu_0 < \bar{x}_2 < \bar{x}_1 < x_1^0 & \text{as } \delta < 0 \\ \text{or} & \nu_0 < \bar{x}_2 < \bar{x}_1 < 0 < x_1^0 & \text{as } \delta > 0. \end{array} \tag{31}$$

Take

$$x_{01} = \begin{cases} \bar{x}_2, & \text{if } \bar{g}(x) \text{ has real zeros for } x > 0, \\ +\infty, & \text{if } \bar{g}(x) \text{ has no real zeros for } x > 0. \end{cases}$$

Similarly, we take $x_{02} = \bar{x}_1$ (or ν_0), $\bar{x}_{01} = \bar{x}_2$ (or ν_0) and $\bar{x}_{02} = -\infty$. In addition, we take $x_0 = x_1^0$, $\bar{x}_0 = x_2^0$ and $\bar{a} = -\lambda$ (in Lemma 11, we may take $\bar{a} = -\lambda$ or 0). Thus, system (13) satisfies conditions (*i*) and (*ii*) of Lemmas 9, 10 and 11.

Next we verify that system (13) satisfies conditions (*iii*) and (*iv*) of Lemma 9, or conditions (*iii*) and (*iv*) of Lemma 10, or condition (*iii*) of Lemma 11 ($\bar{a} = -\lambda$ or $\bar{a} = 0$).

A calculation shows that

$$\begin{array}{ll} \int_0^x f(s)ds = \tilde{F}(x) - \tilde{F}(0) & \text{for } x \in (\nu_0, +\infty) \\ \int_{-\lambda}^x f(s)ds = \tilde{F}(x) - \tilde{F}(-\lambda) & \text{for } x \in (-\infty, \nu_0), \end{array}$$

where

$$\begin{aligned} \tilde{F}(x) &= \bar{F}(x)/(-\lambda)(a\lambda^2 + b\lambda)r_1 r_3 |h(x)|^r, \\ \bar{F}(x) &= (a\lambda^2 + b\lambda)r_3 \bar{f}(x) + r_3 \bar{f}'(x)h(x) - 2f_1 h^2(x), \end{aligned} \tag{32}$$

and $\bar{F}(x)$ is a quadratic polynomial with positive leading coefficient $-(a\lambda^2 + b\lambda)r_1 f_1$.

By (28) and (31), we have

$$\lim_{x\to\infty} \tilde{F}(x) = +\infty, \quad \lim_{x\to\nu_0} \tilde{F}(x) = +\infty.$$

Obviously the equation

$$\int_0^{x_1} f(s)ds = \int_0^{x_2} f(s)ds \quad \text{for } (x_1, x_2) \in D(\text{ or } D') \tag{33}$$

and the equation

$$\int_{-\lambda}^{x_1} f(s)ds = \int_{-\lambda}^{x_2} f(s)ds \quad \text{for } (x_1, x_2) \in D' \tag{34}$$

are equivalent to the equation

$$\tilde{F}(x_1) = \tilde{F}(x_2) \text{ for } (x_1, x_2) \in D(\text{ or } D') \text{ and } (x_1, x_2) \in \bar{D},$$

respectively, where D (D') and $\bar{D}$ are defined as in Lemma 9 and in Lemma 10, respectively. Therefore, the simultaneous equations

$$\int_0^{x_1} f(s)ds = \int_0^{x_2} f(s)ds, \tag{35}$$

$$\frac{f(x_1}{g(x)} = \frac{f(x_2)}{g(x_2)}, \tag{36}$$

for $(x_1, x_2) \in D$ (or D'), and the simultaneous equations

$$\int_{-\lambda}^{x_1} f(s)ds = \int_{-\lambda}^{x_2} f(s)ds, \tag{37}$$

$$\frac{f(x_1)}{g(x_1)} = \frac{f(x_2)}{g(x_2)}, \tag{38}$$

for $(x_1, x_2) \in \bar{D}$ are equivalent to the simultaneous equations

$$\begin{cases} \tilde{F}(x_1) & = \tilde{F}(x_2), \\ \dfrac{\bar{F}(x_1)\bar{f}(x_1)}{x_1(x_1+\lambda)\bar{g}(x_1)} & = \dfrac{\bar{F}(x_2)\bar{f}(x_2)}{x_2(x_2+\lambda)\bar{g}(x_2)}, \end{cases} \tag{39}$$

for $(x_1, x_2) \in D$ (or D') and $(x_1, x_2) \in \bar{D}$, respectively.

First, we discuss the graph of function $y = y(x) = \bar{F}(x)\bar{f}(x)/(x(x+\lambda)\bar{g}(x))$. By (28) and (31), we have

$$\begin{array}{l} y(0^+) = \infty,\ y(0^-) = \infty,\ y(0^+)y(0^-) = -\infty; \\ y(-\lambda^+) = \infty,\ g(-\lambda^-) = \infty,\ y(-\lambda^+)y(-\lambda^-) = -\infty, \\ y(\pm\infty) = (ax^2 + b\lambda)r_1 f_1^2/g_0 := H > 0. \end{array} \tag{40}$$

Obviously, we have the following conclusions which we state as a lemma.

Lemma 20. *The curve $y = y(x) = \bar{F}(x)\bar{f}(x)/(x(x+\lambda)\bar{g}(x))$ has the vertical asymptotes $x = 0$ and $x = -\lambda$, and the positive (negative) sense horizontal asymptote $y = H$. The curve $y = y(x)$ and the line $y = c$ have at most four intersection points.*

By direct calculation, $y(\nu_0) - H = (r_2^2/g_0)c_1$, where c_1 is defined as in (16).

A calculation shows that

$$y = y(x) = \frac{\bar{F}(x)\bar{f}(x)}{x(x+\lambda)\bar{g}(x)} = \frac{h^2(x)P_2(x)}{x(x+\lambda)\bar{g}(x)} + y(\nu_0),$$

where

$$h^2(x)p_2(x) = \bar{F}(x)\bar{f}(x) + \lambda r_3 r_0^2 x(x+\lambda)\bar{g}(x), \tag{41}$$

here $p_2(x)$ is a quadratic polynomial with leading coefficient c_1, we consider separately three possibilities:

Case(a): $c_1 > 0$ $(0 < H < y(\nu_0))$. We consider separately two cases.

(a.1) $P_2(\nu_0) \geq 0$. By (26) and (31), we have $p_2(x_2^0) < 0$. Thus, besides the point $p_0(\nu_0, y(\nu_0))$, the curve $y = y(x)$ and the line $y = y(\nu_0)$ also have the other two intersection points which are denoted by points $M_1(x_{m_1}, y(\nu_0))$ and $M_2(x_{m_2}, y(\nu_0))$ with $x_{m_1} < x_{m_2}$, and hence $x_{m_2} <$

$x_2^0 < x_{m_1} \leq \nu_0$. Therefore, it is easy to see from the graph of the curve $y = y(x)$ and Lemma 20 that $y = y(x) > y(\nu_0)$ for $x \in (x_{02}, 0)$, $y = y(x) < y(\nu_0)$ for $x \in (0, x_{01})$. Thus, the simultaneous equations (35) and (36) have no solution in the region D (or D'), system (13) satisfies condition (iii) of Lemma 11 (taking $\bar{a} = 0$). This completes the proof of subcase (a.1).

(a.2) $p_2(\nu_0) < 0$. Since $p_2(x_2^0) < 0$ and $p_2(\nu_0) < 0$, $x_{m_2} < x_2^0 < \nu_0 < x_{m_1}$. We consider separately four subcases.

(a.21) $-\lambda < x_{m_2} < x_2^0 < \nu_0 < x_{m_1} < 0$. Then, by (41) and (31), $\bar{F}(-\lambda) < 0$.

Since $\bar{F}(\nu_0) = (ax^2 + b\lambda)r_3\bar{f}(\nu_0) > 0$ by (28) and (32), $\bar{F}(0) > 0$ and $\bar{f}(0) > 0$ (i.e. $\delta > 0$) by (32) and (41), respectively. Thus, it is easy to see by Lemma 20 that the curve $y = y(x)$ has a minimal point $Q(x_Q, y(x_Q))$ in the region $D_3 = \{(x, y) \mid \nu_0 < x < x_{m_1},\ H < y < y(\nu_0)\}$, and the curve $y = y(x)$ and the line $y = c$ $(y(x_Q) < c < y(\nu_0))$ have exactly four intersection points which are both on the left hand side of the line $x = x_{m_1}$. Therefore, by Lemma 20 and the graph of the curve $y = y(x)$, it follows that

$$y(x) \geq y(x_Q) \text{ for } x \in (x_{02}, 0); \quad y(x) < y(x_Q) \text{ for } x \in (0, x_{01}),$$

and hence system (14) satisfies condition (iii) of Lemma 11 (taking $\bar{a} = 0$). This proves the theorem in subcase (a.21).

(a.22) $-\lambda < x_{m_2} < x_2^0$, $x_{m_1} > 0$. It may be verified, as in the proof of case (a.21), that

$$\bar{F}(-\lambda) < 0,\ \bar{F}(0) > 0 \text{ and } \bar{f}(0) < 0 \text{ (i.e. } \delta < 0).$$

Thus, by (31) and Lemma 20 it is easy to see that $\bar{g}(x)$ has no real zeros for $x > \nu_0$, and the curve $y = y(x)$ and the line $y = c$ $(H < c < y(\nu_0))$ have exactly four intersection points. Moreover, by Lemma 20 and the graph of the curve $y = y(x)$, we have

$$\begin{aligned} &y = y(x) > H \text{ for } x > 0,\ y = y(x) < y(\nu_0) \text{ for } \nu_0 < x < 0, \\ &\text{and } y'(x) < 0 \text{ as } H < y(x) < y(\nu_0) \text{ and } x > \nu_0. \end{aligned} \tag{42}$$

Therefore, by (42) it follows immediately that system (13) satisfies condition (iii) of Lemma 9.

Since $\bar{F}(0) > 0$, by (28), we get that $\tilde{F}(0) = \bar{F}(0)/(-\lambda)r_1(a\lambda^2+b\lambda)r_3|h(0)|^r > 0$.

A calculation shows that

$$\begin{aligned}\left[\int_0^x f(s)ds\cdot\frac{f(x)}{g(x)}\right]' &= \frac{f^2(x)}{g(x)}+\int_0^x f(s)ds\cdot\left[\frac{f(x)}{g(x)}\right]'\\ &= y'(x)/(-\lambda)(a\lambda^2+b\lambda)r_1r_3-\tilde{F}(0)\left[\frac{f(x)}{g(x)}\right]'.\end{aligned}$$

Thus, by (42) and $\int_0^x f(s)ds>0$ for $x\in(x_{02},x_{20})$ it is easy to show that

$$\left[\int_0^x f(s)ds\cdot\frac{f(x)}{g(x)}\right]'<0,$$

for $x\in(x_{02},x_{20})$, where x_{20} is defined as in Lemma 9. Therefore, system (13) satisfies condition (iv) of Lemma 9 (the case $x_0\in(x_{02},0)$. This proves the theorem in subcase (a.22).

(a.23) $x_{m_2}<-\lambda$, $\nu_0<x_{m_1}<0$. As in the proof of subcase (a.21), by (31) and (41) it follows that

$$\begin{aligned}&\bar{F}(-\lambda)>0,\ y(-\lambda^-)=+\infty,\ y(-\lambda^+)=-\infty,\\ &y(0^+)=-\infty \text{ and } y(0^-)=+\infty.\end{aligned}$$

If the curve $y=g(x)$ has a minimal point $Q(x_Q,y(x_Q))$ in the region D_3 then, by using the same method as in case (a.21), it can be proved that the system (13) satisfies condition (iii) in Lemma 11 (taking $\bar{a}=-\lambda$).

If the curve $y=y(x)$ has no minimal point in the region D_3 and the $\bar{g}(x)$ has real zeros for $x<\nu_0$, then, by using the same method as in case (a.21) it is easy to prove that (13) satisfies condition (iii) in Lemma 11 (taking $\bar{a}=-\lambda$).

If the curve $y=y(x))$ has no minimal point in the region D_3 and $\bar{g}(x)$ have no real zeros for $x<\nu_0$, then, by a similar argument to that in subcase (a.22), it can be proved that

$$y=y(x)>H \text{ for } x<-\lambda,\ y=y(x)<y(\nu_0) \text{ for } x\in(-\lambda,\nu_0),$$

and $y'(x)>0$ as $H<y(x)<y(\nu_0)$ and $x<\nu_0$,

$$\begin{aligned}\left[\int_{-\lambda}^x f(s)ds\cdot\frac{f(x)}{g(x)}\right]' &= \frac{f^2(x)}{g(x)}+\int_{-\lambda}^x f(s)ds\cdot\left[\frac{f(x)}{g(x)}\right]'\\ &= y'(x)/(-\lambda)(a\lambda^2+b\lambda)r_1r_3-\tilde{F}(-\lambda)\left[\frac{f(x)}{g(x)}\right]'>0,\end{aligned}$$

for $x\in(\bar{x}_{10},\nu_0)$, where $\bar{x}_{10}$ is defined as in Lemma 10. Therefore, system (13) satisfies conditions (iii) and (iv) of Lemma 10 (the case $\bar{x}_0\in(\bar{a},\bar{x}_{01})$).

This proves the theorem in subcase (a.23).

(a.24) $x_{m_2} < -\lambda$, $x_{m_1} > 0$. As in the proof of subcase (a.23), it may be proved that

$$\bar{F}(-\lambda) > 0,\ y(-\lambda^-) = +\infty,\ y(-\lambda^+) = -\infty,$$
$$y(0^-) = -\infty \text{ and } y(0^+) = +\infty.$$

Evidently the curve $y = y(x)$ and the horizontal asymptote $y = H$ have exactly three intersection points, and the curve $y = y(x)$ and the line $y = c$ ($H < c < y(\nu_0)$) have exactly four intersection points.

If $\bar{g}(x)$ has real zeros, then, as in proof of the preceding subcase, it can be proved that system (13) satisfies condition (*iii*) of Lemma 11 (taking $\bar{a} = 0$, or taking $\bar{a} = -\lambda$).

If $\bar{g}(x)$ has no real zeros, then it is easy to prove that $y'(x) < 0$ for $H < y < y(\nu_0)$ and $x > \nu_0$, $y'(x) > 0$ for $H < y < y(\nu_0)$ and $x < \nu_0$, and $y = y(x) > H$ for $x > 0$ (this time with $\bar{F}(0) > 0$), or $y = y(x) > H$ for $x < -\lambda$. Thus, by using the same method as in subcases (a.22) or (a.23), it is easy to see that system (13) satisfies conditions (*iii*) and (*iv*) of Lemma 9 (the case $x_0 \in (x_{02}, 0)$), or conditions (*iii*) and (*iv*) of Lemma 10 (the case $\bar{x}_0 \in (\bar{a}, x_{01})$). This proves the theorem in subcase (a.24).

Case (b): $c_1 < 0$ ($0 < y(\nu_0) < H$). Now, the leading coefficient of $p_2(x)$ in (41) is negative. We consider separately two subcases.

(b.1) $p_2(\nu_0) \le 0$. Since $c_1 < 0$ and $p_2(\nu_0) \le 0$, we have that $x_{m_2} < x_{m_1} \le \nu_0$ or $\nu_0 \le x_{m_2} < x_{m_1}$ if x_{m_1} and x_{m_2} exist. Thus, if $\nu_0 \le x_{m_1} < x_{m_1}$, then $y(x) > y(\nu_0)$ for $x \in (\bar{x}_{02}, \lambda)$ and $y(x) < y(\nu_0)$ for $x \in (-\lambda, \bar{x}_{01})$. If as $x_{m_2} < x_{m_1} \le \nu_0$, then $y(x) > y(\nu_0)$ for $x \in (0, x_{01})$ and $y(x) < y(\nu_0)$ for $x \in (x_{02}, 0)$.

If x_{m_1} and x_{m_2} do not exist, then the above conclusions still hold. Therefore, system (13) satisfies condition (*iii*) of Lemma 11 taking $\bar{a} = -\lambda$ or 0). This proves the theorem in subcase (b.1).

(b.2) $p_2(\nu_0) > 0$. Since $p_2(\nu_0) > 0$ and $p_2(x_2^0) < 0$ by (41) and (31), x_{m_1} and x_{m_2} exist, and $-\lambda < x_{m_2} < \nu_0 < x_{m_1}$. We consider separately two subcases.

(b.21) $-\lambda < x_{m_2} < \nu_0 < x_{m_1} < 0$. By a similar argument to that of subcase (a.21), it can be proved that system (13) satisfies condition (*iii*) of Lemma 11 (taking $\bar{a} = -\lambda$ or 0).

(b.22) $-\lambda < x_{m_2} < \nu_0 < 0 < x_{m_1}$. It follows by (31) and (41) that

$$\bar{F}(-\lambda) > 0,\ y(0^-) = +\infty,\ y(0^-) = -\infty,$$
$$y(-x^+) = -\infty,\ \text{and}\ y(-\lambda^-) = +\infty.$$

As in the proof of subcase (a.23), the following conclusions can be proved.

If the curve $y = y(x)$ has a maximal point $R(x_R, y(x_R))$ in the region $D_4 = \{(x_1, y) \mid -\lambda < x < \nu_0,\ y(\nu_0) < y < H\}$, then system (13) satisfies conditions (iii) of Lemma 11 (taking $\bar{a} = -\lambda$).

If the curve $y = y(x)$ has no maximal point $R(x_R, y(x_R))$ in the region D_4 and $\bar{g}(x)$ has no real zeros for $x \in (x_2^0, \nu_0)$, then system (13) satisfies conditions (iii) and (iv) of Lemma 9, or condition (iii) of Lemma 11 (taking $\bar{a} = 0$) if $\bar{g}(x)$ has real zeros for $x > \nu_0$.

If the curve $y = y(x)$ has no maximal point $R(x_R, y(x_R))$ in the region D_4 and $\bar{g}(x)$ has real zeros for $x \in (x_2^0, \nu_0)$, then system (13) satisfies conditions (iii) and (iv) of Lemma 9.

This proves the theorem in case (b.2).

Case (c): $c_1 = 0$.

Then $p_2(x)$ is a degenerate quadratic polynomial. Beside the point $p_0(\nu_0, y(\nu_0))$, the curve $y = y(x)$ and the line $y = y(\nu_0)$ have at most one intersection point, which is denoted by $M_0(x_{m_0}, y(\nu_0))$.

By a similar argument to that in case (b.1), it is easy to prove that the system (13) satisfies condition (iii) in Lemma 11 (taking $\bar{a} = -\lambda$ or 0).

In short, the proof of the theorem is complete.

5. $\delta(\mathbf{m} + \delta) = \mathbf{0}$

The following result appeared in [8, 9, 10]: A quadratic system with a weak focus and a strong focus has at most one limit cycle around one of its two foci.

The next theorem is proved in [11].

THEOREM 21. *A quadratic system with a weak focus and a strong focus has at most one limit cycle around the strong focus.*

Moreover, since a quadratic system has no limit cycle around a 3rd–order weak focus and has at most one limit cycle around a 2nd–order weak focus [12, 13], we have the following result.

THEOREM 22. *A quadratic system with a 2nd–order (3rd–order) weak focus has at most two (one) limit cycles, having a* $(1,1)$*–distribution (*$(0,1)$*-distribution).*

Since system (1) has no limit cycle if $a = 0$ or $m = \delta = 0$, we may assume

$$a > 0,\ \delta \neq 0,\ m + \delta = 0 \text{ (or } m + \delta \neq 0,\ \delta = 0).$$

Moreover, by (21) we can suppose that $O(0, 0)$ is the strong focus and $N(0, 1)$ is the weak focus (i.e. $\delta \neq 0$, $m + \delta = 0$).

In [8] it is proved that system $(1)_{\delta=-m}$ has at most one limit cycle around the strong focus $O(0, 0)$ if $m < 0$, or $m > 0$ and $lb^2 + a(a - mb) > 0$. This result also follows from the proof of Theorem 12. Therefore we need only to consider the following case:

$$m > 0,\ lb^2 + a(a - mb) < 0.$$

As in the proof of Section 4, we take $\lambda = \lambda_1$ (for simplicity λ_1 is rewritten as λ). Now,

$$\begin{aligned}\bar{f}(x) &= ((b + 2l + m\lambda(l - a\lambda))x - m(\lambda^2 + m\lambda + 1))(x + \lambda) \\ &= (\frac{r_0}{\lambda}x + mh(x))(x + \lambda) = (f_1 x + f_0)(x + \lambda) := \bar{\bar{f}}(x)(x + \lambda),\end{aligned}$$

and conditions (28) and (30) still hold. We verify that system (13) satisfies the conditions of Lemma 6, or 7, or 8.

The functions $\bar{\bar{f}}(x)$ and $F(x)$ have respectively a unique real zero $x_0 = -f_0/f_1$ and Δ_0 in the interval $(\nu_0, 0)$ $(\Delta_0 < x_0 < 0)$.

If $\bar{g}(x)$ has real zeros for $x \in (0, +\infty)$ or $x \in (\nu_0, 0)$, then let $\bar{x}_2 = \min\{x/\bar{g}(x) = 0,\ x \in (0, +\infty)\}$ and $\bar{x}_1 = \max\{x/\bar{g}(x) = 0,\ x \in (\nu_0, 0)\}$.

Take

$$x_{01} = \begin{cases} \bar{x}_2, & \text{if } \bar{x}_2 \text{ exists,} \\ +\infty, & \text{if } \bar{x}_2 \text{ does not exist;} \end{cases} \qquad x_{02} = \begin{cases} \bar{x}_1, & \text{if } \bar{x}_1 \text{ exists,} \\ \nu_0, & \text{if } \bar{x}_1 \text{ does not exist.} \end{cases}$$

Since system $(1)_{\delta=-m}$ has no limit cycle around $O(0, 0)$, if $\Delta_0 \leq x_{02}$, we may assume $x_{02} < \Delta_0$. Hence,

$$-\lambda < \nu_0 < x_{02} < \Delta_0 < x_0 < 0 < x_{01}. \tag{43}$$

Thus, it is easy to see that system $(13)_{\delta=-m}$ satisfies conditions (i) of Lemma 6 and conditions (i) and (ii) of Lemma 7 (8).

Next, we verify that system $(13)_{\delta=-m}$ satisfies condition (iii) of Lemma 8, or condition (iii) of Lemma 7, or condition (ii) of Lemma 6.

Since $y = (1/k)F(x_1)$ and $y = F(x_2)$ are strictly monotonous in the interval $(0, x_{01})$ and (x_{02}, Δ_0) respectively, we denote the inverse function of $y = (1/k)F(x_1)$ and $y = F(x_2)$ by $x_1(y)$ and $x_2(y)$ with $x_1(y) > 0$ and

$x_2(y) < 0$, respectively. Let

$$\begin{aligned} V(y) &= -\lambda(a\lambda^2+b\lambda)r_1r_3\frac{F(x_1(y))}{k} + \lambda r_0^2 r_3 \frac{x_1(y)\bar{g}(x_1(y))}{k\overline{\overline{f}}(x_1(y))|h(x_1(y))|^r} - \\ &\left[-\lambda(a\lambda^2+b\lambda)r_1r_3\frac{F(x_2(y))}{k} + \lambda r_0^2 r_3 \frac{x_2(y)\bar{g}(x_2(y))}{k\overline{\overline{f}}(x_2(y))|h(x_2(y))|^r}\right]. \end{aligned} \tag{44}$$

Since $V(0) > 0$, in order to verify condition (iii) of Lemma 8, it is sufficient to show that

$$V'(y) < 0, \ 0 < y < +\infty.$$

By (32) we have

$$F(x) = \frac{1}{-\lambda r_1 r_3(a\lambda^2+b\lambda)}\left[\frac{\bar{F}(x)}{|h(x)|^r} - \frac{\bar{F}(0)}{|h(0)|^r}\right],$$

where

$$\bar{F}(x) = r_3(a\lambda^2+b\lambda)(x+\lambda)\overline{\overline{f}}(x) + r_3((x+\lambda)\overline{\overline{f}}(x))'h(x) - 2f_1h^2(x) \tag{45}$$

Substituting (45) in (44), we obtain that

$$\begin{aligned} V(y) = &\left(\frac{-\bar{F}(0)}{k|h(0)|^r} + \frac{h^2(c_1x_1(y)+c_0)}{k\overline{\overline{f}}(x_1(y))|h(x_1(y))|^r}\right) \\ &- \left(\frac{-\bar{F}(x_2(0))}{|h(x_2(0))|^r} + \frac{h^2(c_1x_2(y)+c_0)}{2\overline{\overline{f}}(x_2(y))|h(x_2(y))|^r}\right), \end{aligned} \tag{46}$$

where $h^2(x)(c_1x+c_0) = \bar{F}(x)\overline{\overline{f}}(x) + \lambda r_0^2 r_3 x\bar{g}(x)$.

Since $h^2(x_0)(c_1x_0+c_0) = \lambda r_0^2 r_3 x_0 \bar{g}(x_0)$, if $c_1x_0+c_0 \geq 0$ then $\bar{g}(x_0) \geq 0$. By (43), this is a contradiction. Therefore

$$c_1x_0 + c_0 < 0. \tag{47}$$

A calculation shows that

$$V'(y) = \frac{h^2(x_1(y))w(x_1(y))}{-\lambda(x_1(y)+\lambda)\overline{\overline{f}}_1^3(x_1(y))} - \frac{h^2(x_2(y))w(x_2(y))}{-\lambda(x_2(y)+\lambda)\overline{\overline{f}}^3(x_2(y_1))} \tag{48}$$

where $w(x(y)) = -r_3(c_1x(y)+c_0)\overline{\overline{f}}(x(y)) + h(x(y))f_1(c_1x_0+c_0)$.

We consider separately three cases.

Case (a): $c_1 > 0$. Since $-r_3 f_1 c_1 < 0$, the leading coefficient of the quadratic polynomial $w(x)$ is negative. By (28),(43) and (47) it follows that

$$\begin{array}{l} w'(x)_{|x=x_0} = (a\lambda^2 + b\lambda) f_1 (c_1 x_0 + c_0) < 0; \\ x\bar{f}(x) < 0 \text{ for } x \in (\nu_0, x_0) \cup (0, +\infty); \\ h(x) < 0 \text{ for } x > \nu_0; \ c_1 x + c_0 \text{ for } x \leq x_0, \end{array} \tag{49}$$

and hence $w(x) < 0$ for $x > \nu_0$. Therefore, $V'(y) < 0$ for $0 < y < +\infty$. This proves the theorem in Case (a).

Case (b): $c_1 = 0$ (now $c_0 < 0$). Since $w'(x) = (a\lambda^2 + b\lambda) c_0 f_1 < 0$ and $w(\nu_0) = -r_3 c_0 \bar{\bar{f}}(\nu_0) < 0$, we have $w(x) < 0$ for $x > \nu_0$, and hence

$$V'(y) < 0 \text{ for } 0 < y < +\infty.$$

This proves the theorem in Case (b).

Case (c): $c_1 < 0$. We verify that system $(14)_{\delta=-m}$ satisfies condition (ii) of Lemma 6, or condition (iii) of Lemma 7. We consider separately two subcases.

(c.1) $\bar{g}(x)$ has two real zeros $\bar{x}_1$ and $\bar{x}_2$ for $x > \nu_0$. By (28), (30) and (43) we have

$$\nu_0 < \bar{x}_2 \leq \bar{x}_1 < x_0 < 0. \tag{50}$$

A calculation shows that

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{|h(x)|^r (m(\lambda^2 + m\lambda + 1) g(x) h(x) + x\bar{\bar{f}}(x) w_2(x))}{h(x)(x\bar{g}(x))^2}$$

where

$$w_2(x) = r_1 \bar{g}(x) - h(x)\bar{g}'(x). \tag{51}$$

Since $(x - x_0)\bar{\bar{f}}(x) < 0$ and $\bar{g}(x)h(x) > 0$ for $x > x_{02}$, to verify condition (iii) of Lemma 7 it is sufficient to show that $w_2(x) < 0$ for $x_{02} < x < +\infty$.

Since the leading coefficient of quadratic polynomial $w_2(x)$ and $\bar{g}(x)$ are all negative, we have $w_2(\bar{x}_1) = -h(\bar{x}_1)\bar{g}'(\bar{x}_1) \leq 0$, $w_2(\bar{x}_2) = -h(\bar{x}_2) g'(\bar{x}_2) \geq 0$ (equalities hold if and only if $\bar{x}_1 = \bar{x}_2$), and $w_2'(\bar{x}_2) = (a\lambda^2 + b\lambda)\bar{g}'(x_2) -$

$h(x_2)g''(x_2) < 0$. Hence $w_2(x) < 0$ for $x \in (x_{02}, +\infty)$. This completes the proof in subcase (c.1).

(c.2) $\bar{g}(x)$ has no real zeros for $x > \nu_0$. We verify that system $(14)_{\delta=-m}$ satisfies condition (ii) of Lemma 6.

By (45) and (46) it is easy to see that the simultaneous equations (5) are equivalent to

$$\begin{cases} \dfrac{\bar{F}(x_1)}{|h(x_1)|^r} & = \dfrac{\bar{F}(x_2)}{|h(x_2)|^r}, \\ \dfrac{\bar{\bar{f}}(x_1)\bar{F}(x_1)}{x_1\bar{g}(x_1)} & = \dfrac{\bar{\bar{f}}(x_2)\bar{F}(x_2)}{x_2\bar{g}(x_2)}, \end{cases}$$

namely

$$\begin{cases} \dfrac{\bar{F}(x_1)}{|h(x_1)|^r} & = \dfrac{\bar{F}(x_2)}{|h(x_2)|^r}, \\ \dfrac{h^2(x_1)(c_1x_1+c_0)}{x_1\bar{g}(x_1)} & = \dfrac{h^2(x_2)(c_1x_2+c_0)}{x_2\bar{g}(x_2)}, \end{cases} \tag{52}$$

where $h^2(x)(c_1x+c_0) = \bar{F}(x)\bar{\bar{f}}(x) + \lambda r_0^2 r_3 x\bar{g}(x)$, $x_1 \in (0, +\infty)$ and $x_2 \in (\nu_0, 0)$.

By $c_1x_0 + c_0 < 0$ and $c_1 < 0$, we have $c_0 < 0$, and hence $c_1x_1 + c_0 < 0$ for $x_1 \in (0, +\infty)$. Since $c_1\nu_0 + c_0 < 0$, the simultaneous equations (52) (i.e. (5)) have no solution in the region D_1 we need only to consider the case $c_1\nu_0 + c_0 > 0$. Obviously the curve $y = h^2(x)(c_1x + c_0)/(x\bar{g}(x))$ and the line $y = c$ have at most three intersection points. From the graph of the curve $y = h^2(x)(c_1x + c_0)/(x\bar{g}(x))$ it is easy to see that

$$\frac{h^2(x_1)(c_1x_1+c_0)}{x_1\bar{g}(x_1)} > \frac{h^2(x_2)(c_1x_2+c_0)}{x_2\bar{g}(x_2)} \quad \text{for } (x_1, x_2) \in D_1.$$

Therefore the simultaneous equations (52) (i.e. (5)) have no solution in the region D_1. This completes the proof in subcase (c.2). In short, the proof of the theorem is completed.

Since all cases have been discussed, this completes the proof of Theorem 1.

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