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Some Lower Bounds for $H(n)$ in Hilbert's 16th Problem

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For some perturbed Z_2-(or Z_4-)equivariant planar Hamiltonian vector field sequnces of degree n ($n = 2^k - 1$ and $n = 3 \times 2^{k-1} - 1, k = 2, 3, \cdots$), some new lower bounds for $H(n)$ in Hilbert's 16th problem and configurations of compound eyes of limit cycles are given, by using the bifurcation theory of planar dynamical systems and the quadruple transformation method given by Christopher and Lloyd. It gives rise to more exact results than Ref.[6].

Key Words: Hilbert's 16th problem, perturbed planar Hamiltonian systems, distributions of limit cycles, second bifurcation.

1. INTRODUCTION

One of the problem posed by Smale[26] in his *"Mathematical Problems for the Next Century"* is **Hilbert's 16th problem**. It is well known that Hilbert's 16th problem consists of two parts. The first part studies the mutual disposition of maximal number (in the sense of Harnack) of separate branches of an algebraic curve, and also the "corresponding investigation" for non-singular real algebraic varieties; and the second part poses the questions of the maximal number and relative dispositions of limit cycles of the planar polynomial vector field:

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$$\frac{dx}{dt} = P_n(x,y), \ \frac{dy}{dt} = Q_n(x,y), \tag{1}$$

where P_n and Q_n are polynomials of degree n (see Hilbert[11], Farkas[8], Ye[29], Zhang et al [31]). As professor Smale said, "Except for the Riemann hypothesis, it seems to be the most elusive of Hilbert's problems." In fact, for the first part, the specialists of the real algebraic geometry usually study the topology of non-singular real planar projective algebraic curves of degree m. Up to now, we know the schemes of mutual arrangement of ovals realized by M-curves only for $m \leq 7$ (see Gudkov[10], Viro[27] and Wilson[28] etc). For the second part, the answer still seems to be far away. Let $H(n)$ be the maximal number of limit cycles of (1). Up to now, we only know that a given system (1) always has a finite number of limit cycles (see Ilyashenko[12]) and that $H(2) \geq 4, H(3) \geq 11$ (see recent discussions in Chan et al[3], Li[14-17], Lloyd[20], Luo[21], Perko[23], Ye[30] and etc). Also by considering a small neighbourhood of a singular point, $H(n) \geq (n^2 + 5n - 20 - 6(-1)^n)/2$ for $n \geq 6$ (see Otrokov[22]). Recently, Christopher and Lloyd[6] showed that $H(2^k - 1) \geq 4^{k-1}(2k - \frac{35}{6}) + 3.2^k - \frac{5}{3}$ (for example $H(7) \geq 25$) by perturbing some families of closed orbits of a Hamiltonian system sequence in small neighbourhoods of some center points and using a "quadruple transformation". The method given by [6] is very interesting. Unfortunately, the computation of a lower bound is not correct (see Section 3: Remark 3.1).

In order to obtain more limit cycles and various configuration patterns of their relative dispositions, one of us indicated in [13]-[17] that an efficient method is to perturb the symmetric Hamiltonian systems having maximal number of centers, i.e., to study the weakened Hilbert's 16th problem posed by V.I.Arnold[2] in 1977 for the symmetric planar polynomial Hamiltonian systems, since bifurcation and symmetry are closely connected and symmetric systems play pivotal roles as a bifurcation point in all planar Hamiltonian system class. To investigate perturbed Hamiltonian systems, we should first know the global behaviour of unperturbed polynomial systems, namely, determine the global property for the families of real planar algebraic curves defined by the Hamiltonian functions. Then by using proper perturbation techniques, we shall obtain the global information of bifurcations for the perturbed non-integrable systems. In this sense, we say that our study method will utilize both parts of Hilbert's 16th problem. On the basis of the method of detection functions posed by Li[13], in our recent papers [4,5,18,19] we have given a method of control parameters in order to obtain more limit cycles for Z_q−equivariant ($q = 2 - 6$) perturbed polynomial Hamiltonian systems of the fifth degree. With the help of numerical analysis (using Maple[1] or Mathematica) we showed that there exist parameter groups such that these systems have at least 17-24 limit cycles having various compound eyes configurations.

In this paper, we shall use previous idea and the method posed in [6] to investigate some perturbed Z_2- (or Z_4-) equivariant planar Hamiltonian vector field sequences of degree n ($n = 2^k - 1$ and $n = 3 \times 2^{k-1} - 1$), some new lower bounds for $H(n)$ in Hilbert's 16th problem and configurations of compound eyes of limit cycles are given.

The paper is divided into five sections. In Section 2, we state 3 lemmas as preliminary knowledge. In Section 3, we discuss the method of Ref.[6] and correct the computation of the lower bound for $H(2^k - 1)$. Section 4 gives a new lower bound for $H(2^k - 1)$ by using a Z_2-equivariant perturbed planar Hamiltonian vector field sequence. In Section 5, we consider Z_2-equivariant Hamiltonian system sequence of degree $3 \times 2^{k-1} - 1$. A lower bound for $H(3 \times 2^{k-1} - 1)$ is obtained.

2. PRELIMINARY LEMMAS

We consider the following perturbed planar polynomial Hamiltonian system

$$
\begin{aligned}
\frac{dx}{dt} &= -\frac{\partial H}{\partial y} + \epsilon R_1(x,y) = f_1(x,y) + \epsilon R_1(x,y), \\
\frac{dy}{dt} &= \frac{\partial H}{\partial x} + \epsilon R_2(x,y) = f_2(x,y) + \epsilon R_2(x,y),
\end{aligned} \tag{2}
$$

where $H(x,y)$ is the Hamiltonian, $0 < \epsilon \ll 1$.

LEMMA 1. *(see [6], p221) (i) Suppose that $R_2(x,y) = 0, p = (x_c, y_c)$ is a non-degenerate center of the unperturbed Hamiltonian system of (2) and let U be a neighbourhood of p. For $n \in Z$, there is ϵ_0 and a polynomial $R_1(x,y)$ of degree $2n+1$ such that the perturbed system (2) has at least n limit cycles in U for $0 < \epsilon < \epsilon_0$. Without loss of generality, suppose that $p = (0, y_c)$ is on the $y-$axis. Then, the perturbation term $R_1(x,y)$ can have the form*

$$R(x) = \sum_{k=0}^{n} (-1)^k \eta_k x^{2(n-k)+1}, \tag{3}$$

where $\eta_0 = 1$ *and* $\eta_r \ll \eta_{r-1}$ *(*$r = 1, \cdots, n$*).*

(ii) Suppose that (2) has N *collinear non-degenerate centers and* $R_2(x, y) = 0$. *Then the* η_k *of (3) can be so chosen that* n *limit cycles appear around each of the centers simultaneusly.*

Suppose the following conditions hold:

(A1) The unperturbed system $(2)_{\epsilon=0}$ defines a Z_q-equivariant Hamiltonian vector field ($q \geq 2$) for which all centers are non-degenerate and all saddle points are hyperbolic.

(A2) When $h \in (-\infty, h_1)$ (or $h \in (h_1, \infty)$), one branch family of the curves $\{\Gamma^h\}$ defined by the Hamiltonian function $H(x, y) = h$ lies in a global period annulus enclosing all finite singular points of $(2)_{\epsilon=0}$. As $h \to h_1, \Gamma^h$ approaches an inner boundary of the period annulus consisting of a heteroclinic (or homoclinic) loop.

We know from Li[13] that the condition (A2) holds if and only if the Hamiltonian H(x,y) of $(2)_{\epsilon=0}$ is positive (or negative) definite at infinity. Let d_0 be the maximal diameter of the area inside the inner boundary and $A > d_0$. For the "quadruple transformation" defined by Ref.[6](p222), we have the following generalized result.

Lemma 2. *Suppose that (A1) and (A2) hold. Then the map*

$$(x, y) \to (X^2 - A, Y^2 - A) \tag{4}$$

transforms (2) to a new system which has the same orbits as

$$\begin{aligned} \frac{dX}{dt} &= -\frac{\partial H_d}{\partial Y} + \epsilon Y R_1(X^2 - A, Y^2 - A), \\ \frac{dY}{dt} &= \frac{\partial H_d}{\partial X} + \epsilon X R_2(X^2 - A, Y^2 - A), \end{aligned} \tag{5}$$

where $H_d(X, Y) = H(X^2 - A, Y^2 - A)$ *is the new Hamiltonian of the unperturbed system* $(5)_{\epsilon=0}$. *Furthermore, we have (i) For the unperturbed system* $(5)_{\epsilon=0}$, *it has four times as many period annuluses as* $(2)_{\epsilon=0}$ *which lie in each quadrant and do not intersect the* $X-$*axis and* $Y-$*axis. At all image points except the origin of the singular points of* $(2)_{\epsilon=0}$, *their Hamiltonian values are preserved. There exist new singular points* $(X_i, 0)$

and $(0, Y_j)$ *on the axes where* X_i *and* Y_j *satisfy* $f_1(X_i^2 - A, -A) = 0$ *and* $f_2(-A, Y_j^2 - A) = 0$*, respectively. There is a global period annulus surrounding all finite singular points of* $(5)_{\epsilon=0}$*.*

(ii) For the perturbed system (5), it has four copies of the existing limit cycles of (2). These limit cycles do not intersect the X *and* Y *axes, if the "shift constant"* A *is moderately large.*

Proof. Notice that for $x > -A, y > -A$, all orbits of $(2)_{\epsilon=0}$ are compact and $X = \pm\sqrt{x+A}, Y = \pm\sqrt{y+A}$ can be determined, hence the conclusions of Lemma 2.2 follow. ▮

As an example to understand Lemma 2.2, we consider a Z_6-equivariant Hamiltonian system of degree 5 (see [18]):

$$\begin{aligned}\frac{dx}{dt} &= -y + 2\delta(x^2+y^2)y - \alpha(x^2+y^2)^2 y \\ &\quad +\beta(5(x^2+y^2)^2 y - 20(x^2+y^2)y^3 + 16y^5), \\ \frac{dy}{dt} &= x - 2\delta(x^2+y^2)x + \alpha(x^2+y^2)^2 x \\ &\quad +\beta(5(x^2+y^2)^2 x - 20(x^2+y^2)x^3 + 16x^5),\end{aligned}$$

or in the polar coordinates:

$$\frac{dr}{dt} = \beta r^5 \sin 6\theta, \ \frac{d\theta}{dt} = 1 - 2\delta r^2 + (\alpha + \beta\cos 6\theta) r^4, \tag{6}$$

which has the Hamiltonian

$$H(r,\theta) = -\frac{1}{2}r^2 + \frac{1}{2}\delta r^4 - \frac{1}{6}(\alpha + \beta\cos 6\theta) r^6.$$

Suppose that $\alpha > \beta > 0, \alpha+\beta > 1$ and $\delta = (\alpha+\beta+1)/2$. From Ref.[18], we know that the system (6) has 25 finite singular points at $(0,0)$ and $(z_1,0),(z_2,0),(z_3,\pi/6),(z_4,\pi/6)$ and their Z_6-equivariant symmetric points, where

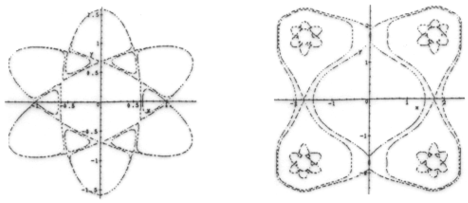
$$z_1 = \frac{1}{\sqrt{\alpha+\beta}}, \ z_2 = 1, \ z_3, \ z_4 = \sqrt{\frac{\delta \mp \sqrt{\delta^2 - \alpha + \beta}}{\alpha - \beta}}.$$

Let $G = (\alpha,\beta,\delta) = (1.4, 0.25, 1.325)$. We have $z_1 = 0.7784989442, z_2 = 1, z_3 = 0.6895372608, z_4 = 1.352363188$ and

$$h_1 = H(z_1, 0) = -0.12090603, \ h_2 = H(z_2, 0) = -0.1125$$

$h_3 = H(z_3, \pi/6) = -0.1085647965,\ h_4 = H(z_4, \pi/6) = 0.1290200579.$

In this case, the phase portrait of (6) is shown in Figure 1 (1) (only homoclinic and heteroclinic orbits are drawn in all phase portraits of this paper). Under the map $(x, y) \to (x^2 - 3, y^2 - 3)$, the new system of degree 11 is Z_2-equivariant. It has 109 finite simple singular points and the phase potrait shown in Figure 1 (2).



(1) A fifth system. **(2) Four copies of (1).**

FIG. 1. Four copies of a Z_6-equivariant Hamiltonian system.

We also need to use the following obvious conclusion.

LEMMA 3. *Suppose that the Hamiltonian function $H(x, y)$ of $(2)_{\epsilon=0}$ is Z_q-invariant, then the Hamiltonian function $H_d(X, Y) = H(X^2 - A, Y^2 - A)$ of $(5)_{\epsilon=0}$ is Z_2-invariant. In other words, the orbits of $(5)_{\epsilon=0}$ have Z_2-equivariant symmetry. Thus, if Γ_i^h is a closed orbit around a center C_i of $(5)_{\epsilon=0}$ on an axis for any $h \in (h_c, h_s)$, then*

$$\begin{aligned} I(h) &= \oint_{\Gamma_i^h} (YR_1(X^2 - A, Y^2 - A)dY - XR_2(X^2 - A, Y^2 - A)dX) \\ &= \int\int_{int\Gamma_i^h} (2XY(\frac{\partial R_1(X^2 - A, Y^2 - A)}{\partial (X^2 - A)} + \frac{\partial R_2(X^2 - A, Y^2 - A)}{\partial (Y^2 - A)}))dXdY \\ &= 0 \qquad (7) \end{aligned}$$

This lemma implies that the perturbation terms of the right hand of (5) do not create any limit cycle around the neighbourhood of a center on an axis.

In the following sections, we shall consider the following perturbed Hamiltonian system sequence:

$$\frac{dx}{dt} = -\frac{\partial H_k}{\partial y} + \epsilon P_k(x, y),$$

$$\frac{dy}{dt} = \frac{\partial H_k}{\partial x} + \epsilon Q_k(x,y), \qquad (PH_k)$$

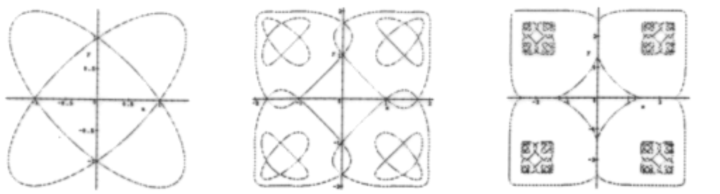
for $k = 2, 3, \cdots$, where $H_{k+1}(x,y) = H_k(x^2 - A^{k-1}, y^2 - A^{k-1}), P_{k+1}(x,y) = P_k(x^2 - A^{k-1}, y^2 - A^{k-1}), Q_{k+1}(x,y) = Q_k(x^2 - A^{k-1}, y^2 - A^{k-1})$.

3. A CORRECTION TO THE LOWER BOUNDS OF $H(2^k - 1)$ GIVEN IN [6]

We first discuss the system given in [6]. Suppose that $H_2(x,y) = (x^2 - 1)^2 + (y^2 - 1)^2$, i.e., we consider the cubic system

$$\begin{aligned} \frac{dx}{dt} &= -4y(y^2-1) + \epsilon(\frac{1}{3}(x-y)^3 - \epsilon(x-y)), \\ \frac{dy}{dt} &= 4x(x^2-1) \end{aligned} \qquad (8)$$

Let (8) be the system (PH_2). Then $(PH_2)_{\epsilon=0}$ is a Z_4-equivariant system which has the phase portrait shown in Figure 2 (1). Since $P_2(x,y) = \frac{1}{3}(x-y)^3 - \epsilon(x-y)$ and $Q_2(x,y) = 0$. By using Lemma 2.1, it follows that there exist at least 3 limit cycles around 3 centers $(-1,-1), (0,0)$ and $(1,1)$ of $(8)_{\epsilon=0}$, respectively.



(1) A cubic system. (2) A system of degree 7. (3) A system of degree 15.

FIG. 2. Copies of a Z_4-equivariant polynomial Hamiltonian vector fields.

We now consider the map: $(x,y) \to (x^2 - 1, y^2 - 1)$. By Lemma 2.2, under this map, the unperturbed system $(PH_3)_{\epsilon=0}$ has the phase portrait shown in Figure 2 (2). For the perturbed system (PH_3), the perturbed terms become $P_3^{(1)}(x,y) = yP_2(x^2-1, y^2-1)$. As the first step, the map creates a new system having at least $12 = 4 \times 3$ limit cycles surrounding the image points of (-1,-1), (0,0) and (1,1), respectively.

As the second step, by using Lemma 2.1, we take $P_3^{(2)}(x) = \eta_0 x^7 - \eta_1 x^5 + \eta_2 x^3 - \eta_3 x$. Thus, around $3 = 2^2 - 1$ centers on the $y-$axis of $(PH_3)_{\epsilon=0}$, at least $9 = 3 \times 3$ limit cycles are created.

Let $P_3(x,y) = P_3^{(1)}(x,y) + P_3^{(2)}(x)$, then the system (PH_3) has at least $S_3 = 4 \times 3 + 3 \times 3 = 21$ limit cycles.

We next consider the map:$(x,y) \to (x^2-2, y^2-2)$. By Lemma 2.2, under this map, the unperturbed system $(PH_4)_{\epsilon=0}$ has the phase portrait shown in Figure 2 (3). The same two-step method as the above shows that the system (PH_4) has at least $S_4 = 4 \times 21 + 7 \times 7 = 133$ limit cycles.

By using inductive method for the system (PH_k), first, taking the map: $(x,y) \to (x^2 - 2^{k-2}, y^2 - 2^{k-2})$, we have the perturbed terms $P_{k+1}^{(1)}(x,y) = yP_k(x^2 - 2^{k-2}, y^2 - 2^{k-2})$. Second, by using Lemma 2.1 to perturb the $2^k - 1$ centers on the $y-$axis of $(PH_{k+1})_{\epsilon=0}$, we obtain the perturbed terms $P_{k+1}^{(2)}(x)$ as (3). It gives $2^k - 1$ limit cycles. Hence, by using $P_{k+1}(x,y) = P_{k+1}^{(1)}(x,y) + P_{k+1}^{(2)}(x,y)$ as the perturbation for (PH_{k+1}), we have

$$S_{k+1} = 4 \times S_k + (2^k - 1)^2.$$

Let $S_k = 4^k \sigma_k$. Then

$$\begin{aligned}
\sigma_{k+1} &= \sigma_k + \frac{1}{4} - \frac{1}{2^{k+1}} + \frac{1}{4^{k+1}} \\
\sigma_k &= \sigma_{k-1} + \frac{1}{4} - \frac{1}{2^k} + \frac{1}{4^k} \\
&= \sigma_2 + \frac{1}{4}(k-2) - (\frac{1}{2^3} + \cdots + \frac{1}{2^k}) + (\frac{1}{4^3} + \cdots + \frac{1}{4^k}) \\
&= \sigma_2 + \frac{1}{4}k - \frac{35}{48} + \frac{1}{2^k} - \frac{1}{3 \times 4^k} \qquad (9)
\end{aligned}$$

Note that $\sigma_2 = \frac{3}{16}$. Thus,

$$S_k = 4^{k-1}(k - \frac{13}{6}) + 2^k - \frac{1}{3} \qquad (10)$$

Remark 4. It was stated in Ref.[6] (p223) that "We take $R(x,y)$ to be of the form $yR_1(x) + R_2(y), \cdots$. We then construct $R_2(y), \cdots$, to be a polynomial of degree $2^{k+1} - 1$ so that $2^k - 1$ limit cycles appear near each of the $2^k - 1$ centers on the $x-$axis." The last conclusion is incorrect!

Because $\frac{\partial R_2(y)}{\partial x} \equiv 0$, under the perturbed terms given in [6] (for which $R_2(x,y) \equiv 0$ in (2)), it has no contribution to the divergence of the vector field. Therefore, the term $R_2(y)$ cannot create any limit cycle from the centers on the $x-$axis.

Write that $n = 2^k - 1$. Then $k = \log_2(n+1) = \frac{\ln(n+1)}{\ln 2} \approx (1.442695)\ln(n+1)$. From (10), we obtain

PROPOSITION 5. *By using the Z_4-equivariant systems (PH_k) to create limit cycles, where $Q_k(x,y) = 0$ and (PH_2) is (11), we have*

$$H(n) \geq \frac{1}{4}(n+1)^2((1.442695)\ln(n+1) - \frac{13}{6}) + n + \frac{2}{3} \tag{11}$$

This Proposition is the correction of Theorem 3.4 of Ref.[6].

4. A NEW LOWER BOUND FOR $H(2^k - 1)$

In this section, we consider the perturbed Z_2-equivariant vector field (see Li and Huang [14]):

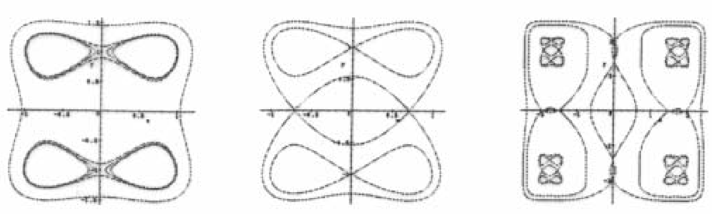
$$\begin{aligned} \frac{dx}{dt} &= y(1-y^2) + \epsilon x(y^2 - x^2 - \lambda), \\ \frac{dy}{dt} &= -x(1-2x^2) + \epsilon y(y^2 - x^2 - \lambda), \end{aligned} \tag{12}$$

where $0 < \epsilon \ll 1$. The system $(12)_{\epsilon=0}$ has Hamiltonian

$$H_2(x,y) = -2x^4 - y^4 + 2(x^2 + y^2) \tag{13}$$

There exist 9 finite singular points of $(12)_{\epsilon=0}$ which are the intersection points of the straight lines $x = 0, x = \pm\frac{1}{\sqrt{2}}$ and $y = 0, y = \pm 1$. The phase portrait of $(12)_{\epsilon=0}$ is shown in Figure 3 (2).

Let (12) be the system (PH_2) and suppose that $-4.80305 + O(\epsilon) < \lambda < -4.79418 + O(\epsilon)$. We know from [14] that the system (PH_2) has at least 11 limit cycles having the configuration shown in Figure 3 (1). By taking the map: $(x,y) \to (x^2 - 3, y^2 - 3)$, the new unperturbed system $(PH_3)_{\epsilon=0}$ has 49 finite singular points which are intersection points of the straight lines $x = 0, x = \pm\sqrt{3 \pm \frac{1}{\sqrt{2}}}, x = \pm\sqrt{3}$ and $y = 0, y = \pm\sqrt{2}, y = \pm\sqrt{3}, y = \pm 2$. The phase portrait of $(PH_3)_{\epsilon=0}$ is shown in Figure 3 (3). The first



(1) 11 limit cycles. **(2) The cubic system $(12)_0$.** **(3) A system of degree 7.**

FIG. 3. Four copies of the Z_2-equivariant Hamiltonian system $(12)_{\epsilon=0}$

perturbed terms of (PH_3) have the forms:

$$P_3^{(1)}(x,y) = y(x^2-3)((y^2-3)^2-(x^2-3)^2-\lambda),$$

$$Q_3^{(1)}(x,y) = x(y^2-3)((y^2-3)^2-(x^2-3)^2-\lambda).$$

These are polynomials of degree 7. Hence, firstly, we have from Lemma 2.2 that there exist $4\times 11 = 44$ limit cycles of (PH_3) under the first perturbations $P_3^{(1)}$ and $Q_3^{(1)}$. By Lemma 2.3, the above perturbations do not create limit cycle around the centers on the $y-$axis. Thus, secondly, we use Lemma 2.1 to add new perturbation terms $P_3^{(2)}$ and $Q_3^{(2)} = 0$ such that 3×3 limit cycles appear around the $3 = 2^2-1$ centers of $(PH_3)_{\epsilon=0}$ on the $y-$axis. To sum up, two sets of perturbations give rise to $S_3 = 4\times 11 + 3\times 3 = 53$ limit cycles of (PH_3).

By using inductive method, similar to that in Section 3, from the "quadruple transformation" $(x,y)\to(x^2-3^{k-1}, y^2-3^{k-1})$ and the bifurcations of small amplitude limit cycles around the centers on the $y-$axis for the system $(PH_{k+1}), k = 3,4,\cdots$, we have

$$S_{k+1} = 4\times S_k + (2^k-1)^2 \tag{14}$$

limit cycles. Note that $S_2 = 11$. Thus we obtain from (14) and (9) that

$$S_k = 4^{k-1}(k-\frac{1}{6}) + 2^k - \frac{1}{3} \tag{15}$$

PROPOSITION 6. *By using the Z_2-equivariant systems (PH_k) to yield limit cycles, where (PH_2) is (12), we have*

$$H(n) \geq \frac{1}{4}(n+1)^2((1.442695)\ln(n+1) - \frac{1}{6}) + n + \frac{2}{3} \tag{16}$$

5. A LOWER BOUND FOR $H(3 \times 2^{k-1} - 1)$

In this section, we consider the perturbed Z_2-equivariant vector field of degree 5 (see Ref.[19]):

$$\begin{aligned} \frac{dx}{dt} &= -y(1 - by^2 + y^4) - \epsilon x(px^4 + qy^4 + gx^2y^2 + mx^2 + ny^2 - \lambda), \\ \frac{dy}{dt} &= x(1 - ax^2 + x^4) - \epsilon y(px^4 + qy^4 + gx^2y^2 + mx^2 + ny^2 - \lambda) \end{aligned} \tag{17}$$

or its polar coordinate form:

$$\begin{aligned} \frac{dr}{dt} &= \frac{1}{4}\sin 2\theta((b-a) - (b+a)\cos 2\theta + 2r^2\cos 2\theta)r^3 \\ &- \epsilon r(r^4(p\cos^4\theta + q\sin^4\theta + g\cos^2\theta\sin^2\theta) + r^2(m\cos^2\theta + n\sin^2\theta) - \lambda), \\ \frac{d\theta}{dt} &= 1 - \frac{1}{8}(3(a+b) + 4(a-b)\cos 2\theta + (a+b)\cos 4\theta)r^2 + \frac{1}{8}(5 + 3\cos 4\theta)r^4, \end{aligned} \tag{18}$$

where $a > b > 2$. $(17)_{\epsilon=0}$ and $(18)_{\epsilon=0}$ have the Hamiltonian functions as follows:

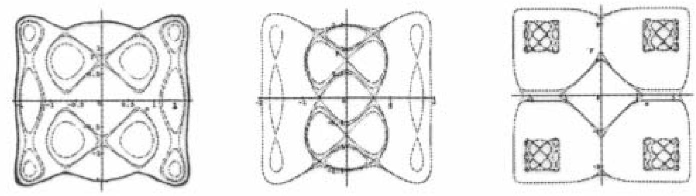
$$H(x,y) = -\frac{1}{2}(x^2+y^2) + \frac{1}{4}(ax^4 + by^4) - \frac{1}{6}(x^6 + y^6), \tag{19}$$

$$\begin{aligned} H_1(r,\theta) &= -\frac{1}{2}r^2 + \frac{1}{32}(3(a+b) + 4(a-b)\cos 2\theta \\ &\quad +(a+b)\cos 4\theta)r^4 - \frac{1}{48}(5 + 3\cos 4\theta)r^6 \end{aligned} \tag{20}$$

Denote that

$$\xi_1 = \sqrt{(a - \sqrt{a^2-4})/2},\ \xi_2 = \sqrt{(a + \sqrt{a^2-4})/2},$$

$$\eta_1 = \sqrt{(b - \sqrt{b^2-4})/2},\ \eta_2 = \sqrt{(b + \sqrt{b^2-4})/2}$$



(1) 23 limit cycles. (2) The fifth system $(17)_0$. (3) A system of degree 11.

FIG. 4. Four copies of the Z_2-equivariant Hamiltonian system $(17)_{\epsilon=0}$

It is easy to see that the system (17) has 13 centers at

$$(0,0), (\xi_1,\eta_1), (\xi_1,-\eta_1), (\xi_2,0), (\xi_2,\eta_2), (\xi_2,-\eta_2), (0,\eta_2)$$

and their Z_2-equivariant symmetric points, 12 saddle points at

$$(0,\eta_1), (\xi_1,0), (\xi_1,\eta_2), (\xi_1,-\eta_2), (\xi_2,\eta_1), (\xi_2,-\eta_1)$$

and their Z_2-equivariant symmetric points. We have from (19) that

$$h_0^c = H(0,0) = 0,$$

$$h_1^c = H(\xi_1,\eta_1) = H(\xi_1,-\eta_1) = -\frac{1}{24}(6(a+b)-(a^3+b^3)+(a^2-4)^{\frac{3}{2}}+(b^2-4)^{\frac{3}{2}}),$$

$$h_2^c = H(\xi_2,0) = -\frac{1}{24}(6a - a^3 - (a^2-4)^{\frac{3}{2}}),$$

$$h_3^c = H(0,\eta_2) = H(0,-\eta_2) = -\frac{1}{24}(6b - b^3 - (b^2-4)^{\frac{3}{2}}),$$

$$h_4^c = H(\xi_2,\eta_2) = H(\xi_2,-\eta_2) = -\frac{1}{24}(6(a+b)-(a^3+b^3)-(a^2-4)^{\frac{3}{2}}-(b^2-4)^{\frac{3}{2}});$$

and

$$h_1^s = H(\xi_1, 0) = -\frac{1}{24}(6a - a^3 + (a^2-4)^{\frac{3}{2}}),$$

$$h_2^s = H(0, \eta_1) = -\frac{1}{24}(6b - b^3 + (b^2-4)^{\frac{3}{2}}),$$

$$h_3^s = H(\xi_1, \eta_2) = -\frac{1}{24}(6(a+b) - (a^3+b^3) + (a^2-4)^{\frac{3}{2}} - (b^2-4)^{\frac{3}{2}}),$$

$$h_4^s = H(\xi_2, \eta_1) = -\frac{1}{24}(6(a+b) - (a^3+b^3) - (a^2-4)^{\frac{3}{2}} + (b^2-4)^{\frac{3}{2}})$$

Suppose that $(a, b) = (2.5, 2.3)$. We have

$$\xi_1 = 0.7071067812,\ \xi_2 = 1.415213562,\ \eta_1 = 0.762960789,\ \eta_2 = 1.310683347$$

and

$$h_1^c = -0.2436732647,\ h_2^s = -0.1290899314,\ h_3^s = -0.1215767351,\ h_1^s = -0.1145833333,$$

$$h_3^c = -0.0069934018,\ h_4^s = 0.03757673603,\ h_4^c = 0.1596732652,\ h_2^c = 0.16666666667,$$

$$-\infty < h_1^c < h_2^s < h_3^s < h_1^s < h_3^c < 0 < h_4^s < h_4^c < h_2^c.$$

The unperturbed system $(17)_{\epsilon=0}$ has the phase portrait of Figure 4 (2). In Ref.[19], we showed that when

$$(p, q, g, m, n) = (-.144543, 1.157350656, -3.328234861, 3.014502, 6.564525872),$$

$$\lambda \in (\lambda_1(h_2^s), min(max(\lambda_1(h)), max(\lambda_6(h)))) \approx (9.319050412, 9.319051762),$$

the system (17) has at least 23 limit cycles having the configuration shown in Figure 4 (1).

We now take (17) as (PH_2). Under the map $(x, y) \to (x^2-3, y^2-3)$, the new system $(PH_3)_{\epsilon=0}$ has the phase portrait shown in Figure 4 (3). There exist 121 finite singular points of $(PH_3)_{\epsilon=0}$ consisting of the intersection points of the straight lines $x = 0, x = \pm\sqrt{3 \pm \xi_i}, x = \sqrt{3}$ and $y = 0, y = \pm\sqrt{3 \pm \eta_i}, y = \sqrt{3}, (i = 1, 2)$. There are $5 = 3 \times 2^{2-1} - 1$ centers on the $y-$axis. By Lemma 2.2, the perturbed terms

$$\begin{aligned} P_3^{(1)}(x,y) &= y(x^2-3)(p(x^2-3)^4+q(y^2-3)^4 \\ &\quad +g(x^2-3)^2(y^2-3)^2+m(x^2-3)^2+n(y^2-3)^2-\lambda) \\ Q_3^{(1)}(x,y) &= x(y^2-3)(p(x^2-3)^4+q(y^2-3)^4 \\ &\quad +g(x^2-3)^2(y^2-3)^2+m(x^2-3)^2+n(y^2-3)^2-\lambda) \end{aligned}$$

quadruple the number of limit cycles of (PH_2), i.e., there exist $92 = 4 \times 23$ limit cycles of (PH_3). Next, by using Lemma 2.1 to perform secondary perturbation for 5 centers on the $y-$axis of $(PH_3)_{\epsilon=0}$, we have $P_3^{(2)} = \eta_0 x^{11} - \eta_1 x^9 + \cdots + \eta_4 x^3 - \eta_5 x, Q_3^{(2)}(x,y) = 0$. It give rise to $5^2 = (3 \times 2^{2-1} - 1)^2 = 25$ limit cycles. Thus, the system (PH_3) has at least 92+25=117 limit cycles.

Again by using inductive method, suppose that the system (PH_k) has S_k limit cycles. First, transform the system (PH_k) by the quadruple map: $(x,y) \to (x^2 - 3^{k-1}, y^2 - 3^{k-1})$. Then perform secondary perturbation to the centers on the $y-$axis of $(PH_{k+1})_{\epsilon=0}$. We have

$$S_{k+1} = 4 \times S_k + (3 \times 2^{k-1} - 1)^2. \tag{21}$$

Also let $S_k = 4^k \sigma_k$. Similar to the computation of (8), we have

$$\begin{aligned} \sigma_k &= \sigma_2 + \frac{9}{16}(k-2) - \frac{3}{2}(\frac{1}{2^3} + \cdots + \frac{1}{2^k}) + (\frac{1}{4^3} + \cdots + \frac{1}{4^k}) \\ &= \sigma_2 + \frac{9}{16}k - \frac{71}{48} + \frac{3}{2^{k+1}} - \frac{1}{3 \times 4^k} \end{aligned} \tag{22}$$

Notice that $\sigma_2 = \frac{23}{16}$. Thus,

$$S_k = 4^{k-1}(\frac{9}{4}k - \frac{1}{6}) + 3 \times 2^{k-1} - \frac{1}{3} \tag{23}$$

Let $n = 3 \times 2^{k-1} - 1$. Then, $k - 1 = \log_2(\frac{n+1)}{3}) = \frac{\ln(n+1) - \ln 3}{\ln 2} \approx (1.442695)(\ln(n+1) - \ln 3) \approx (1.442695)\ln(n+1) - 1.5849625$. We have from (23) that

PROPOSITION 7. *By considering the Z_2-equivariant systems (PH_k) to yield limit cycles, where (PH_2) is (17), we have*

$$H(n) \geq \frac{1}{4}(n+1)^2((1.442695)\ln(n+1) + \frac{25}{27} - \frac{\ln 3}{\ln 2}) + n + \frac{2}{3}$$

Denote that $\mu = \frac{1}{4\ln 2} \approx 0.360673$. To sum up, the propositions 3.2, 4.1 and 5.1 imply that

THEOREM 5.2. *There are two sequences of $n = 2^k - 1$ and $n = 3 \times 2^{k-1} - 1, k = 2, 3, \cdots$, and a constant $\mu = (4 \times \ln 2)^{-1}$ such that the number $H(n)$ of limit cycles of the systems (PH_k) grows at least as rapidly as $\mu(n+1)^2 \ln(n+1)$.*

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