QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **3**, 285–307 (2002)
ARTICLE NO. 40

Versal Unfoldings for Rank-2 Singularities of Positive Quadratic Differential Forms

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In [5], local problems around a class of rank-2 singular points called simple such as normal forms, finite determinacy, and versal unfoldings are studied for smooth positive quadratic differential forms on surfaces, as well as for their associated pair of foliations (with singularities). To extend this study to the class of rank-2 singular points, two cases of rank-2 singular points remain to be treated, namely that of type C and that of type E(λ), with $\lambda \geq 1$. Using the theory of normal forms for singularities of positive quadratic differential forms, we obtain the phase portrait and a versal unfolding for type C singular points proving that their codimension is three.

Key Words: Quadratic differential forms, one-dimensional foliations, versal unfolding.

1. INTRODUCTION

This paper focuses on local problems around rank-2 singular points of smooth positive quadratric differential forms. We deal with normal forms, finite determinacy and versal unfoldings. For a class of generic singular

* Part of this work was supported by FONDECYT Grant # 1980778, and by USACH Grant # 049633GM.
† Part of this work was supported by UTFSM Grants # 120121.

points called *simple* similar problems are studied in [5]. We propose to complete our study to any rank-2 singular point.

We now recall standard definitions and give an overview of the main results of [5].

A C^r– quadratic differential form on an oriented, connected, smooth two–manifold M is an element of the form $\omega = \sum_{k=1}^{n} \phi_k \psi_k$, where ϕ_k and ψ_k are 1-forms on M of class C^r. Therefore, for each point p in M, we have that $\omega(p)$ is a quadratic form on the tangent space T_pM. We say that ω is *positive* if, for every point p in M, the subset $\omega(p)^{-1}(0)$ of T_pM is either the union of two transversal lines or all T_pM. In the former case, p will be called a *regular point* of ω. In the latter case, p will be called a *singular point* of ω.

If p is a regular point of such a ω, we call $L_1(\omega)(p)$ (resp. $L_2(\omega)(p)$) the line of $\omega(p)^{-1}(0)$ which is characterized as follows. Let C be a positively oriented circle around the origin of T_pM. By definition, $q \in C \cap L_1(\omega)(p)$ (resp. $q \in C \cap L_2(\omega)(p)$) if there exists a small open arc (q_1, q_2) on C containing q such that $\omega(p)$ is positive (resp. negative) on (q_1, q) and negative (resp. positive) on (q, q_2). Thus we associate to each positive C^r-quadratic differential form ω on M a triplet $C(\omega) = \{ f_1(\omega) , f_2(\omega) , Sing(\omega)\}$ which is called the *configuration associated to* ω, where $Sing(\omega)$ is the set of singular points of ω, and $f_1(\omega)$ and $f_2(\omega)$ are the two transversal C^r *one-dimensional foliations* defined on $M - Sing(\omega)$ whose tangent lines at each regular point p are $L_1(\omega)(p)$ and $L_2(\omega)(p)$, respectively.

Two positive C^∞-quadratic differential forms ω_1 and ω_2 will be called *equivalent* if there exists a homeomorphism h of M such that $h(C(\omega_1)) = C(\omega_2)$. In other words, h maps $Sing(\omega_1)$ onto $Sing(\omega_2)$, and leaves of $f_1(\omega_1)$ (resp. $f_2(\omega_1)$) onto leaves of $f_1(\omega_2)$ (resp. $f_2(\omega_2)$). A positive C^∞-quadratic differential form ω will be called *structurally stable* if any positive C^∞-quadratic differential form $\tilde{\omega}$, which is C^1–sufficiently close to ω, is equivalent to ω. A complete characterization of structurally stable positive C^∞-quadratic differential forms with the C^2-topology is obtained in [1] and [3].

In order to introduce the type of singular points studied here, we recall that a singular point p of a positive quadratic differential form ω is said to be of *rank-k*, for $k = 0, 1, 2$, if there are local coordinates (x, y) taking p to the origin such that if $(x, y)^*(\omega) = \omega = a(x, y)dy^2 + 2b(x, y)dxdy + c(x, y)dx^2$, then the Jacobian matrix of the map $g = (a, 2b, c)$ at the origin has rank-k. A rank-2 singular point p as above is called *simple* if the origin is a nondegenerate minimum of the map $h = b^2 - ac$; otherwise p is called *semi-simple*.

A normal form for a simple singular point is

$$\omega = a(x, y)dy^2 + 2b(x, y)dxdy + c(x, y)dx^2 ,$$

with

$$\begin{aligned} a(x,y) &= y + M_1(x,y) \\ b(x,y) &= b_1 x + b_2 y + M_2(x,y) \\ c(x,y) &= -y + M_3(x,y) \end{aligned} \tag{1}$$

where $b_1 \neq 0$ and $M_i(x,y) = O((x^2+y^2)^{\frac{1}{2}})$, for $i = 1,2,3$.

A singular point as above is called *hyperbolic* if $b_2^2 - 2b_1 + 1 \neq 0$ and $b_1 \neq \frac{1}{2}$. These singular points are C^1-locally stable, that is, their codimension is zero (see [1, Proposition 6.2]). There are three types of hyperbolic singular points: type D_1 if $b_2^2 - 2b_1 + 1 < 0$, type D_2 if $b_2^2 - 2b_1 + 1 > 0$, $b_1 > 0$, $b_1 \neq \frac{1}{2}$ and type D_3 if $b_1 < 0$. A corresponding versal unfolding is

$$ydy^2 + 2xdxdy - ydx^2\,, \quad ydy^2 + \frac{1}{2}xdxdy - ydx^2\,, \quad ydy^2 - 2xdxdy - ydx^2.$$

Figure 1 shows their phase portraits.

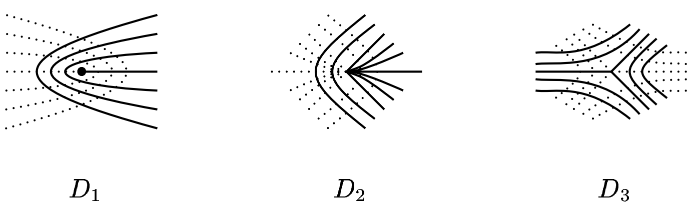


Figure 1.

A simple singular point which admits a normal form as in (1), with $b_2 \neq 0$ and either $b_2^2 - 2b_1 + 1 = 0$ or $b_1 = \frac{1}{2}$, is called of type D_{12}. Its codimension is one and a versal unfolding is

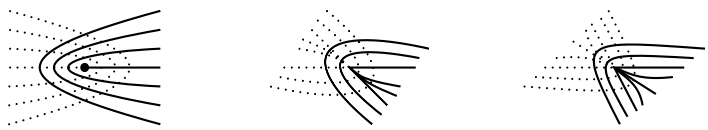
$$ydy^2 + 2((1-\lambda)x + y)dxdy - ydx^2\,,$$

with $\lambda \in I\!R$ such that $\lambda < \frac{1}{2}$ (see [5, Theorem 5.7]). The corresponding bifurcation diagram is shown in Figure 2.

A simple singular point which admits a normal form as in (1), with $b_2 = 0$ and $b_1 = \frac{1}{2}$, is said to be of type $\tilde{D}_1$. It is of codimension 2 and a versal unfolding is

$$ydy^2 + xdxdy + (\lambda_1 x + (-1+\lambda_2)y)dx^2,$$

with $\lambda_1, \lambda_2 \in I\!R$ small (see [5, Theorem 5.9]). Figure 3 shows the configuration around the singular point, as well as its bifurcation diagram.



$\lambda < 0 : D_1$ $\lambda = 0 : D_{12}$ $\lambda > 0 : D_2$

Figure 2.

Figure 3.

When p is a semi-simple singular point of ω, the coordinates (x, y) may be chosen in such a way that the Jacobian matrix of the map $g = (a, 2b, c)$ at the origin is one of the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \lambda & 0 \\ 0 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 \\ \lambda & 0 \\ 0 & 0 \end{pmatrix},$$

with $\lambda \neq 0$. (See [3, Proposition 2.9, p. 7].)

These singularities are called, respectively, of type A, B, C, E(λ) and F(λ). Only those of type A, B, E(λ), with $\lambda < 0$ and $0 < \lambda < 1$, as well as those of type F(λ) with $\lambda \neq 0$, $\frac{1}{4}$, are C^1–locally stable (see [3, Theorem B, p. 8]). The singular points of type F($\frac{1}{4}$) are topologically determined by their linear part. They are of codimension 1 (see [4, Proposition 4.3, p. 13]) and a versal unfolding, which is equivalent to a versal unfolding of a D_{12}-singular point, is

$$(x + y)dy^2 + \frac{1}{4}(1 + \lambda)xdxdy - (x^3 + y^3)dx^2 ,$$

with $\lambda \in I\!R$.

We consider in this paper one of the two cases of rank-2 that remain to be treated, namely the case in which the singular point is of type C under a *nonflatness condition* which we next define. The last case to be treated, E(λ), with $\lambda \geq 1$, will be presented in a forthcoming article.

Finally, after a linear change of coordinates, the associated matrix of a positive quadratic differential form at a singular point of type C is

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

In what follows, we will use this linear part.

The paper is organized as follows:

Section 2 contains general results on rank-2 singular points, as well as on smooth k-parameter families of positive quadratic differential forms passing through a form with a rank-2 singular point.

In Section 3 via an analytic change of coordinates, we find a normal form for any family $\omega(\mu)$, with parameter $\mu \in I\!R^k$, such that $\omega(0)$ has a type C singular point (see Theorem 3.1 of Section 3). This implies that if ω has a type C singular point at the origin, then there is an analytic change of coordinates such that if

$$\omega = a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2 ,$$

then

$$\begin{aligned} a(x,y) &= -x + y + M_1(x,y) , \\ b(x,y) &= \frac{1}{2}y + b_2x^2 + b_3x^3 + \cdots + b_nx^n + M_2(x,y) , \\ c(x,y) &= c_3x^3 + c_4x^4 + \cdots + c_nx^n + M_3(x,y) , \end{aligned}$$

with $M_1 = O(|\ (x,y)\ |)$ and $M_2, M_3 = O(|\ (x,y)\ |^n)$.

Using the normal form above, we define the nonflatness condition as follows. We require that the maps $b(x,0)$ or $c(x,0)$ be nonflat in $x = 0$. Since ω is positive, this condition implies that there exist an integer $k \geq 2$ and a pair $(b_0, c_0) \neq (0,0)$, with $c_0 \geq 0$, such that

$$\begin{aligned} a(x,y) &= -x + y + M_1(x,y) \\ b(x,y) &= \frac{1}{2}y + b_0x^k + M_2(x,y) \\ c(x,y) &= c_0x^{2k-1} + M_3(x,y) , \end{aligned}$$

with $M_1(x,y) = O(|\ (x,y)\ |)$, $M_2(x,y) = O(|\ (x,y)\ |^k)$ and $M_3(x,y) = O(|\ (x,y)\ |^{2k-1})$ (see Remark 3.2 of Section 3).

In Section 4, we determine the local phase portrait of the foliations associated with a positive quadratic differential form around a nonflat type

C singular point and show that it is always topologically equivalent to the one shown in Figure 4 (see Theorem 4.1).

Figure 4.

Section 5 contains our main result. For a nonflat type C singular point, we find a versal unfolding and show that its codimension is three. Further, we present its bifurcation diagram.

2. PRELIMINARY RESULTS

In this section, we give some general results on rank-2 singular points and smooth k-parameter families of positive quadratic differential forms passing through a form with a rank-2 singular point. These results are used in the subsequent sections, as well as in a forthcoming article concerning singular points of type E(λ), with $\lambda \geq 1$.

We first record a proposition that appears in [4, Proposition 2], which asserts that the rank-2 singular points are persistent by small perturbations of the positive quadratic differential form. In other words,

PROPOSITION 1. *Let $\omega_0 \in \mathcal{F}(M)$, and let $p \in Sing(\omega_0)$. If p is a rank-2 singular point of ω_0, then there exist neighborhoods $\mathcal{N}$ of ω_0 and V of p as well as a C^∞ –map $\rho : \mathcal{N} \to V$ such that, for every $\omega \in \mathcal{N}$, the point $\rho(\omega)$ is the unique singular point of ω in V. Moreover, this singular point is of rank-2.*

Our next result asserts that, for a smooth family $\omega(\mu)$ of positive C^∞-quadratic differential forms with parameter $\mu \in I\!R^k$, in such that $\omega(0)$ has a rank-2 singular point at the origin, we may assume, without loss of generality, that the origin is a singular point of $\omega(\mu)$, for all $|\mu|$ small. In other words,

LEMMA 2. *Let $\omega(\mu)$ be an arbitrary smooth family of positive C^∞-quadratic differential forms with parameter $\mu \in I\!R^k$ such that $\omega(0)$ has a rank-2 singular point at the origin. Then there exists a change of coordinates of the form $(x, y, \mu) = (h(u, v, \mu), \mu)$ such that, for all μ with $|\mu|$*

small, the origin is a singular point of

$$(u, v)^*(\omega(\mu)).$$

Proof. By hypothesis, $(a, b, c)((0, 0), \bar{0})) = (0, 0)$. Assume that

$$D_1(a, b)((0, 0), \bar{0}) = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

is nonsingular. Since the map $(a, b) : \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}^2$ is smooth, by the implicit function theorem there exists a smooth map S defined in a small neighborhood of $\bar{0} \in \mathbb{R}^k$ such that $S(\bar{0}) = (0, 0)$ and

$$(a, b)(S(\mu), \mu) = (0, 0)\,,$$

for all μ in such a neighborhood.

Since $\omega(\mu)$ is positive for all μ, we have $(a, b, c)(S(\mu), \mu) = (0, 0, 0)$. Hence in this case the lemma follows by using the change of coordinates

$$(x, y, \mu) = (u, v, \mu) - (S(\mu), \bar{0}).$$

If the matrix $D_1(a, b)((0, 0), \bar{0})$ is singular, then $D_1(b, c)((0, 0), \bar{0})$ is nonsingular and the proof of the lemma is analogous. ▮

The following gives us a normal form for the linear part of $\omega(\mu)$ at rank-2 singular points which are of two classes: one class includes the singular points of type C, while the other class includes the singular points of type E(λ).

Lemma 3. *Let $\omega(\mu)$ be a smooth family of positive C^∞-quadratic differential forms with parameter $\mu \in \mathbb{R}^k$. Assume that $\omega(0)$ has a rank-2 singular point p_0. Let $U \subset M$ be a neighborhood of p_0, and let $V \subset \mathbb{R}^k$ be a neighborhood of the origin such that $Sing(\omega(\mu)) \cap U = \{p(\mu)\}$, for every $\mu \in V$, where $p(\mu)$ is the C^∞ continuation of the singular point $p_0 = p(0)$ in U.*

Let $(x, y) : (U, p_0) \to (\mathbb{R}^2, (0, 0))$ be a local chart such that

$$\begin{aligned}(x, y)^*(\omega_0) \;&=\; (a_1 x + a_2 y + M_1(x, y))dy^2 + 2(b_1 x + b_2 y + M_2(x, y))dxdy \\ &\quad +(c_1 x + c_2 y + M_3(x, y))dx^2\,,\end{aligned}$$

with $M_i(x, y) = O((x^2 + y^2)^{1/2})$, for $i = 1, 2, 3$.

There exists a local chart $\phi : (U_0 \times V_0, (p_0, 0)) \to (\mathbb{R}^2 \times \mathbb{R}^k, (0, 0, 0))$ of the form $\phi(p, \mu) = (x(p, \mu), y(p, \mu), \mu)$, with $\phi(p, 0) = (x(p), y(p), 0)$ for all $p \in U$ and $\phi(p(\mu), \mu) = (0, 0, \mu)$ for all $\mu \in V$, such that: if $(x, y)^*(\omega(\mu)) = a(x, y, \mu)dy^2 + b(x, y, \mu)dxdy + c(x, y, \mu)dx^2$, *(a) and if* $a_1, a_2 \neq 0$ *and* $a_1 + 2b_2 = 0$*, then*

$$\begin{aligned} a(x,y,\mu) &= a_1 x + a_2 y + M_1(x,y,\mu)\,, \\ b(x,y,\mu) &= (b_1 + B_1(\mu))x - \frac{1}{2}a_1 y + M_2(x,y,\mu) \quad \textit{and} \\ c(x,y,\mu) &= (c_1 + C_1(\mu))x + (c_2 + C_2(\mu))y + M_3(x,y,\mu), \end{aligned}$$

with $B_1(0) = C_1(0) = C_2(0) = 0$*;*
(b) and if $b_1, 2b_1 + c_2 \neq 0$ *and* $c_1 = 0$*, then*

$$\begin{aligned} a(x,y,\mu) &= a_1 x + (a_2 + A_2(\mu))y + M_1(x,y,\mu)\,, \\ b(x,y,\mu) &= b_1 x + (b_2 + B_2(\mu))y + M_2(x,y,\mu) \quad \textit{and} \\ c(x,y,\mu) &= (c_2 + C_2(\mu))y + M_3(x,y,\mu)\,, \end{aligned}$$

with $A_2(0) = B_2(0) = C_2(0) = 0\,,$

where in both cases $M_k(0,0,\mu) = \frac{\partial M_k}{\partial x}(0,0,\mu) = \frac{\partial M_k}{\partial y}(0,0,\mu) = 0$*, for all* (k,μ) *in* $\{1,2,3\} \times V_0$*.*

Proof. According to the preceding lemma and taking $U \times V$ smaller if necessary, we may assume that $U \times V$ is open and that there exists a local chart $\phi : (U \times V, (p_0, 0)) \to (\mathbb{R}^2 \times \mathbb{R}^k, (0,0,0))$ of the form $\phi(p,\mu) = (x(p,\mu), y(p,\mu), \mu)$ such that $\phi(p,0) = (x(p), y(p), 0)$ for every $p \in U$ and $\phi(p(\mu),\mu) = (0,0,\mu)$, for every $\mu \in V$.

Thus if $(x,y)^*(\omega(\mu)) = a(x,y,\mu)dy^2 + b(x,y,\mu)dxdy + c(x,y,\mu)dx^2$, then

$$\begin{aligned} a(x,y,\mu) &= a_1(\mu)x + a_2(\mu)y + M_1(x,y,\mu)\,, \\ b(x,y,\mu) &= b_1(\mu)x + b_2(\mu)y + M_2(x,y,\mu)\,, \quad \text{and} \\ c(x,y,\mu) &= c_1(\mu)x + c_2(\mu)y + M_3(x,y,\mu)\,, \end{aligned}$$

with

$$(a_1, b_1, c_1)(\mu) = (a_1, b_1, c_1) + (A_1, B_1, C_1)(\mu)\,,$$

$$(a_2, b_2, c_2)(\mu) = (a_2, b_2, c_2) + (A_2, B_2, C_2)(\mu)\,,$$

and

$$M_k(0,0,\mu) = \frac{\partial M_k}{\partial x}(0,0,\mu) = \frac{\partial M_k}{\partial y}(0,0,\mu) = 0\,,$$

for all $(k,\mu) \in \{1,2,3\} \times V$.

Let $A_{(\alpha,\beta,\gamma,\delta,\mu)} : \mathbb{R}^2 \to \mathbb{R}^2$ be the family of linear isomorphisms with parameter $(\alpha,\beta,\gamma,\delta,\mu) \in \mathbb{R}^4 \times V$ such that if $A = A_{(\alpha,\beta,\gamma,\delta,\mu)}$, then its inverse is given by

$$A^{-1}(x,y,\mu) = ((1+\alpha)x + \beta y, \gamma x + (1+\delta)y, \mu).$$

Under these conditions, we have that $\bar{\phi} = A \circ \phi$ is of the form $\bar{\phi}(p,\mu) = (u(p,\mu), v(p,\mu), \mu)$. Therefore, if $(u,v)^*(\omega(\mu)) = \bar{a}(u,v,\mu)dv^2 + \bar{b}(u,v,\mu)dudv + \bar{c}(u,v,\mu)du^2$, then

$$\begin{aligned}
\bar{a}(u,v,\mu) = {} & \bar{a}_1(\mu)u + \bar{a}_2(\mu)v + \bar{M}_1(u,v,\mu)\\
\bar{b}(u,v,\mu) = {} & \bar{b}_1(\mu)u + \bar{b}_2(\mu)v + \bar{M}_2(u,v,\mu)\\
\bar{c}(u,v,\mu) = {} & \bar{c}_1(\mu)u + \bar{c}_2(\mu)v + \bar{M}_3(u,v,\mu)
\end{aligned}$$

where

$$\begin{aligned}
\bar{a}_1(\mu) = {} & (1+\delta)^2[(1+\alpha)a_1(\mu) + \gamma a_2(\mu)] +\\
& 2\beta(1+\delta)[(1+\alpha)b_1(\mu) + \gamma b_2(\mu)] +\\
& \beta^2[(1+\alpha)c_1(\mu) + \gamma c_2(\mu)]\,,\\
\bar{a}_2(\mu) = {} & (1+\delta)^2[\beta a_1(\mu) + (1+\delta)a_2(\mu)] +\\
& 2\beta(1+\delta)[\beta b_1(\mu) + (1+\delta)b_2(\mu)] +\\
& \beta^2[\beta c_1(\mu) + (1+\delta)c_2(\mu)]\,,\\
\bar{b}_1(\mu) = {} & \gamma(1+\delta)[(1+\alpha)a_1(\mu) + \gamma a_2(\mu)] +\\
& [(1+\alpha)(1+\delta) + \beta\gamma][(1+\alpha)b_1(\mu) + \gamma b_2(\mu)] +\\
& (1+\alpha)\beta[(1+\alpha)c_1(\mu) + \gamma c_2(\mu)]\,,\\
\bar{b}_2(\mu) = {} & \gamma(1+\delta)[\beta a_1(\mu) + (1+\delta)a_2(\mu)] +\\
& [(1+\alpha)(1+\delta) + \beta\gamma][(\beta b_1(\mu) + (1+\delta)b_2(\mu)] +\\
& (1+\alpha)\beta[\beta c_1(\mu) + (1+\delta)c_2(\mu)]\,,\\
\bar{c}_1(\mu) = {} & \gamma^2[(1+\alpha)a_1(\mu) + \gamma a_2(\mu)] +\\
& 2(1+\alpha)\gamma[(1+\alpha)b_1(\mu) + \gamma b_2(\mu)] +\\
& (1+\alpha)^2[(1+\alpha)c_1(\mu) + \gamma c_2(\mu)] \quad \text{and}\\
\bar{c}_2(\mu) = {} & \gamma^2[\beta a_1(\mu) + (1+\delta)a_2(\mu)] +\\
& 2(1+\alpha)\gamma[\beta b_1(\mu) + (1+\delta)b_2(\mu)] +\\
& (1+\alpha)^2[\beta c_1(\mu) + (1+\delta)c_2(\mu)]\,.
\end{aligned}$$

Remark that

$$(\bar{a}_1, \bar{b}_1, \bar{c}_1)(0) = (a_1, b_1, c_1)$$

and that

$$(\bar{a}_2, \bar{b}_2, \bar{c}_2)(0) = (a_2, b_2, c_2)\,.$$

Case $\mathbf{a_1}$, $\mathbf{a_2 \neq 0}$ and $\mathbf{a_1 + 2b_2 = 0}$. We first assume $a_1(\mu) + 2b_2(\mu) \equiv 0$, for all μ, with $\mid \mu \mid$ small. We leave this condition invariant by setting $\beta \equiv \gamma \equiv 0$, and we obtain

$$\begin{aligned}
\bar{a}_1(\mu) &= (1+\alpha)(1+\delta)^2(a_1 + A_1(\mu))\\
\bar{a}_2(\mu) &= (1+\delta)^3(a_2 + A_2(\mu))\,.
\end{aligned}$$

By the implicit function theorem, there exist $\alpha = \alpha(\mu)$ and $\delta = \delta(\mu)$, with $\alpha(0) = \delta(0) = 0$, such that $\bar{a}_2(\mu) \equiv a_2$ and $\bar{a}_1(\mu) \equiv a_1$. The latter identity implies $\bar{b}_2(\mu) \equiv -\frac{1}{2}a_1$.

Now if $a_1(\mu)+2b_2(\mu)\not\equiv 0$, setting $\alpha\equiv\beta\equiv\delta\equiv 0$ we obtain

$$\bar{a}_1(\mu)+2\bar{b}_2(\mu)=A_1(\mu)+2B_2(\mu)+3\gamma[a_2+A_2(\mu)]$$

and the result follows from the implicit function theorem.

Case $\mathbf{b_1}$, $\mathbf{2b_1+c_2\neq 0}$ and $\mathbf{c_1=0}$. We first assume $c_1(\mu)\equiv 0$, for all μ, with $\mid\mu\mid$ small. We leave this condition invariant by setting $\gamma\equiv 0$, and we obtain

$$\begin{aligned}\bar{a}_1(\mu) &= (1+\delta)^2(1+\alpha)a_1(\mu)+2\beta(1+\delta)(1+\alpha)b_1(\mu)\\ \bar{b}_1(\mu) &= (1+\alpha)^2(1+\delta)b_1(\mu)\,.\end{aligned}$$

We next set $\alpha\equiv 0$. Then according to the implicit function theorem, there exist $\delta=\delta(\mu)$ and $\beta=\beta(\mu)$, with $\delta(0)=\beta(0)=0$, such that $\bar{a}_1(\mu)\equiv a_1$ and $\bar{b}_1(\mu)\equiv b_1$.

Now if $c_1(\mu)\not\equiv 0$, setting $\alpha\equiv\beta\equiv\delta\equiv 0$ we obtain

$$\bar{c}_1(\mu)=\gamma^3a_2(\mu)+\gamma^2(a_1(\mu)+2b_2(\mu))+\gamma(2b_1(\mu)+c_2(\mu))+c_1(\mu)$$

and the result follows from the implicit function theorem. The proof is now complete. ▌

3. NORMAL FORMS

Let

$$\omega(\mu)=a(x,y,\mu)dy^2+2b(x,y,\mu)dxdy+c(x,y,\mu)dx^2$$

be an arbitrary smooth family, with parameter $\mu\in I\!R^k$, such that $\omega_0=\omega(0)$ has a type C singular point at the origin.

We may assume

$$\begin{aligned}a(x,y,\mu) &= -x+y+M_1(x,y,\mu)\,,\\ b(x,y,\mu) &= b_1(\mu)x+\frac{1}{2}y+M_2(x,y,\mu)\,,\\ c(x,y,\mu) &= c_1(\mu)x+d_1(\mu)y+M_3(x,y,\mu)\,,\end{aligned}$$

with $b_1(0)=c_1(0)=d_1(0)=0$, $M_i(x,y,\mu)=O(\mid(x,y)\mid)$, for $i=1,2,3$.

Theorem 4. *Let k be an integer greater than or equal to two. Then there exists a local chart $\phi:(U_0\times V_0,((0,0),0))\to(I\!R^2\times I\!R^k,((0,0),0))$ of the form $\phi(p,\mu)=(u(p,\mu),v(p,\mu),\mu)$, with $\phi(p,0)=(u(p),v(p),0)$ for all $p\in U_0$, and $\phi((0,0),\mu)=(0,0,\mu)$, for all $\mu\in V_0$. Moreover, such a ϕ*

may be found such that if $(u,v)^*(\omega(\mu)) = \tilde{a}(u,v,\mu)dv^2 + \tilde{b}(u,v,\mu)dudv + c(u,v,\mu)du^2$*, then*

$$\begin{aligned}
\tilde{a}(u,v,\mu) &= -u + v + N_1(u,v,\mu)\,,\\
\tilde{b}(u,v,\mu) &= b_1(\mu)u + \frac{1}{2}v + b_2(\mu)u^2 + \cdots + b_k(\mu)u^k + N_2(u,v,\mu)\,,\ \textit{and}\\
\tilde{c}(u,v,\mu) &= c_1(\mu)u + d_1(\mu)v + c_2(\mu)u^2 + \cdots + c_k(\mu)u^k + N_3(u,v,\mu)\,,
\end{aligned}$$

with $b_1(0) = c_1(0) = d_1(0) = 0$*,* $N_1(u,v,\mu) = O(|\ (u,v)\ |)$*, and* $N_i(u,v,\mu) = O(|\ (u,v)\ |^k)$*, for* $i = 2,3$.

Proof. Let $2 \le m \le k$ be an integer. Let

$$\begin{aligned}
p_m(x,y,\mu) &= A_{m,0}(\mu)x^m + A_{m-1,1}(\mu)x^{m-1}y + \cdots + A_{0,m}(\mu)y^m\,,\\
q_m(x,y,\mu) &= B_{m,0}(\mu)x^m + B_{m-1,1}(\mu)x^{m-1}y + \cdots + B_{0,m}(\mu)y^m\,, \quad \text{and}\\
r_m(x,y,\mu) &= C_{m,0}(\mu)x^m + C_{m-1,1}(\mu)x^{m-1}y + \cdots + C_{0,m}(\mu)y^m
\end{aligned}$$

be the homogeneous parts of degree m of the maps $a(.,.,\mu)$, $b(.,.,\mu)$, and $c(.,.,\mu)$, respectively.

In a neighborhood of the origin, we consider a change of coordinates of the form $(x,y,\mu) = (u,v,\mu) + (\alpha(x,y,\mu), \beta(x,y,\mu), 0)$, with

$$\begin{aligned}
\alpha(x,y,\mu) &= \alpha_{m,0}(\mu)x^m + \alpha_{m-1,1}(\mu)x^{m-1}y + \cdots + \alpha_{0,m}(\mu)y^m \quad \text{and}\\
\beta(x,y,\mu) &= \beta_{m,0}(\mu)x^m + \beta_{m-1,1}(\mu)x^{m-1}y + \cdots + \beta_{0,m}(\mu)y^m
\end{aligned}$$

whose coefficients we will determine. Now, as in the proof of Proposition 3.1 of [5], if

$$(u,v)^*(w(\mu)) = \tilde{a}(u,v,\mu)dv^2 + 2\tilde{b}(u,v,\mu)dudv + \tilde{c}(u,v,\mu)du^2\,,$$

then

$$\begin{aligned}
(\tilde{a},\tilde{b},\tilde{c})(u,v,\mu) &= (-u+v, b_1(\mu)u + \frac{1}{2}v, c_1(\mu)u + d_1(\mu)v) +\\
&\quad (p_2,q_2,r_2)(u,v,\mu) + \cdots + (p_m,q_m,r_m)(u,v,\mu) +\\
&\quad ad_m(\alpha(u,v,\mu),\beta(u,v,\mu)) + (\tilde{P}_m,\tilde{Q}_m,\tilde{R}_m)(u,v,\mu)\,,
\end{aligned}$$

with $\tilde{P}_m(u,v,\mu)$, $\tilde{Q}_m(u,v,\mu)$, $\tilde{R}_m(u,v,\mu) = O(|\ (u,v)\ |^m)$.

Therefore, the terms of degree less than m are preserved, whereas the resulting ones of degree m are

$$(\tilde{p}_m,\tilde{q}_m,\tilde{r}_m)(u,v,\mu) = (p_m,q_m,r_m)(u,v,\mu) + ad_m(\alpha(u,v,\mu),\beta(u,v,\mu))\,,$$

where $ad_m = (ad_m^1, ad_m^2, ad_m^3)$ with

$$\begin{aligned} ad_m^2(\alpha(u,v,\mu),\beta(u,v,\mu)) &= b_1(\mu)\alpha(\mu) + \frac{1}{2}\beta(\mu) + (-u+v)\frac{\partial\beta}{\partial u} + \\ &\quad (b_1(\mu)u + \frac{1}{2}v)(\frac{\partial\alpha}{\partial u} + \frac{\partial\beta}{\partial v}) + \\ &\quad (c_1(\mu)u + d_1(\mu)v)\frac{\partial\alpha}{\partial v}, \\ ad_m^3(\alpha(u,v,\mu),\beta(u,v,\mu)) &= c_1(\mu)\alpha(\mu) + d_1(\mu)\beta(\mu) + \\ &\quad (2b_1(\mu)u + v)\frac{\partial\beta}{\partial u} + 2(c_1\mu u + d_1(\mu)v)\frac{\partial\alpha}{\partial u}\,\cdot \end{aligned}$$

Let

$$\tilde{q}_m(u,v,\mu) = \sum_{i=0}^{m} \tilde{B}_{m-i,i}(\mu)u^{m-i}v^i \;\;,\;\; \tilde{r}_m(u,v,\mu) = \sum_{i=0}^{m} \tilde{C}_{m-i,i}(\mu)u^{m-i}v^i$$

be the homogeneous parts of degree m of the maps $\tilde{b}(.,.,\mu)$, $\tilde{c}(.,.,\mu)$, respectively.

A simple calculation yields

$$\begin{aligned} ad_m^2(\mu)(u^{m-i}v^i, 0) &= ((1+m-i)b_1(\mu) + id_1(\mu))u^{m-i}v^i + \\ &\quad \frac{1}{2}(m-i)u^{m-i-1}v^{i+1} + ic_1(\mu)u^{m-i+1}v^{i-1} \quad \text{and} \\ ad_m^2(\mu)(0, u^{m-i}v^i) &= ib_1(\mu)u^{m-i+1}v^{i-1} + (\frac{1}{2} - m + \frac{3}{2}i)u^{m-i}v^i + \\ &\quad (m-i)u^{m-i-1}v^{i+1}, \end{aligned}$$

which imply that, for $i = 1, \cdots, m-1$,

$$\begin{aligned} \tilde{B}_{m-i,i}(\mu) &= B_{m-i,i}(\mu) + \frac{1}{2}(m+1-i)\alpha_{m+1-i,i-1}(\mu) + \\ &\quad ((m+1-i)b_1(\mu) + id_1(\mu))\alpha_{m-i,i}(\mu) + \\ &\quad (i+1)c_1(\mu)\alpha_{m-i-1,i+1}(\mu) + (m+1-i)\beta_{m+1-i,i-1}(\mu) + \\ &\quad \frac{1}{2}(1+3i-2m)\beta_{m-i,i}(\mu) + (i+1)b_1(\mu)\beta_{m-i-1,i+1}(\mu), \end{aligned}$$

as well as that

$$\begin{aligned} \tilde{B}_{m,0}(\mu) &= B_{m,0}(\mu) + (1+m)b_1(\mu)\alpha_{m,0}(\mu) + c_1(\mu)\alpha_{m-1,1}(\mu) + \\ &\quad \frac{1}{2}(1-2m)\beta_{m,0}(\mu) + b_1(\mu)\beta_{m-1,1}(\mu), \end{aligned}$$

$$\begin{aligned}\tilde{B}_{0,m}(\mu) &= B_{0,m}(\mu)+\frac{1}{2}\alpha_{1,m-1}(\mu)+(b_1(\mu)+md_1(\mu))\alpha_{0,m}(\mu)+\\ &\quad \beta_{1,m-1}(\mu)+\frac{1}{2}(1+m)\beta_{0,m}(\mu)\,.\end{aligned}$$

Also,

$$\begin{aligned}ad_m^3(\mu)(u^{m-i}v^i,0) &= (1+2(m-i))c_1(\mu)u^{m-i}v^i+\\ &\quad 2(m-i)d_1(\mu)u^{m-i-1}v^{i+1} \quad \text{and}\\ ad_m^3(\mu)(0,u^{m-i}v^i) &= (d_1(\mu)+2(m-i)b_1(\mu))u^{m-i}v^i+\\ &\quad (m-i)u^{m-i-1}v^{i+1}\,,\end{aligned}$$

which imply that, for $i=1,\cdots,m$,

$$\begin{aligned}\tilde{C}_{m-i,i}(\mu) &= C_{m-i,i}(\mu)+(1+2(m-i))c_1(\mu)\alpha_{m-i,i}(\mu)+\\ &\quad 2(m-i+1)d_1(\mu)\alpha_{m-i+1,i-1}(\mu)+(d_1(\mu)+\\ &\quad 2(m-i)b_1(\mu))\beta_{m-i,i}(\mu)+(m+1-i)\beta_{m+1-i,i-1}(\mu)\,,\end{aligned}$$

as well as that

$$\tilde{C}_{m,0}(\mu)=C_{m,0}(\mu)+(1+2m)c_1(\mu)\alpha_{m,0}(\mu)+(d_1(\mu)+2mb_1(\mu))\beta_{m,0}(\mu)\,.$$

We set $\alpha_{0,m}=\beta_{0,m}=0$. Then

$$\begin{pmatrix}\tilde{B}_{m-1,1}\\ \tilde{B}_{m-2,2}\\ \vdots\\ \tilde{B}_{1,m-1}\\ \tilde{B}_{0,m}\\ \tilde{C}_{m-1,1}\\ \tilde{C}_{m-2,2}\\ \vdots\\ \tilde{C}_{1,m-1}\\ \tilde{C}_{0,m}\end{pmatrix}(\mu)=\begin{pmatrix}B_{m-1,1}\\ B_{m-2,2}\\ \vdots\\ B_{1,m-1}\\ B_{0,m}\\ C_{m-1,1}\\ C_{m-2,2}\\ \vdots\\ C_{1,m-1}\\ C_{0,m}\end{pmatrix}(\mu)+M(\mu)\begin{pmatrix}\alpha_{m,0}\\ \alpha_{m-1,1}\\ \vdots\\ \alpha_{2,m-2}\\ \alpha_{1,m-1}\\ \beta_{m,0}\\ \beta_{m-1,1}\\ \vdots\\ \beta_{2,m-2}\\ \beta_{1,m-1}\end{pmatrix}(\mu)\,,$$

with

$$M(\mu)=\begin{pmatrix}F(\mu) & G(\mu)\\ H(\mu) & J(\mu)\end{pmatrix}\,,$$

where $F(\mu),G(\mu),H(\mu)$, and $J(\mu)$ are smooth $m\times m$ matrices such that

$$J(0)=\text{Diagonal}\{m,m-1,\cdots,2,1\},\quad H(0)=0 \quad \text{and} \quad F(0)=\frac{1}{2}J(0)\,.$$

Therefore, the matrix $M(\mu)$ is nonsingular for all $|\ \mu\ |$ small, and the coefficients $\alpha_{m-i,i}(\mu)$ and $\beta_{m-i,i}(\mu)$, for $i = 0, \cdots, m-1$, may be chosen in such a way that $\tilde{B}_{m-j,j}(\mu)$ and $\tilde{C}_{m-j,j}(\mu)$ vanish in a neighborhood of $\mu = 0$, for all $j = 1, \cdots, m$. ▌

Remark 5. Consider $\tilde{a}, \tilde{b}, \tilde{c}$ as in Theorem 4. Let k be an integer greater than one such that $c_1(0) = \cdots = c_{k-1}(0) = 0$ and $c_k(0) \neq 0$. Since the quadratic differential form is positive, k is odd and $c_k(0) > 0$. Hence, assuming $\tilde{b}(u, 0)$ or $\tilde{c}(u, 0)$ is nonflat at $u = 0$, there exists an integer $k \geq 2$ such that $b_1(0) = \cdots = b_{k-1}(0) = 0$, $c_1(0) = \cdots = c_{2k-2}(0) = 0$, and $(b_k(0), c_{2k-1}(0)) \neq (0, 0)$ with $c_{2k-1}(0) \geq 0$.

4. FINITE DETERMINACY

In the next Theorem, we show that the local configuration around a nonflat type C singular point is topologically equivalent to the one shown in Figure 4.

THEOREM 6. *Let $k \geq 2$ be an integer, and let $(b_0, c_0) \neq (0, 0)$ be a pair of real numbers, with $c_0 \geq 0$. Consider a positive quadratic differential form $\omega = a(x, y)dy^2 + 2b(x, y)dxdy + c(x, y)dx^2$ such that*

$$\begin{aligned} a(x, y) &= -x + y + M_1(x, y) \\ b(x, y) &= \frac{1}{2}y + b_0x^k + M_2(x, y) \\ c(x, y) &= c_0x^{2k-1} + M_3(x, y), \end{aligned}$$

with $M_1(x, y) = O(|\ (x, y)\ |)$, $M_2(x, y) = O(|\ (x, y)\ |^k)$ and $M_3(x, y) = O(|\ (x, y)\ |^{2k-1})$. Then the local phase portrait around the origin of the foliations associated with ω is topologically equivalent to the one shown in Figure 4.

The proof of Theorem 4.1 is consequence of the next two lemmas. We use the blowing-up technique (compare [3, Section 4]).

Now since the roots of the polynomial

$$\begin{aligned} S(\omega) &= da_{(0,0)}(x, y)y^2 + 2db_{(0,0)}(x, y)xy + da_{(0,0)}(x, y)x^2 \\ &= y^3 \end{aligned}$$

correspond to the possible directions of asymptotic convergence to the singular point for the leaves of the foliations $f_1(\omega)$ and $f_2(\omega)$, we only need to make a blowing-up in the x-direction.

LEMMA 7. *Consider the blowing-up* $(x, y) = G(u, v) = (u, u^j v)$, *for* $1 \leq j \leq k$. *Then* $(u, v)^*(\omega) = u^{2j-1}\omega_j$, *with*

$$\omega_j = u^2 A_j(u, v)dv^2 + 2uB_j(u, v)dudv + C_j(u, v)du^2.$$

Moreover:

a) If $j < k$, *then the unique singular point of* ω_j *over the line* $u = 0$ *is the origin and* $(B_j^2 - A_j C_j)(0, 0) = 0$.

b) If $j = k$, *then* $Sing(w_k) \cap \{u = 0\} = \{(0, v_1), (0, v_2)\}$, *with* $v_1 \leq 0 \leq v_2$ *and* $v_1 < v_2$. *Further,* $(B_k^2 - A_k C_k)(0, v_i) > 0$ *for* $i = 1, 2$. *If* k *is even (resp. odd), then the local configuration of* $(u, v)^*(\omega)$ *at the origin is topologically equivalent to the one shown in Figure 5 (resp. Figure 6).*

$f_1((u, v)^*(\omega))$ $f_2((u, v)^*(\omega))$

Figure 5.

$f_1((u, v)^*(\omega))$ $f_2((u, v)^*(\omega))$

Figure 6.

Proof. As in [5, Section 3], if $G(u,v) = (u, u^j v)$, then in the new coordinates we have that ω takes the form

$$(u,v)^*(\omega) = (du\, dv) \begin{pmatrix} \tilde{c}(u,v) & \tilde{b}(u,v) \\ \tilde{b}(u,v) & \tilde{a}(u,v) \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix},$$

with

$$\begin{pmatrix} \tilde{c}(u,v) & \tilde{b}(u,v) \\ \tilde{b}(u,v) & \tilde{a}(u,v) \end{pmatrix} = DG(u,v)^T M(G(u,v)) DG(u,v)$$

where $DG(u,v)$ is the Jacobian matrix of G in (u,v), its transpose matrix is $DG(u,v)^T$, and $M(u,v)$ is the matrix

$$M(u,v) = \begin{pmatrix} c(u,v) & b(u,v) \\ b(u,v) & a(u,v) \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \tilde{a}(u,v) &= u^{2j} a(u, u^j v), \\ \tilde{b}(u,v) &= j u^{2j-1} v a(u, u^j v) + u^j b(u, u^j v), \quad \text{and} \\ \tilde{c}(u,v) &= j^2 u^{2j-2} v^2 a(u, u^j v) + 2j u^{j-1} v b(u, u^j v) + c(u, u^j v), \end{aligned}$$

which imply

$$\begin{aligned} A_j(u,v) &= -1 + u^{j-1} v + u N_1(u,v), \\ B_j(u,v) &= b_0 u^{k-j} + (j - \frac{1}{2}) v + j u^{j-1} v^2 + u N_2(u,v), \\ C_j(u,v) &= j(1-j) v^2 + 2j b_0 u^{k-j} v + c_0 u^{2(k-j)} + u^{j-1} v^3 + u N_3(u,v). \end{aligned}$$

Thus, for $j < k$, we have that $C_j(0,v) = 0$ if and only if $v = 0$. Since $B_j(0,0) = 0$, part a) of the lemma follows.

Concerning b), we assume $j = k$. The singular points of ω_k over the line $u = 0$ are $(0, v_1)$ and $(0, v_2)$, where v_1, v_2 are the solutions of the equation

$$k(1-k)v^2 + 2j b_0 v + c_0 = 0.$$

If $v_1 < v_2$, we have that $v_1 \leq 0 \leq v_2$ and that

$$B_k(0, v_i) = \begin{cases} b_0 & \text{if } v_i = 0 \\ -\frac{k^2 v_i^2 + c_0}{2k v_i} & \text{if } v_i \neq 0, \end{cases}$$

hence $B_k(0, v_1) > 0 > B_k(0, v_2)$.

In order to obtain the local configuration around the origin, we consider for $i = 1, 2$ the vector fields

$$\begin{aligned} X_i(u,v) &= (u^2 A_k(u,v), -uB_k(u,v) + (-1)^i \sqrt{u^2 H_k(u,v)}\,), \\ Y_i(u,v) &= (uA_k(u,v), -B_k(u,v) + (-1)^i \sqrt{H_k(u,v)}\,), \quad \text{and} \\ Z_i(u,v) &= (u(B_k(u,v) + (-1)^i \sqrt{H_k(u,v)}), -C_k(u,v)\,)\,, \end{aligned}$$

where $H_k(u,v) = B_k(u,v)^2 - A_k(u,v)C_k(u,v)$.

Recall that, for $i = 1, 2$, the vector field X_i is tangent to the foliation $f_i(\omega_1)$ and that the vector field Y_i is tangent to the vector field Z_i. However, Y_i is tangent to X_i only for u positive and, for u negative, Y_i is tangent to X_{3-i}.

Since

$$\begin{aligned} (-B_k - \sqrt{H_k})(0, v_1) &< 0 \\ (-B_k - \sqrt{H_k})(0, v_2) &= 0 \\ (-B_k + \sqrt{H_k})(0, v_1) &= 0 \\ (-B_k + \sqrt{H_k})(0, v_2) &> 0\,, \end{aligned}$$

we have:

- The point $(0, v_1)$ is regular for Y_1; it is singular for Y_2 and a hyperbolic saddle of Z_2.
- The point $(0, v_2)$ is regular for Y_2; it is singular for Y_1 and a hyperbolic saddle of Z_1.

Therefore, the local phase portraits of the vector fields Y_i and X_i around the line $u = 0$ are

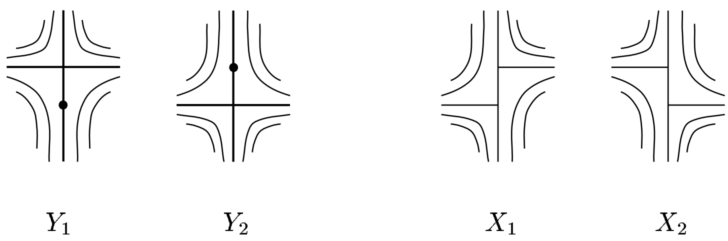


Figure 7.

Finally, since the determinant of the Jacobian matrix of the change of coordinates is $\det J(u,v) = u^k$ and $(u,v)^*(\omega) = u^{2k-1}\omega_k$, the phase portrait of the foliation $f_i((u,v)^*(\omega))$, for $i = 1, 2$, is topologically equivalent to the phase portrait of Y_i (resp. X_i) if k is even (resp. if k is odd) (cf. [3, Remark 4.1]); the proof of part b) now follows. ▮

To complete our analysis we must perform, for each $1 \le j \le k$, the blowing-up $(x,y) = (uv, u^j v^{j+1})$, and study what occurs at the origin.

Lemma 8. *Let* $(x,y) = (uv, u^{j-1}v^j)$, *with* $2 \le j \le k$. *Then we have* $(u,v)^*(\omega) = u^{2j-3}v^{2j-1}\omega_j$, *with* $\omega_j = u^2 A_j(u,v)dv^2 + 2uvB_j(u,v)dudv + v^2 C_j(u,v)du^2$, *and the local configurations of the foliations associated to* $(u,v)^*(\omega)$ *around the origin are topologically equivalent to the one shown in Figure 8 for* $j = 2$, *and for* $j > 2$, *to the one in Figure 9.*

$f_1((u,v)^*(\omega))$ $f_2((u,v)^*(\omega))$

Figure 8.

$f_1((u,v)^*(\omega))$ $f_2((u,v)^*(\omega))$

Figure 9.

Proof. We have that

$$\begin{aligned} A_j(u,v) &= j(1-j) + N_1(u,v), \\ B_j(u,v) &= j(2-j) - \frac{1}{2} + N_2(u,v)\,, \quad \text{and} \\ C_j(u,v) &= (j-1)(2-j) + u^{j-2}v^{j-1} + N_3(u,v)\,, \end{aligned}$$

with $N_i(0,0) = 0$ for $i = 1, 2, 3$, and $N_3(0, v) = 0$.

The corresponding vector fields become

$$\begin{aligned} X_i(u,v) &= (u^2 A_j(u,v), -uvB_j(u,v) + (-1)^i \sqrt{u^2v^2H_j(u,v)}\,)\,, \\ Y_i(u,v) &= (uA_j(u,v), v(-B_j(u,v) + (-1)^i \sqrt{H_j(u,v)})\,)\,, \text{ and} \\ Z_i(u,v) &= (u(B_j(u,v) + (-1)^i \sqrt{H_j(u,v)}), -vC_j(u,v)\,)\,, \end{aligned}$$

where $H_j = B_j^2 - A_jC_j$.

Since $(B_j^2 - A_jC_j)(0,0) = \frac{1}{4}$, we have that

$$DY_1(0,0) = \begin{pmatrix} j(1-j) & 0 \\ 0 & j(j-2) \end{pmatrix}$$

and that

$$DY_2(0,0) = \begin{pmatrix} j(1-j) & 0 \\ 0 & j(j-2)+1 \end{pmatrix}.$$

Then if $j > 2$, the origin is a hyperbolic saddle for Y_1 and Y_2, and the local configuration around the origin of $(u,v)^*(\omega)$ is topologically equivalent to the one shown in Figure 9.

When $j = 2$, the origin is a hyberbolic saddle for Y_2; it is a saddle-node for Y_1 and Z_1, with

$$DZ_1(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Z_1(0,v) = (0, -v^2)\,.$$

Therefore, the local phase portrait around the origin of the vector fields Y_i and X_i are

Y_1 Y_2 X_1 X_2

Figure 10.

Finally, since $(u,v)^*(\omega) = uv^3\omega_2$, and since the determinant of the Jacobian matrix of the change of coordinates is $\det J(u,v) = uv^2$, we have that the local configuration around the origin of $(u,v)^*(\omega)$ is topologically equivalent to the one shown in Figure 8. The proof is now complete. ▮

Proof of Theorem 4.1. The proof is a direct consequence of the two previous lemmas. For example, for the foliation $f_1(\omega)$, after k blowing-ups, for k even as well as for k odd, we obtain Figure 11 below. The proof is now complete. ▮

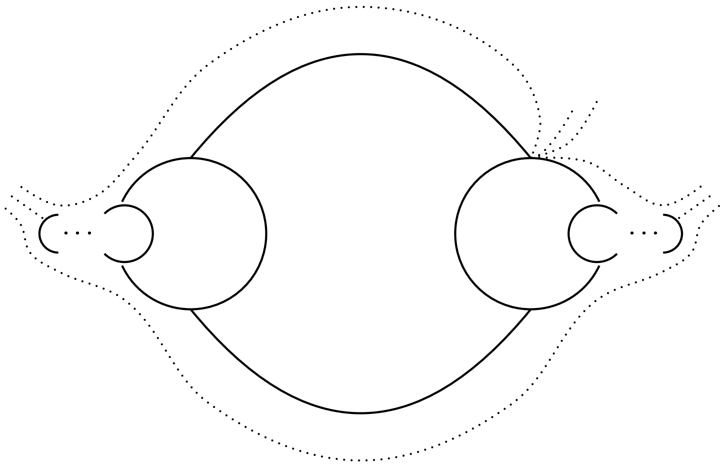


Figure 11.

5. VERSAL UNFOLDINGS

In this section, we find a versal unfolding of a nonflat type C singular point. We first recall the corresponding definitions.

5.1. Definitions

Definition 9. Two smooth families (ω_μ) and $(\tilde{\omega}_\mu)$ of positive C^∞-quadratic differential forms with (the same) parameter $\mu \in \mathbb{R}^k$ are called *C^0–equivalent* (over the identity) if there exist homeomorphisms $h_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ such that, for each $\mu \in \mathbb{R}^k$, we have that h_μ is a C^0–equivalence between the forms ω_μ and $\tilde{\omega}_\mu$.

Remark 10. For local families around the origin of $\mathbb{R}^2 \times \mathbb{R}^k$, we impose the conditions that $h_0(0,0) = (0,0)$, that $h_\mu(x,y)$ only be defined for $((x,y),\mu)$ which belongs to a neighborhood $V \times W$ of $((0,0),0)$ in $I\!R^2 \times I\!R^k$, and that $\{(h_\mu(x,y),\mu)/((x,y),\mu) \in V \times W\}$ be a neighborhood of $((0,0),0)$.

DEFINITION 11. Let $U \subset I\!R^k$ and $V \subset I\!R^l$ be neighborhoods of the origin. If $\phi : (V,0) \to (U,0)$ is a smooth map and $(\ \omega_\mu\ )$ a smooth family of positive C^∞-quadratic differential forms with parameter $\mu \in U$, the family $(v_\alpha) = (\ \omega_{\phi(\alpha)}\ )$, with parameter $\alpha \in V$, is called *family* C^∞ *–induced* by ϕ.

Recall that an *unfolding of a smooth positive quadratic differential form* ω is any smooth family ω_μ of positive C^∞-quadratic differential forms with $\omega_0 = \omega$; we thus have the following Definition.

DEFINITION 12. An unfolding ω_μ of ω_0 is called a *versal unfolding* of ω_0 if all unfoldings of ω_0 are C^0–equivalent to an unfolding C^∞ –induced from ω_μ .

5.2. Main Result

THEOREM 13. *A versal unfolding of a nonflat type C singular point is*

$$(-x+y)dy^2 + (\lambda_1 x + y + 2x^3)dxdy + (\lambda_2(x-y) + \lambda_3(\lambda_1 x + y) + x^3)dx^2\,,$$

with $\lambda_1 > -1$ *and* $\lambda_2 - \lambda_3^2 \geq 0$.

Proof. Let $\omega(\mu)$ be any smooth family with parameter $\mu \in I\!R^k$ such that $\omega(0)$ has a nonflat type C singular point at the origin.
We may assume

$$\omega(\mu) = a(x,y,\mu)dy^2 + 2b(x,y,\mu)dxdy + c(x,y,\mu)dx^2\,,$$

with

$$\begin{aligned}
a(x,y,\mu) &= -x + y + N_1(x,y,\mu)\,,\\
b(x,y,\mu) &= b_1(\mu)x + \frac{1}{2}y + b_2(\mu)x^2 + b_3(\mu)x^3 + \cdots + b_k(\mu)x^k\\
&\quad +N_2(x,y,\mu)\,,\\
c(x,y,\mu) &= c_1(\mu)x + d_1(\mu)y + c_2(\mu)x^2 + c_3(\mu)x^3 + \cdots + c_{2k-1}(\mu)x^{2k-1}\\
&\quad +N_3(x,y,\mu)\,,
\end{aligned}$$

and $b_1(0) = \cdots = b_{k-1}(0) = 0$, $c_1(0) = \cdots = c_{2k-2}(0) = 0$, $d_1(0) = 0$, $(b_k, c_{2k-1})(0) = (b_0, c_0) \neq (0,0)$, $N_1(x,y,\mu) = O(|\ (x,y)\ |)$, and $N_2(x,y,\mu) = O(|\ (x,y)\ |^k)$, $N_3(x,y,\mu) = O(|\ (x,y)\ |^{2k-1})$.
Consider the map $\Delta(\mu) = 4(2b_1(\mu)+d_1(\mu))^3+27c_1(\mu)^2$. Then the origin is a singular point of $\omega(\mu)$ of type

a) D_1 , if $\Delta(\mu) > 0$;
b) D_2 , if $\Delta(\mu) < 0$;
c) D_{12} , if $\Delta(\mu) = 0$ and $c_1(\mu) \neq 0$;
d) $\tilde{D}_1$, if $(2b_1 + d_1, c_1)(\mu) = (0,0)$ and $b_1(\mu) \neq 0$;
e) C , if $(b_1, c_1, d_1)(\mu) = (0,0,0)$.

This implies that our family is C^0–equivalent to the family

$$\tilde{\omega}(\mu) = (-x+y)dy^2 + 2(b_1(\mu)x + \frac{1}{2}y + x^3)dxdy + (c_1(\mu)x + d_1(\mu)y + x^3)dx^2 .$$

Finally, if

$$\phi(\mu) = (2b_1(\mu), \frac{c_1(\mu) - 2b_1(\mu)d_1(\mu)}{1 + 2b_1(\mu)}, \frac{c_1(\mu) + d_1(\mu)}{1 + 2b_1(\mu)})$$

and

$$\begin{aligned} \upsilon(\lambda_1, \lambda_2, \lambda_3) &= (-x + y)dy^2 + (\lambda_1 x + y + 2x^3)dxdy + \\ &\quad (\lambda_2(x - y) + \lambda_3(\lambda_1 x + y) + x^3)dx^2 , \end{aligned}$$

then the unfolding induced by ϕ from the family $\upsilon(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_1, \lambda_2, \lambda_3 \in I\!R$ is precisely the family $\tilde{\omega}(\mu)$. The proof is now complete. ▌

The corresponding bifurcation diagram is shown in Figure 12.

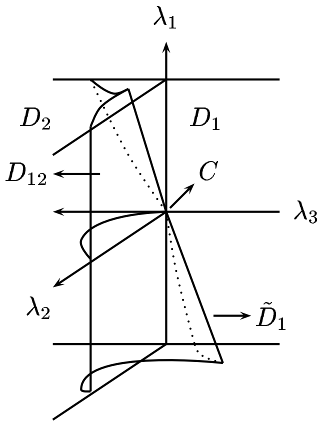


Figure 12.

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