

About Homotopy Classes of non-Singular Vector Fields on the Three-Sphere

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Generically, the set of points along which two non-singular vector fields on the three-sphere are positively (resp. negatively) collinear form a link. We prove that the two vector fields are homotopic if and only if the linking number of those links is zero. We use this criterion to give a new proof of a result of Yano: every non-singular vector field on the three-sphere is homotopic to a non-singular Morse-Smale vector field.

Key Words: Dynamical systems, three-manifolds, Morse-Smale vector fields

INTRODUCTION

The study of surface homeomorphisms up to isotopy, following Nielsen and Thurston, has led to many exciting results. For example, the work of Franks on area-preserving diffeomorphisms of the annulus, of Handel about fixed point of planar homeomorphisms, of Bestvina & Handel for surface homeomorphisms, of Gambaudo, van Strien & Tresser and Llibre & MacKay about the forcing problem . . . The cornerstone of this theory is the classification result of Thurston (see [13]). In [9], MacKay proposes to study the same problem for non-singular vector fields on three-dimensional manifolds. The main goal of this project is to understand the relationship between the geometry of the manifold (in the sense of Thurston, [12]) and the possible vector fields in each homotopy class (homotopy via non-singular vector fields). The problem of homotopy of non-singular vector fields has already been studied by (among others) Asimov, Yano, Gompf and G. Kuperberg [1, 17, 6, 7]. In the last two papers, Gompf and G.

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Kuperberg give a complete classification of non-singular vector fields on a three-manifold in term of Euler class, spin structure, . . . Those studies were done using a trivialization of the tangent bundle of the manifold; it seems that they lead to very poor results on the dynamical point of view.

One key point of the programme of Thurston is to find a "simplest representative" in each isotopy class. For three-dimensional vector fields, as remarked by MacKay, the notion of "simplest" is still unclear for one could not expect to define this notion with respect to the topological equivalence of vector fields. Indeed, the work of K. Kuperberg on the Seifert conjecture (see [8, 5]) implies that each non-singular vector field on a three-manifold is homotopic to a vector field without periodic orbit. However, it is still interesting to know if there is an homotopy to a well-known class of vector fields (e.g. Morse-Smale, volume preserving, pseudo-Anosov flows, . . .). One early work in this direction, after the work of Asimov, is the work of Yano, [17], where conditions for the existence of a Morse-Smale vector field in a given homotopy class on a graph manifold (or on a manifold that admits a Morse-Smale vector field) are given in term of Euler class.

The situation on the three-sphere is in many ways simpler than on other three-manifolds. We will study the problem of homotopy on the sphere, expecting to generalise our results and methods to arbitrary three-manifolds.

One purpose of this paper is to give a criterion to decide whether two non-singular vector fields are homotopic or not, directly computable from the vector fields. Generically, the set of points on which two non-singular vector fields X and Y on the three-sphere are positively (resp. negatively) collinear form a link C_+ (resp. C_-). The linking number of C_+ with C_- is well-defined up to its sign.

Criterion: *the vector fields X and Y are homotopic if and only if the absolute linking number of C_+ with C_- is zero.*

We show that the absolute linking number of C_+ and C_- is the distance in homotopy classes between X and Y . Lee Rudolph pointed out that this result is also proved in [11, Proposition 1.1].

We expect this criterion to be extended to arbitrary three-manifold, using previous work of Kuperberg and Gompf, [7, 6] and the extension of Hopf's ideas by Pontryagin.

The last part of the paper is devoted to the study of Morse-Smale vector fields on the three-sphere. We show that every vector field on the sphere is homotopic to a Morse-Smale one, using the criterion. For that purpose, it is sufficient to find a non-singular Morse-Smale vector field in each homotopy class. This has already been done by Yano, [17], using action groupe arguments. Our construction is more geometric than the construction of Yano. The dynamic of the Morse-Smale vector fields is given by a vector field on the two-sphere and is known explicitly.

1. DEFINITIONS AND FIRST RESULTS

1.1. Hopf fibration and Hopf invariant

In this section we give the classical definition of the Hopf fibration of the 3-sphere, we give an alternative way to obtain this fibration and finally, we recall the definition of the Hopf invariant.

We consider S^3 as the unit sphere of \mathbf{R}^4 with the standard metric: $S^3 = \{(x_1, x_2, x_3, x_4) \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$. Identifying \mathbf{R}^4 with \mathbf{C}^2 , by $(z_1, z_2) = (x_1 + ix_2, x_3 + ix_4)$, the sphere S^3 is the set $\{|z_1|^2 + |z_2|^2 = 1\}$. Each complex line of \mathbf{C}^2 intersects S^3 along a great circle. The union of those circles is the 3-sphere and if two of those circles have a non-empty intersection, they are equal. We obtain a fibration from S^3 to $\mathbf{C}P(1) \simeq S^2$ which associate to each point of the 3-sphere the direction of the corresponding complex line in $\mathbf{C}P(1)$.

DEFINITION 1. The map described above, $\mathcal{H}: S^3 \rightarrow S^2$, is called the Hopf fibration.

Let us now identify \mathbf{R}^4 with \mathbf{H} , the field of quaternions. With this identification, the canonical basis of \mathbf{R}^4 is denoted by $\{1, i, j, k\}$. We identify $\mathbf{R} \cdot 1$ with \mathbf{R} and $\mathbf{R}i + \mathbf{R}j + \mathbf{R}k$ with \mathbf{R}^3 ; this subset of \mathbf{H} is called the set of pure quaternions. One can write every quaternion $q = \mathcal{R}(q) + \mathcal{P}(q)$ with $\mathcal{R}(q) \in \mathbf{R}$ and $\mathcal{P}(q) \in \mathbf{R}^3$, and define the conjugate $\bar{q} = \mathcal{R}(q) - \mathcal{P}(q)$. We obtain the canonical norm on \mathbf{R}^4 : $\|q\| = \sqrt{q\bar{q}}$.

The 3-sphere is the set $S^3 = \{q \mid \|q\| = 1\}$ and it has a multiplicative group structure, induced by the multiplication of quaternions. For a given point s in S^3 , one can consider the inner product: $\rho'_s: S^3 \rightarrow S^3$, defined by $\rho'_s(q) = sqs^{-1}$.

PROPOSITION 2 ([2], Corollary 8.9.3). Let s be a point of S^3 , the map ρ'_s leaves \mathbf{R}^3 invariant and its restriction $\rho_s = \rho'_s|_{\mathbf{R}^3}$ belongs to $SO(3)$.

Moreover the map $s \mapsto \rho_s$ from S^3 to $SO(3)$ is a surjective group homomorphism with kernel $\{\pm 1\}$. In particular, $SO(3)$ and $\mathbf{R}P(3)$ are isomorphic.

Let \star be a point of S^2 , the unit sphere of \mathbf{R}^3 , we define the map

$$f: S^3 \rightarrow S^2$$

$$s \mapsto \rho_s(\star).$$

LEMMA 3. For $\star = (0, 1, 0, 0)$, f is the Hopf fibration defined above.

Proof. Let y be a point of S^2 , we have to prove that $f^{-1}(y)$ is a circle included in a complex line of \mathbf{C}^2 . Proposition 8.9.4 in [2] asserts that given $s = \alpha + t$ in S^3 , with $t \in \mathbf{R}^3 \setminus 0$ and $\alpha \in \mathbf{R}$, the axis of the rotation ρ_s is

the line $\mathbf{R}t$, and its angle θ is given by the relation $\tan \frac{\theta}{2} = \frac{\|t\|}{|\alpha|}$ if $\alpha \neq 0$, and $\theta = \pi$ if $\alpha = 0$.

For $u \in [0, 2\pi]$, let us define $c_u = (\cos(u), \sin(u), 0, 0)$ a point in S^3 . The rotations associated to c_u by the Proposition 2 are the only ones keeping \star invariant.

Let us denote $y = (0, a, b, c)$, with $a^2 + b^2 + c^2 = 1$ and let s_y be the point of S^3 with coordinates $(a, 0, c, -b)$. We have $\rho_{s_y}(y) = \star$, therefore $\rho_{c_u} \circ \rho_{s_y}(y) = \rho_{c_u \cdot s_y}(y) = \star$. Hence, we get

$$f^{-1}(y) = \{c_u \cdot s_y, u \in [0, 2\pi]\}.$$

Let $u \in [0, 2\pi]$, $c_u \cdot s_y = (a \cos(u), a \sin(u), b \sin(u) + c \cos(u), c \sin(u) - b \cos(u))$, therefore, $f^{-1}(y)$ is circle, invariant by the antipodal map and included in the complex line of slope $\frac{c-ib}{a}$. \blacksquare

Remark 4. For a different choice of point \star , we obtain a new map $f' : S^3 \rightarrow S^2$, and there exists a diffeomorphism g of the 2-sphere such that $f' = g \circ \mathcal{H}$ (the map g could be the restriction of an isometry of \mathbf{R}^3 to S^2).

Let f be a smooth map from S^3 to S^2 (we assume that the spheres are oriented) and let y and z be two regular values of f . The sets $f^{-1}(y)$ and $f^{-1}(z)$ are oriented links (a link is disjoint union of knots) and the linking number: $\text{link}(f^{-1}(y), f^{-1}(z))$ does not depend on the choice of y and z .

DEFINITION 5. The *Hopf invariant* of f is $H(f) = \text{link}(f^{-1}(y), f^{-1}(z))$.

The Hopf invariant does not depend on the choice of f in its homotopy class.

EXAMPLE 6. The Hopf invariant of the Hopf fibration is 1.

Let f be a map from S^3 to S^2 , $h : S^3 \rightarrow S^3$ and $g : S^2 \rightarrow S^2$ we have the equalities:

$$H(f \circ h) = \text{deg}(h)H(f)$$

$$H(g \circ f) = \text{deg}^2(g)H(f).$$

We deduce from the second formula that the Hopf invariant of any map f as above (see before lemma 3) is 1.

1.2. Combing and framings

Let us define some geometrical objects on an arbitrary compact oriented 3-manifold M . In this section we follow [15, 7, 6, 4]. We denote by TM the tangent bundle of M .

DEFINITION 7. A *combing* of M is a section of the tangent bundle TM .

A *framing* of M is a trivialization of the tangent bundle into a product: $TM \simeq M \times \mathbf{R}^3$.

A *combing* is a non-singular vector field on M . As every compact oriented 3-manifold has a vanishing Euler characteristic, there always exists a combing on M . Every combing is homotopic to a unitary combing for a given Riemannian metric on M .

A framing consists of three linearly independent combings whose orientation gives the orientation of the manifold. We remark that with a Riemannian metric on M , two linearly independent combings are sufficient to define a framing and every framing is homotopic to an orthonormal framing. It is more difficult to show that there always exists a framing on a three-manifold. We will restrict ourselves to the case $M = S^3$ which is a Lie group, therefore it is easy to see that S^3 admits a framing. The next lemma is classical.

LEMMA 8. *On the three-sphere, one can always complete a combing into a framing. Moreover, two such framings are homotopic through framings which complete the initial combing.*

Therefore, to each non-singular vector field X on S^3 , we associate a framing τ_X which is well-defined up to homotopy.

Let X and Y be two unitary non-singular vector fields on the three-sphere and denote by τ_X and τ_Y respectively some associated orthonormal framings. We have the following.

- With respect to τ_X , the vector field Y is a map Y_{τ_X} from S^3 to S^2 , we can associate to this map its Hopf invariant: $H_X(Y)$;
- at each point x in S^3 , we can associate a unique linear map of $SO(3)$, denoted by $(\tau_X - \tau_Y)(x)$, mapping $X(x)$ to $Y(x)$ and the two others combings of τ_X to the combings of τ_Y . The map $(\tau_X - \tau_Y)$ from S^3 to $SO(3)$ has a well-defined degree: $[\tau_X - \tau_Y]$.

Remark 9. As $SO(3)$ is homeomorphic to $\mathbf{RP}(3)$ and S^3 is simply connected, $[\tau_X - \tau_Y]$ is an even integer (see proof of lemma 10).

LEMMA 10. *The quantity H_X gives an isomorphism between the homotopy classes of non-singular vector fields on the three-sphere and the integers.*

Two non-singular vector fields on the three-sphere, X and Y , are homotopic if and only if $H_X(Y) = 0$.

Moreover, we have the formula: $H_X(Y) = \frac{1}{2}[\tau_X - \tau_Y]$.

Proof. Let X be a non-singular vector field on S^3 and τ_X an associated framing. The homotopy class of a non-singular vector field Y is uniquely determined by the homotopy class of $Y_{\tau_X} : S^3 \rightarrow S^2$. The Hopf invariant

gives an isomorphism between $\pi_3(S^2)$ and \mathbf{Z} then it gives an isomorphism between the homotopy classes of non-singular vector fields on S^3 and \mathbf{Z} .

A continuous map from S^3 to S^2 is homotopic to a constant if and only if its Hopf invariant is zero. The map X_{τ_X} is constant, then X and Y are homotopic if and only if $H_X(Y) = 0$.

Let us prove now the last formula. Let \star be a point of S^2 and define the map $M: SO(3) \rightarrow S^2$ which associate to a matrix A the point $A(\star)$ of S^2 . Moreover, denote by $p: S^3 \rightarrow \mathbf{R}P(3) \simeq SO(3)$ the standard projection. With $X_{\tau_X} \equiv \star$, there exists $f: S^3 \rightarrow S^3$ such that we have the following commutative diagram.

$$\begin{array}{ccc}
 & S^3 & \xrightarrow{Y} & S^2 \\
 & \uparrow f & & \uparrow M \\
 S^3 & \xrightarrow{p} & SO(3) & \\
 & \downarrow \tau_X - \tau_Y & & \\
 & S^3 & &
 \end{array}$$

The projection p is of degree two then the integer $[\tau_X - \tau_Y]$ is even and the degree of f is exactly $\frac{1}{2}[\tau_X - \tau_Y]$.

Using this diagram, we see that $Y = M \circ p \circ f$ then $H_X(Y) = H(M \circ p \circ f) = \text{deg}(f)H(M \circ p)$. The map $M \circ p$ is exactly the map constructed in section 1.1 and, $H(M \circ p) = H(\mathcal{H}) = 1$. Therefore we have $H_X(Y) = \text{deg}(f) = \frac{1}{2}[\tau_X - \tau_Y]$. ■

Remark 11. The isomorphism we obtain is not well-defined, it depends on the choice of a preferred combing (or framing) on the three-sphere.

Remark 12. If we exchange X and Y we get: $H_X(Y) = -H_Y(X)$.

If X, Y and Z are three non-singular vector fields on S^3 , we have the following equality, for x in S^3 : $(\tau_X - \tau_Y)(x) \cdot (\tau_Y - \tau_Z)(x) = (\tau_X - \tau_Z)(x)$.

Hence, $[\tau_X - \tau_Y] + [\tau_Y - \tau_Z] = [\tau_X - \tau_Z]$ and $H_X(Y) + H_Y(Z) = H_X(Z)$.

These remarks lead to the following definition.

DEFINITION 13. The *distance in homotopy classes* between X and Y , denoted by $\mathcal{D}(X, Y)$, is the absolute value $|H_X(Y)| = |H_Y(X)|$.

Following Theorem 5.4 and Theorem A.4 of [3], a diffeomorphism of the three-sphere is either isotopic to the identity (orientation-preserving) or isotopic to $R(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4)$ (orientation-reversing). We will see in section 1.4 that the vector field \mathcal{H}_+ , tangent to the Hopf fibration, satisfies $\mathcal{D}(\mathcal{H}_+, R_*(\mathcal{H}_+)) = 1$.

DEFINITION 14. We associate to a vector field X on the three-sphere its *homotopy number*:

$$\mathcal{I}(X) = \frac{(\mathcal{D}(X, \mathcal{H}_+) + \mathcal{D}(X, R_*(\mathcal{H}_+)) - 1)}{2}.$$

It is clear that $\mathcal{D}(X, Y) = \mathcal{D}(g_*(X), g_*(Y))$ for any vector fields X and Y , and for any diffeomorphism g . Therefore we have the formula $\mathcal{I}(X) = \mathcal{I}(R_*(X))$. The next lemma is straightforward.

LEMMA 15. *The homotopy number gives a well-defined isomorphism between the homotopy classes (up to diffeomorphism) of non-singular vector fields on the three-sphere and the natural integers \mathbf{N} .*

In particular, we get the formula: $\mathcal{D}(X, R_*(X)) = 2\mathcal{I}(X) + 1$ and we obtain that there is a unique homotopy class (up to diffeomorphism) such that $\mathcal{D}(X, R_*(X)) = 1$.

Remark 16. If we pay attention to the homotopy classes of vector fields up to diffeomorphism, the not well-defined isomorphism of remark 11 become the well-defined isomorphism of lemma 15.

1.3. Intrinsic definition of $\mathcal{D}(X, Y)$

Given two vector fields X and Y , it is not easy to compute $H_X(Y)$, because we have to complete X into a framing and to express Y in the coordinates of this framing. We will investigate another way, more direct, to compute $\mathcal{D}(X, Y)$.

Let X and Y be two non-singular vector fields on the three-sphere. We denote by C_+ (resp. C_-) the set of points of S^3 where X and Y are positively (resp. negatively) collinear.

$$C_+ = \{x \in S^3 / X(x) = \lambda Y(x) \text{ with } \lambda > 0\}$$

$$C_- = \{x \in S^3 / X(x) = \lambda Y(x) \text{ with } \lambda < 0\}$$

LEMMA 17. *Generically, C_+ and C_- are embedded links in S^3 (possibly empty).*

Proof. The vector fields X and Y are homotopic to unitary vector fields for a given Riemannian metric. Fixing any orthonormal framing τ of S^3 , the vector fields X and Y give a map (X, Y) from S^3 to $S^2 \times S^2$. The diagonal D_+ (resp. the anti-diagonal D_-) of $S^2 \times S^2$ is the subset of points of the form (x, x) (resp. $(x, -x)$) with x in S^2 . The set C_+ (resp. C_-) is the reciprocal image $(X, Y)^{-1}(D_+)$ (resp. $(X, Y)^{-1}(D_-)$).

The spheres D_+ and D_- are codimension 2 submanifolds of $S^2 \times S^2$. The transversality theorem asserts that a small perturbation of every map $g: S^3 \rightarrow S^2 \times S^2$ is transverse to D_+ and D_- . Therefore a small perturbation of X and Y makes the map (X, Y) transverse to D_+ and D_- . Hence $(X, Y)^{-1}(D_+)$ and $(X, Y)^{-1}(D_-)$ are either empty or compact submanifolds of S^3 , of codimension 2. Then if C_+ and C_- are non-empty, they are disjoint unions of embedded circles in S^3 . ■

Remark 18. Lemma 17 is true on arbitrary three-manifold.

Although we made use of a framing in the previous proof, the sets C_+ and C_- do not depend on a framing (indeed one can prove lemma 17 by a local argument).

Let M be a compact three-manifold, and f and g be two maps from M to $S^2 \subset \mathbf{R}^3$. If the points $f(x)$ and $g(x)$ are never antipodal, that is if the set $\{x \in M \mid f(x) = -g(x)\}$ is empty or alternatively if $\|f(x) - g(x)\| < 2\pi$ for all x in S^2 , then f and g are homotopic. This leads to the following lemma.

LEMMA 19. *If either C_+ or C_- is empty, then X and Y are homotopic.*

Remark 20. The previous argument shows that a perturbation of a vector field leading to a vector field in the same homotopy class can be very large.

Our goal is to compute the linking number between C_+ and C_- . For that purpose, we need to give them an orientation; in fact we will give two possible orientations of those links.

Let us denote by C_0 one component of C_+ , and let D be a small disc, transverse to X and Y at a point x of C_0 . The vector $X(x)$ gives an orientation of this disc. We consider the Gauss map from D to S^2 (with a fixed orientation) which associates to a point of D the direction of the vector field Y at this point. We oriente C_0 such that the orientation of D and the orientation of C_0 at x gives the orientation of S^3 if the Gauss map is orientation-preserving or gives the opposite of the orientation of S^3 otherwise. This choice of orientation does not depend on the choice of D or of x . We oriente this way each component of C_+ and C_- .

If we take the Gauss map associated to X instead, we obtain the opposite orientation on all the components. We recall that the linking number with the empty set is zero.

DEFINITION 21. The *absolute linking number* of C_+ and C_- is the absolute value of the linking number of C_+ and C_- with one of the above orientations: $|\text{link}(C_+, C_-)|$.

Remark 22. If C_+ and C_- are linked by only two components (one for each link), any orientation of those links would give the same linking number, up to sign.

If C_+ and C_- are orbits of X (and of Y), all their components are oriented as orbits of X (or of Y).

The intrinsic definition of $\mathcal{D}(X, Y)$ is given by the following.

LEMMA 23. *The distance in homotopy classes between two vector fields X and Y is given by the absolute linking number of C_+ and C_- .*

$$|\text{link}(C_+, C_-)| = \mathcal{D}(X, Y)$$

Proof. Let τ_x be a framing associated to X , and Y_{τ_x} be the vector field Y expressed in this framing. Let \star be the point of S^2 such that $X_{\tau_x} \equiv \star$. We can perturb Y , staying in the same homotopy class, such that \star and $-\star$ are regular values of Y_{τ_x} .

As C_+ is $Y_{\tau_x}^{-1}(\star)$ and C_- is $Y_{\tau_x}^{-1}(-\star)$, we can orientate C_+ and C_- with respect to the orientation given by the map $Y_{\tau_x}^{-1}$ and the orientation of S^3 and S^2 . This orientation fits with one of the orientations of C_+ and C_- given above, depending on the orientation of S^2 . By the definition of the Hopf invariant, we have $\text{link}(C_+, C_-) = H_X(Y)$.

The choice for the orientation of C_+ and C_- is not canonical, as well as the choice of orientation of S^2 , we obtain the result: $|\text{link}(C_+, C_-)| = \mathcal{D}(X, Y)$. ■

We obtain the criterion stated in the introduction as a corollary of the previous lemma and lemma 10.

1.4. First examples

In this section, we use our criterion to show that \mathcal{H}_+ and $R_\star(\mathcal{H}_+)$ are in adjacent homotopy classes, we give examples of vector fields homotopic to \mathcal{H}_+ and $R_\star(\mathcal{H}_+)$ and finally we give a construction of a vector field in each homotopy class.

Let $\mathcal{H}: S^3 \rightarrow S^2$ be the Hopf fibration. Choosing an orientation on S^3 and on S^2 gives a natural orientation of the fibres. We obtain this way a vector field \mathcal{H}_+ , tangent to the fibres of \mathcal{H} , such that the linking number between two different orbits is $+1$. The equations of this vector field are: $\mathcal{H}_+(z_1, z_2) = (iz_1, iz_2)$ if we identify \mathbf{R}^4 with \mathbf{C}^2 as in the first section.

In a similar way, we define \mathcal{H}_- such that the linking number of two orbits is -1 : $\mathcal{H}_-(z_1, z_2) = (iz_1, i\bar{z}_2)$. We remark that $\mathcal{H}_- = R_\star(\mathcal{H}_+)$. The following lemma was proved with an other method in [15].

LEMMA 24. *The two vector fields above, \mathcal{H}_+ and \mathcal{H}_- , lie in adjacent homotopy classes: $\mathcal{D}(\mathcal{H}_+, \mathcal{H}_-) = 1$.*

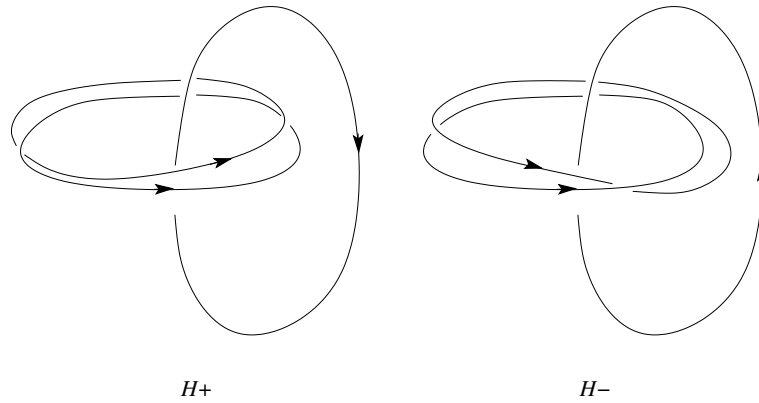


Figure 1. \mathcal{H}_+ and \mathcal{H}_-

Proof. The vector fields \mathcal{H}_+ and \mathcal{H}_- are positively tangents along C_+ , the unit circle of $\mathbf{C} \times \{0\}$ in \mathbf{C}^2 and they are negatively tangent along C_- , the unit circle of $\{0\} \times \mathbf{C}$. The circles C_+ and C_- form a Hopf-link, their absolute linking number is 1. Using lemma 23, we have $\mathcal{D}(\mathcal{H}_+, \mathcal{H}_-) = 1$. ■

As a corollary, we obtain that $\mathcal{I}(\mathcal{H}_+) = \mathcal{I}(\mathcal{H}_-) = 0$.

DEFINITION 25. Given two relatively prime numbers $p \neq 0$ and $q > 0$, a (p, q) -Seifert fibration of the three-sphere is a map from $S^3 \subset \mathbf{C}^2$ to $S^2 \simeq \mathbf{C}P(1)$ defined by:

$$\mathcal{S}_{p,q}: (z_1, z_2) \mapsto \begin{bmatrix} z_2^q \\ \frac{z_2}{z_1^p} \end{bmatrix}$$

A Seifert fibration is a continuous map from the three-sphere to the two-sphere, such that the reciprocal image of each point is a circle. Each of those circles are (p, q) -knots of the tori $T_{a,b} = \{(z_1, z_2) \mid |z_1| = a, |z_2| = b\}$ except the reciprocal image of the North and the South poles which are the cores of those tori and give a Hopf link. Let us denote by $H_{p,q}$ the unitary vector field tangent to the fibres of the (p, q) -Seifert fibration. We remark that $H_{1,1} = \mathcal{H}_+$ and $H_{-1,1} = \mathcal{H}_-$.

LEMMA 26 ([15], corollary 2.4). *The vector fields $H_{p,q}$ are homotopic to \mathcal{H}_+ if p is positive, and to \mathcal{H}_- if p is negative.*

Proof. This is an easy corollary of lemma 19. For $p > 0$, the vector fields $H_{p,q}$ and \mathcal{H}_+ are never negatively tangents and it is the same for $H_{p,q}$ and \mathcal{H}_- whenever $p < 0$. ■

Let us consider the link consisting of two regular fibres L_1 and L_2 of a (p, q) -Seifert fibration. The absolute linking number between L_1 and L_2 is the product $|pq|$. We will construct a vector field X_n for each n in $\mathbf{N} \setminus \{0\}$ which is positively tangent to $H_{n,1}$ along L_1 and negatively tangent to $H_{n,1}$ along L_2 (and the vector fields are transverse everywhere else). Such a vector field satisfies: $\mathcal{D}(X_n, H_{n,1}) = \mathcal{D}(X_n, \mathcal{H}_+) = n$.

LEMMA 27. *There exists such a vector field X_n for each n in $\mathbf{N} \setminus \{0\}$.*

Proof. Let τ be a trivialization of the tangent bundle of the three-sphere such that $H_{1,n}$ is a constant map from S^3 to S^2 . Let T_1 (resp. T_2) be a tubular neighbourhood of L_1 (resp. L_2), and define $X_n(x) = H_{n,1}(x)$ if $x \in L_1$ and $X_n(x) = -H_{n,1}(x)$ if $x \in L_2$. Let \star be the point of S^2 such that $H_{n,1} \equiv \star$ and let \dagger denote a different point of S^2 .

Let $\gamma_t, t \in [0, 1]$ be a smooth path on the two-sphere, such that $\gamma^{-1}(\star) = \{0\}, \gamma^{-1}(\dagger) = \{1\}$ and $\gamma_t \neq -\star$ for all t . The solid torus T_1 is diffeomorphic to $D^2 \times S^1$, where $D^2 = \{(r, \theta), 0 \leq r \leq 1, \theta \in [0, 2\pi]\}$. We define X_n on $D^2 \times \omega$ by $X_n(r, \theta, \omega) = \gamma(r)$. Therefore, X_n is equal to $H_{1,n}$ on L_1 , it is equal to \dagger on ∂T_1 , and is transverse to $H_{1,n}$ in $T_1 \setminus L_1$.

Similarly, we define X_n on T_2 such that $X_n \equiv \dagger$ on the boundary of T_2 . Therefore one can complete X_n outside $T_1 \cup T_2$ by $X_n|_{S^3 \setminus (T_1 \cup T_2)} \equiv \dagger$.

With a small perturbation of X_n along the boundaries of T_1 and T_2 , we can assume that X_n is smooth, always transverse to $H_{n,1}$ except on L_1 and L_2 where the vector field are positively and negatively tangents, respectively. ■

We have therefore a vector field in each homotopy class: $\mathcal{I}(X_n) = n - 1$ for $n > 0$.

2. HOMOTOPY TO MORSE-SMALE FLOWS

In this section, we will exhibit a non-singular Morse-Smale vector field in each homotopy class of vector fields on the sphere. This result has already been obtained by Yano, [17] after previous work of Wilson, [16]. Yano constructs a non-singular Morse-Smale vector field X_n such that $\mathcal{I}(X_n) = n$ for each n , and X_n has exactly $2n$ periodic orbits. The main interest of our construction is that all the vector fields (except those in the homotopy class of \mathcal{H}_+ and \mathcal{H}_-) have exactly 4 periodic orbits. This shows therefore that the number of those orbits is not relevant (as suspected in [17, Remark 5.2]). Using our criterion, we see that their linking number is important.

DEFINITION 28. A non-singular vector field is a *Morse-Smale* vector field if its flow has

- a finite set of periodic orbits which are all hyperbolic,

- the intersections of the invariant manifolds of those orbits are transversal and,
- its nonwandering set consists entirely of those orbits.

A periodic orbit of a Morse-Smale vector field is either an attractor (index 0), a saddle (index 1) or a repeller (index 2). Following [10, 1], Wada, [14], has classified all the indexed links that are realisable as the set of periodic orbits of a non-singular Morse-Smale flow on the three-sphere.

We present here a well-known construction of a Morse-Smale flow in the homotopy class of \mathcal{H}_+ , for ours will be similar in other homotopy classes.

Let X_0 be the $S - N$ -gradient vector field pictured in Figure 2.

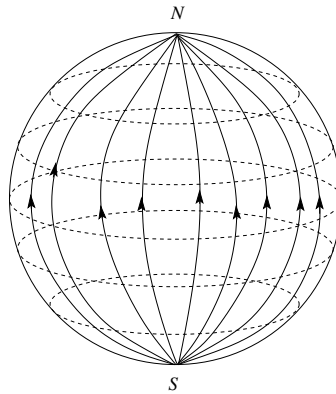


Figure 2. Phase portrait of X_0

The Hopf fibration \mathcal{H} is smooth on S^3 and each point of S^2 is a regular value of \mathcal{H} . We can lift X_0 to a vector field of S^3 , orthogonal to the orbits of \mathcal{H}_+ . The vector field $X_0 + \mathcal{H}_+$ is a Morse-Smale vector field around the $(0, 2)$ -Hopf link, homotopic to \mathcal{H}_+ . The image by R of this vector field gives a Morse-Smale vector field homotopic to \mathcal{H}_- .

We will construct explicitly a Morse-Smale vector field in each homotopy class on the three-sphere. We remark that our vector fields can be obtained using once the fifth or the fourth Wada operation on the $(0, 2)$ -Hopf link, [14]. We will make use of the following remark in our construction.

Remark 29. For a given non-singular Morse-Smale vector field, it is always possible to change the orientation of a periodic orbit, staying in the Morse-Smale class.

The (p, q) -Seifert fibration, $\mathcal{S}_{p,q}$, is smooth on the three-sphere, except along the singular fibres L_N and L_S : $\mathcal{S}_{p,q}$ is smooth on $\mathcal{S}_{p,q}^{-1}(S^2 \setminus \{N, S\})$ where N and S stand for the North and South poles respectively. Every

point of S^2 , different from N and S is a regular value for $\mathcal{S}_{p,q}$. This fibration maps the tori $T_{a,b}$ on circles parallel to the equateur of S^2 .

Let X_1 be a vector field with two sources (at the poles), one sink and one saddle (that lie on the same parallel), X_1 is tangent to this circle (2 heteroclinic orbits) and transverse to every other parallel circles. The phase portrait of X_1 is given on Figure 3.

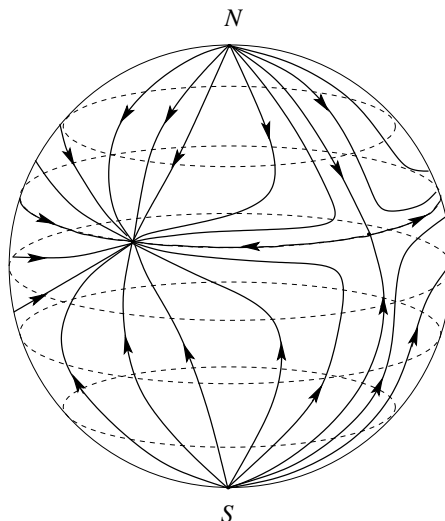


Figure 3. Phase portrait of X_1

Let p be a strictly positive integer and let us consider the following vector field Y_p on the three-sphere. For each point in $S^3 \setminus \{L_N, L_S\}$, one can lift the vector field X_1 on the three-sphere, to a vector field A orthogonal to the fibres of $\mathcal{S}_{p,1}$. We complete A into a smooth vector field on S^3 by $A|_{L_N} = A|_{L_S} = 0$. We define $Y_p(x) = H_{p,1}(x) + A(x)$. The vector field Y_p is a Morse-Smale vector field with 4 periodic orbits (corresponding to the singularities of X_1), it is positively collinear to $H_{p,1}$ along those orbits, and transverse everywhere else. In fact, it is transverse to every tori $T_{a,b}$ except the invariant one. On the invariant torus, it is not difficult to make it transverse to $H_{p,1}$ (on the complement of the periodic orbits).

Let \mathcal{M}_{2p+1} be a Morse-Smale vector field, with the same periodic orbits as Y_p but with the opposite orientation along the orbit of index 0. A slight modification of Y_p leads to \mathcal{M}_{2p+1} , keeping the transversality with $H_{p,1}$ on the complement of the periodic orbits. Similarly, let \mathcal{M}_{2p+2} be a Morse-Smale vector field, with the same periodic orbits as \mathcal{M}_{2p+1} but with the opposite orientation along the orbit corresponding to the North pole of the two-sphere.

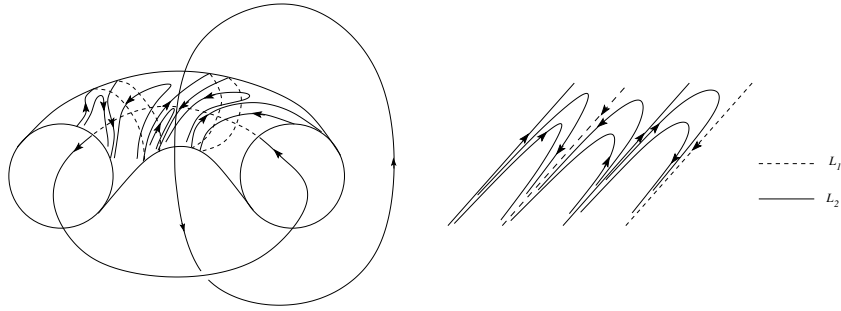


Figure 4. The vector field \mathcal{M}_n in S^3 and on the invariant torus

As p is strictly positive, we obtain a vector field \mathcal{M}_n for each $n \geq 3$. We define \mathcal{M}_2 as follow (the idea is to do the same with " $H_{0,1}$ "): it has 4 periodic orbits, two repellers form a Hopf link, one saddle and one attractor that are linked to one of the repellers, with linking number 1 for the saddle and -1 for the attractor. The saddle and the attractor are on an invariant torus $T_{a,b}$, such that $T_{a,b}$ is the union of the stable manifold of the saddle and the attractor (in particular, the saddle and the attractor are not linked together).

Remark 30. The vector field \mathcal{M}_2 is obtained using the fourth Wada operation, [14], and is similar to the vector field constructed by Yano in [17].

All the other vector fields, \mathcal{M}_n with $n \geq 3$ are obtained using once the fifth Wada operation.

Finally, we define \mathcal{M}_1 to be the Morse-Smale vector field constructed around the $(0, 2)$ -Hopf link (see the begining of this section).

LEMMA 31. *For $n \geq 1$, the vector field \mathcal{M}_n belongs to the $(n-1)^{th}$ class of homotopy of non-singular vector fields on the three-sphere: for $n \geq 1$, we have $\mathcal{I}(\mathcal{M}_n) = n - 1$.*

Proof. The lemma is true for $n = 1$ by construction.

If n is equal to 2, \mathcal{M}_2 is tangent to every $H_{p,q}$ with p and q strictly positive, along a Hopf link and two loops on the invariant torus, parallel to the periodic orbits of \mathcal{M}_2 . On one of those loops, $H_{p,q}$ and \mathcal{M}_2 are negatively tangent. Then we obtain: $|\text{link}(C_+, C_-)| = 1$ and $\mathcal{I}(\mathcal{M}_2) = 1$.

If $n \geq 3$ is an odd integer, $n = 2p + 1$, the set C_+ , where \mathcal{M}_n and $H_{p,1}$ are positively collinear, is the union of the orbits of \mathcal{M}_n of index 1 and 2, namely L_N, L_S and L_1 . The set C_- is the orbit of index 0: L_0 . We oriente those orbits with respect to $H_{p,1}$ and we obtain that:

- $\text{link}(L_0, L_1) = \text{link}(L_0, L_S) = p,$
- $\text{link}(L_0, L_N) = 1.$

Therefore, with this orientation, $\text{link}(C_-, C_+) = 2p + 1$. By lemma 23 and lemma 26, we have $\mathcal{D}(\mathcal{M}_n, \mathcal{H}_+) = \mathcal{D}(\mathcal{M}_n, H_{p,1}) = 2p + 1$.

Similarly, if $n \geq 4$ is even, $n = 2p + 2$, we have: C_+ is the union of L_S and L_1 , and C_- is the union of L_N and L_0 . Once again, we oriente those orbits with respect to $H_{p,1}$ and we obtain that:

- $\text{link}(L_S, L_0) = \text{link}(L_1, L_0) = p$,
- $\text{link}(L_1, L_N) = \text{link}(L_S, L_N) = 1$.

We obtain that $\mathcal{D}(\mathcal{M}_n, \mathcal{H}_+) = 2p + 2$.

Using the same technique and the picture of \mathcal{M}_n on the invariant torus, we obtain that $\mathcal{D}(\mathcal{M}_{2p+1}, \mathcal{H}_-) = 2p$ and $\mathcal{D}(\mathcal{M}_{2p}, \mathcal{H}_-) = 2p + 1$. Therefore, for each $p > 0$, we have $\mathcal{I}(\mathcal{M}_{2p+1}) = 2p = n - 1$ and $\mathcal{I}(\mathcal{M}_{2p+2}) = 2p + 1 = n - 1$, then for each $n \geq 1$, $\mathcal{I}(\mathcal{M}_n) = n - 1$. ■

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