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Topological Equivalence for Saddle Connections.*

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We prove the existence of three classes of topological equivalence for saddle connections adapted to a exceptional divisor depending on the ratio of eigenvalues.

Key Words: Topological equivalence, compatible foliation, real blowing-up.

1. INTRODUCTION

In this paper we give a complete topological classification for saddle connections of three dimensional analytic real vector fields that respect a normal crossings divisor. This kind of connections may appear after blowing-up of degenerate singularities, the divisor being the exceptional divisor of the blowing-ups sequence. So, we are leaded to look at them in order to study the topological equivalence locally at degenerated singularities. On the other hand, the "adapted" saddle connections that we treat here correspond exactly to the case not considered in the work [1] of Bonatti-Dufraine.

Our main result is that one gets three classes of topological equivalence, depending on the fact the first ratio of eigenvalues is lower, equal or bigger than the second one.

The technic is to extend a topological equivalence in a "chimney" from a given homeomorphism on the top. This is in contrast with the usual way [3] that gives a topological equivalence from a homeomorphism in the "fence".

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Obviously, not all homeomorphisms in the top can be extended. To deal with this, we use the idea of compatible foliations (similar to the one found in [1]). After this, we get standard models of compatible foliations by means of a blowing-up technics. Finally, our result follows from the study of the compatibility of a homeomorphism with two standard foliations.

2. STATEMENT OF THE RESULT

Definition 1. Let $\Gamma \subset \mathbf{R}^3$ be the segment $y = z = 0$, $0 \le x \le 1$. We say that $\mathcal{C} = (\xi, \Gamma, V, H_0, H_1, D, E)$ is an *adapted saddle-connection along* Γ (adapted to H_0, H_1, D, E) iff:

1. V is an open set of $\mathbf{R}^3$ containing Γ.
2. ξ is an analytic vector field on V.
3. $H_0 = \{x = 0\}$, $H_1 = \{x = 1\}$, $D = \{z = 0\}$ and $E = \{y = 0\}$ are invariant planes for ξ.
4. $P_0 = (0, 0, 0)$ and $P_1 = (1, 0, 0)$ are the only singular points of ξ in V.
5. Let λ_0, μ_0, δ_0, λ_1, μ_1, δ_1 be the eigenvalues of ξ at P_0 and P_1 corresponding to the invariant lines $x = y = 0$, $x = z = 0$, $y = z = 0$ (for P_0) and $\{x = 1, y = 0\}$,$\{x = 1, z = 0\}$, $y = z = 0$ (for P_1). Then λ_0, μ_0, $\delta_1 < 0$ and λ_1, μ_1, $\delta_0 > 0$.

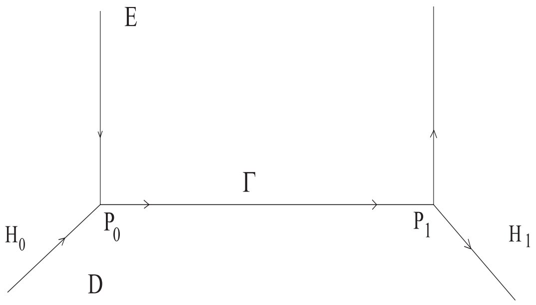

FIG. 1. Saddle connection along Γ.

Given an open set W such that $\Gamma \subset W \subset V$, we define the restriction $\mathcal{C} |_W$ in the evident way. We say that two degenerate saddle connections $\mathcal{C}$ and $\mathcal{C}'$ are *topologically equivalent* iff there exists a homeomorphism $h : V \to V'$ such that $h(H_i) = H_i'$, $h(D) = D'$, $h(E) = E'$ and h sends oriented trajectories of ξ into oriented trajectories of ξ'. Let us denote $\alpha_0(\mathcal{C}) = \lambda_0/\mu_0$, $\alpha_1(\mathcal{C}) = \lambda_1/\mu_1$ and $\varepsilon(\mathcal{C}) = \alpha_0(\mathcal{C})/\alpha_1(\mathcal{C})$. We will prove here the following result.

THEOREM 1. *Given two adapted saddle connections $\mathcal{C}$ and $\mathcal{C}'$ the following statements are equivalent:*

a) There are open sets $\Gamma \subset W \subset V$, $\Gamma' \subset W' \subset V'$ such that $\mathcal{C}\mid_W$ and $\mathcal{C}'\mid_{W'}$ are topologically equivalent.
b) One of the next possibilities holds:

1) $\varepsilon(\mathcal{C}) = \varepsilon(\mathcal{C}') = 1$.
2) $\varepsilon(\mathcal{C}) < 1$ and $\varepsilon(\mathcal{C}') < 1$.
3) $\varepsilon(\mathcal{C}) > 1$ and $\varepsilon(\mathcal{C}') > 1$.

3. STANDARD FOLIATIONS ON THE DISC

Let $\mathbb{D}$ be the unit disc $\mathbb{D}=\{(x,y); x^2+y^2=1\}$ on $\mathbf{R}^2$. Given a real number $\alpha > 0$, we define the singular foliation $\mathcal{F}_\alpha$ on $\mathbb{D}$ to be determined by the form $\omega_\alpha = xdy - \alpha ydx$. The leaves of $\mathcal{F}_\alpha$ on the first quadrant $\mathbb{D}_{++} = \mathbb{D} \cap (\mathbf{R}_{\geq 0})^2$ are given by $y = \lambda x^\alpha$, $\lambda \in \mathbf{R}_{\geq 0}$; the axes $x = 0$ and $y = 0$ are invariant by ω_α. We obtain the rest of the leaves on $\mathbb{D}$ by symmetry. Given a point $q \in \mathbb{D}_{++}$, $q \neq 0$, we define the point $r_\alpha(q)$ to be the only point in $\mathbb{S}_{++} = \mathbb{S}^1 \cap (\mathbf{R}_{\geq 0})^2$ contained in the leaf $\mathcal{L}_\alpha(q)$ of $\mathcal{F}_\alpha$ passing through q.

Remark. Given α, $\beta > 0$ and $q \in \mathbb{D}_{++}$, the following statements are equivalent:

a) $\alpha < \beta$
b) If $r_\alpha(q) = (x(r_\alpha(q)), y(r_\alpha(q)))$ then $y(r_\alpha(q)) < y(r_\beta(q))$.

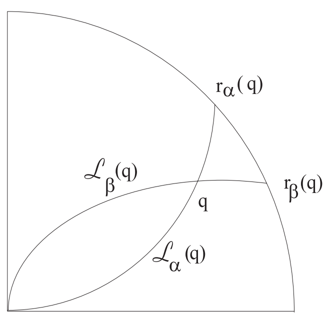

FIG. 2. Standard foliations.

If $\alpha < \beta$ and $p_1, p_2 \in \mathbb{S}_{++}$ with $y(p_1) < y(p_2)$, there is a single point $q \in \mathbb{D}_{++}$ such that $r_\alpha(q) = p_1$ and $r_\beta(q) = p_2$. If $\alpha < \beta$ and $y(p_1) > y(p_2)$ there is no $q \in \mathbb{D}_{++}$ such that $r_\alpha(q) = p_1$ and $r_\beta(q) = p_2$.

We are interested in detecting homeomorphisms $h : \mathbb{D} \to \mathbb{D}$ that send a pair of foliations $(\mathcal{F}_\alpha, \mathcal{F}_\beta)$ onto another pair of foliations $(\mathcal{F}_{\alpha'}, \mathcal{F}_{\beta'})$ respecting the origin and the axes.

PROPOSITION 2. *Consider $\alpha, \alpha', \beta, \beta' > 0$. The following statements are equivalent:*

a) There exists a homeomorphism $h : \mathbb{D}_{++} \to \mathbb{D}_{++}$ such that $h_*\mathcal{F}_\alpha = \mathcal{F}_{\alpha'}$, $h_*\mathcal{F}_\beta = \mathcal{F}_{\beta'}$ and $h(\{x = 0\}) = \{x = 0\}$, $h(\{y = 0\}) = \{y = 0\}$.
b) One of the next possibilities holds:

b.1) $\alpha = \beta$ and $\alpha' = \beta'$.
b.2) $\alpha < \beta$ and $\alpha' < \beta'$.
b.3) $\alpha > \beta$ and $\alpha' > \beta'$.

Proof. b)⇒a) Assume first that b.1) holds. Given $q \neq 0$, put $\mathcal{L}_\alpha(q) = \{y = K_\alpha(q)x^\alpha\}$, $0 \leq K_\alpha(q) \leq +\infty$. We define

$$h(q) = \{y = K_\alpha(q)x^{\alpha'}\} \cap \{q'; \mid q' \mid = \mid q \mid\}.$$

By putting $h(0) = 0$ we get the desired homeomorphism, that induces the identity on the axes. Assume now that b.2) holds (the case b.3) is analogous). Let us define first a homeomorphism $h : \mathbb{S}_{++} \to \mathbb{S}_{++}$ by

$$h(q) = \{y = K_\alpha(q)x^{\alpha'}\} \cap \mathbb{S}_{++}.$$

Now, we extend h to $\overset{\circ}{\mathbb{D}}$ by $h(q) = \mathcal{L}_{\alpha'}(h(r_\alpha(q))) \cap \mathcal{L}_{\beta'}(h(r_\beta(q)))$. Note that $h(q)$ is well defined since $y(h(r_\alpha(q))) < y(h(r_\beta(q)))$ and $\alpha' < \beta'$. Moreover $h_*\mathcal{F}_\alpha = \mathcal{F}_{\alpha'}$ and $h_*\mathcal{F}_\beta = \mathcal{F}_{\beta'}$ by construction. It remains to show that h extends continuously to the axes. By an elementary computation we get that

$$x(h(q))^{\beta'-\alpha'} = (x(q)/x(r_\beta(q)))^{\beta-\alpha}(x((h(r_\beta(q))))^{\beta'-\alpha'}.$$

Since $\lim_{y(q)\to 0}(r_\beta(q)) = \lim_{y(q)\to 0} h(r_\beta(q)) = (1, 0)$, we get that

$$\lim_{q\to(x_0,0)} x(h(q)) = x_0{}^{(\beta-\alpha)/(\beta'-\alpha')}.$$

Analogously with the y-axis. Hence h extends continuously to the axes.

a)⇒b) If $\alpha = \beta$ and $\alpha' \neq \beta'$ it is obvious that h does not exist. Assume that $\alpha < \beta$ and $\alpha' > \beta'$. Given a point $q \in \mathbb{D}_{++}$, then $y(r_\alpha(q)) < y(r_\beta(q))$, this necessarily implies that

$$y(h(r_\alpha(q))) < y(h(r_\beta(q)))$$

(recall that $h(1,0) = (1,0)$ and $h(0,1) = (0,1)$). Then

$$\mathcal{L}_{\alpha'}(h(r_\alpha(q))) \cap \mathcal{L}_{\beta'}(h(r_\beta(q))) = \varnothing$$

and we get no image for q.

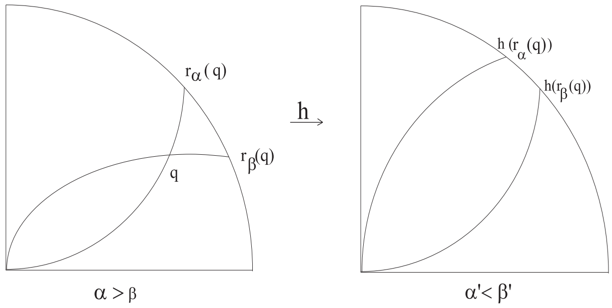

FIG. 3. Homeomorphisms on the disc.

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4. REAL SADDLES AND CHIMNEYS

Let ξ be an analytic vector field in $\mathbf{R}^3$ such that the origin $0 \in \mathbf{R}^3$ is an isolated singular point of saddle type. Assume that $z = 0$ is the stable manifold and $\{x = 0, y = 0\}$ the unstable manifold. Denote by $B(a,b) = \{0 \le z \le b; x^2 + y^2 = a^2\}$. Taking a and b small enough we have that

$$D(x^2 + y^2)(\xi) < 0 \qquad \text{on } B(a,b) \setminus \{x = y = 0\}$$

$$dz(\xi) > 0 \qquad \text{on } B(a,b) \setminus \{z = 0\}.$$

Let ρ be such that $0 < \rho \le a$ and denote $\Omega = \{z = b; x^2 + y^2 \le \rho^2\}$, $\Delta = \{z = 0; x^2 + y^2 \le a^2\}$. By a *chimney* $C_\xi(a,b,\rho)$ we mean the set

$$C_\xi(a,b,\rho) = Sat_{\xi,B(a,b)}(\Omega) \cup \Delta,$$

where $Sat_{\xi,B(a,b)}(\Omega)$ is the union of the trajectories of ξ in $B(a,b)$ that contain a point in Ω. The disc Ω is called *the top* of the chimney and

$$\Sigma = C_\xi(a,b,\rho) \cap \{x^2 + y^2 = a^2\}$$

is the *fence* of $C_\xi(a,b,\rho)$.

Remark. Up to change the sign of ξ or to change the coordinates, we get a "downstairs" or a "turned" chimney, that we continue to call "a chimney".

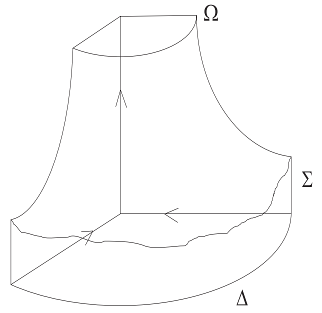

FIG. 4. A chimney.

Let σ be the curve $\sigma = Sat_{\xi,B(a,b)}(\delta\Omega) \cap \{x^2+y^2=a^2\}$. Then σ is the graph of an analytic function $c : \delta\Delta \to (0,b]$ and we can write the fence Σ as follows

$$\Sigma = \{x^2+y^2=a^2; 0 \le z \le c(x,y)\}.$$

Remark. The fence Σ is homeomorphic to $\mathbb{S}^1 \times [0,1]$ by a homeomorphism $\psi : \Sigma \to \mathbb{S}^1 \times [0,1]$ such that $\psi(\delta\Delta) = \mathbb{S}^1 \times \{0\}$ and $\psi(\sigma) = \mathbb{S}_1 \times \{1\}$. To see this, define ψ by

$$\psi(x,y,z) = (\frac{x}{a}, \frac{y}{a}, \frac{z}{c(x,y)})$$

Obviously, the homeomorphism ψ is not unique.

Remark. Given a chimney $C_\xi(a,b,\rho)$ and a subset $\Omega^* \subset \Omega$ homeomorphic to a disc, the set

$$C_\xi(a,b,\Omega^*) = Sat_{\xi,B(a,b)}(\Omega^*) \cup \Delta$$

is called a *non rectified subchimney of* $C_\xi(a,b,\rho)$. We get the same kind of properties for $C_\xi(a,b,\Omega^*)$ as for the chimneys $C_\xi(a,b,\rho)$.

DEFINITION 3. We say that $\mathcal{S} = (\xi, C_\xi(a,b;\Omega^*), H, D, E)$ is an *adapted real saddle* iff

a) The vector field ξ has an isolated singular point at $0 \in \mathbf{R}^3$ of saddle type and we have a chimney $C_\xi(a,b,\rho)$ for it and $C_\xi(a,b,\Omega^*)$ is a subchimney.

b) The planes $E = \{x = 0\}$, $D = \{y = 0\}$, $H = \{z = 0\}$ are invariant for ξ where H is the stable manifold and $D \cap E$ is the unstable manifold.

We say that two *adapted real saddles* $\mathcal{S}, \mathcal{S}'$ are *topologically equivalent* iff there exists a homeomorphism

$$h : C_\xi(a, b, \Omega^*) \to C_{\xi'}(a', b', \Omega^{*\prime})$$

sending oriented trajectories into oriented trajectories and preserving the invariant planes H, D, E (we write $h : \mathcal{S} \to \mathcal{S}'$). We get an induced homeomorphism

$$h \mid_{\Omega^*} : \Omega^* \to \Omega^{*\prime}, \qquad h(0, 0, b) = (0, 0, b')$$

between the tops of the chimneys.

We are interested in getting conditions to extend a homeomorphism between the tops to a topological equivalence between adapted real saddles.

Given a homeomorphism $\psi : \Sigma \to \mathbb{S}^1 \times [0, 1]$ such that $\psi(\delta\Delta) = \mathbb{S}^1 \times \{0\}$ and $\psi(\sigma) = \mathbb{S}^1 \times \{1\}$, we define the foliation $\mathcal{L}_\psi$ on Σ by considering the pull-back of the vertical lines on $\mathbb{S}^1 \times [0, 1]$. The flow of ξ induces an analytic isomorphism $\eta : \Sigma \setminus \delta\Delta \to \Omega \setminus \{(0, 0, b)\}$ that extends to a continuous map $\eta : \Sigma \to \Omega$ with $\eta(\delta\Delta) = \{(0, 0, b)\}$. By taking the direct image we get a singular topological foliation $\mathcal{F}_\psi = \eta_* \mathcal{L}_\psi$ on Ω.

Definition 4. The foliations of the form $\mathcal{F}_\psi$ are called *compatible* with the adapted real saddle $\mathcal{S}$.

Proposition 5. *Consider two adapted real saddles $\mathcal{S}$ and $\mathcal{S}'$ and let $h : \Omega \to \Omega'$ be a homeomorphism preserving the lines $E \cap \Omega$ and $D \cap \Omega$. The following statements are equivalent:*

a) There are homeomorphisms $\psi_0 : \Sigma \to \mathbb{S}^1 \times [0, 1]$ and $\psi'_0 : \Sigma' \to \mathbb{S}^1 \times [0, 1]$ such that $h_* \mathcal{F}_{\psi_0} = \mathcal{F}_{\psi'_0}$.

b) Given a homeomorphism $\psi : \Sigma \to \mathbb{S}^1 \times [0, 1]$, there is $\psi' : \Sigma' \to \mathbb{S}^1 \times [0, 1]$ such that $h_* \mathcal{F}_\psi = \mathcal{F}_{\psi'}$

c) There is a topological equivalence $H : \mathcal{S} \to \mathcal{S}'$ that extends h.

Proof. a)⇒b) Denote $\widetilde{h} = \eta' \circ h \circ \eta^{-1} : \Sigma \setminus \delta\Delta \to \Sigma' \setminus \delta\Delta'$ the homeomorphism induced by h in a way compatible with the flows of ξ and ξ'. Then $\widetilde{h}$ extends to a homeomorphism $\widetilde{h} : \Sigma \to \Sigma'$. To see this, consider $\bar{h} = \psi'_0 \circ \widetilde{h} \circ {\psi_0}^{-1}$ that is an automorphism of $\mathbb{S}^1 \times (0, 1]$ respecting the vertical foliation. Extend $\bar{h}$ to $\mathbb{S}^1 \times \{0\}$ and put $\psi' = \bar{h} \circ \psi \circ \widetilde{h}^{-1}$.

b)⇒a) Evident.

b)⇒c) We know [3] that a homeomorphism $\widetilde{h} : \Sigma \to \Sigma'$ of the fences extends to a topological equivalence $H : \mathcal{S} \to \mathcal{S}'$. By compatibility with the flow, the restriction of H to Ω is exactly h.

c)⇒b) Put $\widetilde{h} = H\,|_{\Sigma}\colon \Sigma \to \Sigma'$ and $\psi' = \psi \circ \widetilde{h}^{-1}$.

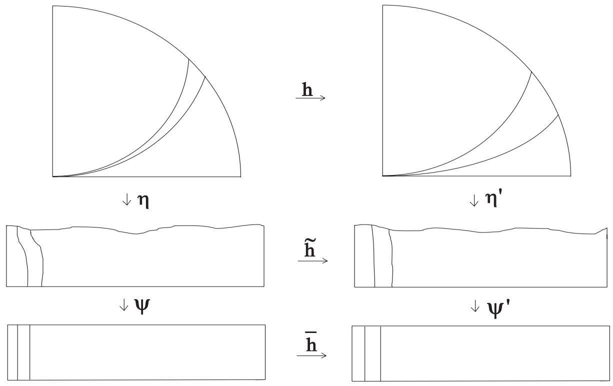

FIG. 5. Homeomorphisms on the fence.

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Remark. We get the same kind of result for non rectified chimneys.

5. BLOWING UP A SADDLE

Let $\mathcal{S} = (\xi, C_\xi(a,b,\rho), H, D, E)$ be an adapted real saddle. Denote by λ, μ and δ the eigenvalues corresponding respectively to $H \cap E, H \cap D$ and $D \cap E$. Recall that $\lambda, \mu < 0$ and $\delta > 0$. Take coordinates (x,y,z) such that $H = x = 0$, $E = y = 0$, $D = z = 0$. Put $\alpha = \frac{\lambda}{\mu}$. Let us consider the map

$$\begin{aligned}\pi_\alpha : \mathbf{R} \times (\mathbb{S}^1 \times \mathbf{R}_{\geq 0}) &\longrightarrow \mathbf{R} \times \mathbf{R}^2 \\ (x,\theta,r) &\longrightarrow (x, r\cos\theta, r^\alpha \sin\theta)\end{aligned}$$

if $\alpha \geq 1$ and $\pi_\alpha(x,\theta,r) = (x, r^{1/\alpha}\cos\theta, r\sin\theta)$ if $0 < \alpha < 1$. From now on we shall assume that $\alpha \geq 1$, the case $\alpha < 1$ is treated in an analogous way.

LEMMA 6. *There is a unique $\mathcal{C}^1$-vector field $\widetilde{\xi}$ defined on the inverse image $\widetilde{C} = \pi_\alpha^{-1}(C_\xi(a,b,\rho))$ that coincides with ξ on $\widetilde{C} \setminus \pi_\alpha^{-1}(D \cap E)$ under the isomorphism*

$$\pi_\alpha : \widetilde{C} \setminus \pi_\alpha^{-1}(D \cap E) \longrightarrow C_\xi(a,b,\rho) \setminus D \cap E$$

Proof. Let us write

$$\xi = A(x,y,z)x\frac{\partial}{\partial x} + B(x,y,z)y\frac{\partial}{\partial y} + C(x,y,z)z\frac{\partial}{\partial z}$$

where A, B and C are analytic functions such that

$$A(0) = \delta; B(0) = -1; C(0) = -\alpha$$

(up to multiply ξ by a positive constant). Now, put $\widetilde{\xi} = \widetilde{A}\frac{\partial}{\partial x} + \widetilde{B}\frac{\partial}{\partial \theta} + \widetilde{C}\frac{\partial}{\partial r}$ where

$$\begin{aligned}
\widetilde{A} &= (A \circ \pi_\alpha)x. \\
\widetilde{B} &= \tfrac{\sin\theta\cos\theta}{\alpha\sin^2\theta + \cos^2\theta}(-\alpha(B \circ \pi_\alpha) + (C \circ \pi_\alpha)). \\
\widetilde{C} &= \tfrac{r}{\alpha\sin^2\theta + \cos^2\theta}(\cos^2\theta(B \circ \pi_\alpha) + \sin^2\theta(C \circ \pi_\alpha)).
\end{aligned}$$

Since $\alpha \geq 1$ we have that $(A \circ \pi_\alpha), (B \circ \pi_\alpha)$ and $(C \circ \pi_\alpha)$ are $\mathcal{C}^1$- functions. We have also that $d\pi_\alpha(\widetilde{\xi}) = \xi \circ \pi_\alpha$. The uniqueness is evident by continuity. ▮

Remark. The vector field $\widetilde{\xi}$ is tangent to the exceptional divisor of the blowing-up $\Lambda = {\pi_\alpha}^{-1}(D \cap E) = \{r = 0\}$. In fact

$$\widetilde{\xi}\mid_{\{r=0\}} = G_1(x)x\frac{\partial}{\partial x} - \frac{\sin\theta\cos\theta}{\alpha\sin^2\theta + \cos^2\theta}G_2(x)\frac{\partial}{\partial \theta}$$

where $G_1(x)$ and $G_2(x)$ are analytic functions with $G_1(0) = \delta$ and $G_2(0) = 0$. Moreover, the vector field $\widetilde{\xi}$ is also tangent to $\widetilde{H} = {\pi_\alpha}^{-1}(H \setminus D \cap E) = \{x = 0\}$, $\widetilde{D} = \{\theta = \frac{\pi}{2}\}$ and $\widetilde{E} = \{\theta = 0\}$. We have

$$\begin{aligned}
\widetilde{\xi}\mid_{\{x=0\}} = &\tfrac{\sin\theta\cos\theta}{\alpha\sin^2\theta + \cos^2\theta} rF_1(0,\theta,r)\tfrac{\partial}{\partial\theta} \\
+ &\tfrac{r}{\alpha\sin^2\theta + \cos^2\theta}(-1 + rF_2(0,\theta,r))\tfrac{\partial}{\partial r}
\end{aligned}$$

where $F_1(x)$ and $F_2(x)$ are functions with $F_1(0) = F_2(0) = 0$. The only singular points of $\widetilde{\xi}$ are the points of the axis $\{r = x = 0\} = {\pi_\alpha}^{-1}(0)$ and the linear part of $\widetilde{\xi}$ on them is given by

$$\widetilde{\xi}_0 = \delta x\frac{\partial}{\partial x} - r\frac{\partial}{\partial r} + \frac{\sin\theta\cos\theta}{\alpha\sin^2\theta + \cos^2\theta}(k_1 x - k_2 r\cos\theta)\frac{\partial}{\partial\theta}$$

where k_1, k_2 are real constants.

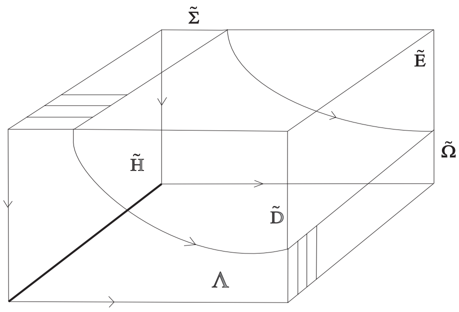

FIG. 6. Blowing-up a chimney.

In order to simplify notation, let us denote $C = C_\xi(a, b, \rho)$, $\widetilde{C} = {\pi_\alpha}^{-1}(C)$, $\widetilde{\Sigma} = \pi_\alpha^{-1}(\Sigma)$, and $\widetilde{\Omega} = \overline{{\pi_\alpha}^{-1}(\Omega \setminus D \cap E)}$. We know that $\widetilde{\Sigma}$ is isomorphic to Σ under the restriction of π_α; let us denote by $\tau : \widetilde{\Sigma} \longrightarrow \Sigma$ this isomorphism. Moreover, there is a homeomorphism $\kappa : \widetilde{\Omega} \longrightarrow \mathbb{S}^1 \times [0, 1]$ given by $\kappa(b, \theta, r) = (\theta, \frac{r}{\rho})$.

Lemma 7. *Let $\varphi : \Sigma \longrightarrow \mathbb{S}^1 \times [0, 1]$ be a homeomorphism such that if $\widetilde{\varphi} = \kappa^{-1} \circ \varphi \circ \tau : \widetilde{\Sigma} \longrightarrow \widetilde{\Omega}$ we have the following property:*

"For each point $Q \in \widetilde{\Sigma} \setminus \pi_\alpha^{-1}(H)$ then $\widetilde{\varphi}(Q)$ is in the trajectory of $\widetilde{\xi}$ passing through Q."

Then $\mathcal{F}_\alpha = \mathcal{F}_\varphi$.

Proof. Since $\mathcal{F}_\alpha$ and $\mathcal{F}_\varphi$ are singular foliations with the origin $P = \Omega \cap (D \cap E)$ as singular point, we have only to prove that

$$\mathcal{F}_\alpha \mid_{\Omega \setminus \{P\}} = \mathcal{F}_\varphi \mid_{\Omega \setminus \{P\}} .$$

Let $\chi : \widetilde{\Omega} \setminus \Lambda \longrightarrow \Omega \setminus \{P\}$ be the isomorphism obtained as a restriction of the blowing-up π_α. Denote by $\mathcal{L}$ the vertical foliation on $\mathbb{S}^1 \times [0, 1]$. Looking at the diagram of isomorphisms

$$\begin{array}{ccc} \widetilde{\Omega} \setminus \Lambda & \xrightarrow{\chi} & \Omega \setminus \{P\} \\ \kappa \big\downarrow & & \\ \mathbb{S}^1 \times (0, 1] & & \end{array}$$

we get that $\mathcal{F}_\alpha = (\kappa \circ \chi^{-1})^* \mathcal{L}$. Let $\eta : \Sigma \setminus H \longrightarrow \Omega \setminus \{P\}$ be the isomorphism induced by the flow of ξ. Then, the stated property is equivalent to say that the following diagram is commutative

$$\begin{array}{ccccc}
 & & \widetilde{\Sigma} \setminus {\pi_\alpha}^{-1}(H) & & \\
 & \swarrow^{\tau} & & \searrow^{\widetilde{\varphi}} & \\
\Sigma \setminus H & \xrightarrow{\eta} & \Omega \setminus \{P\} & \xleftarrow{\chi} & \widetilde{\Omega} \setminus \Lambda \\
 & \searrow^{\varphi} & & \swarrow_{\kappa} & \\
 & & \mathbb{S}^1 \times (0,1] & &
\end{array}$$

Hence $\mathcal{F}_\alpha = (\kappa \circ \chi^{-1})^* \mathcal{L} = (\varphi \circ \eta^{-1})^* \mathcal{L} = \eta_* \mathcal{L}_\varphi = \mathcal{F}_\varphi$. ∎

Lemma 8. *The isomorphism* $\widetilde{\varphi} : \widetilde{\Sigma} \setminus {\pi_\alpha}^{-1}(H) \longrightarrow \widetilde{\Omega} \setminus \Lambda$ *induced by the flow of* $\widetilde{\xi}$ *extends to* $\widetilde{\varphi} : \widetilde{\Sigma} \longrightarrow \widetilde{\Omega}$.

Proof. Recall that $\widetilde{\xi} \mid_{\widetilde{H}} = r\xi_1$, where ξ_1 is non singular on $0 \leq r \leq a$ and transversal to the levels $r = cte$. Hence we have an isomorphism $\nu : \widetilde{\Sigma} \cap \widetilde{H} \longrightarrow {\pi_\alpha}^{-1}(0)$ given by the flow of ξ_1. In the same way, we have that

$$\widetilde{\xi} \mid_{\Lambda} = x\xi_2$$

where ξ_2 is non singular on $0 \leq x \leq b$ and transversal to the levels $x = cte$. Thus we get an isomorphism

$$\mu : \widetilde{\Omega} \cap \Lambda \longrightarrow {\pi_\alpha}^{-1}(0)$$

given by the flow of ξ_2. Consider a point $Q \in \widetilde{\Sigma} \cap \widetilde{H}$ and a sequence of points $Q_n \in \widetilde{\Sigma} \setminus \widetilde{H}$ such that

$$\lim_{n \to \infty} Q_n = Q$$

Notice that $\widetilde{\varphi} : \widetilde{\Sigma} \setminus \widetilde{H} \longrightarrow \widetilde{\Omega} \setminus \Lambda$ is given by the flow of $\widetilde{\xi}$. In order to prove our result it is enough to prove that:

$$\lim_{n \to \infty} \widetilde{\varphi}(Q_n) = \widetilde{Q}$$

where $\widetilde{Q} = \mu^{-1}(\nu(Q))$. Then, we put $\widetilde{\varphi}(Q) = \widetilde{Q}$. Let us prove it. In view of the description of the linear part of $\widetilde{\xi}$ along the only stationary points $(\{r = x = 0\} = {\pi_\alpha}^{-1}(0))$ we have that ${\pi_\alpha}^{-1}(0)$ is the central variety of all its points. By applying the "reduction theorem to the center manifold" [8, 10, 11] there is a neighborhood $\mathcal{V}$ of $\nu(Q)$ and a topological equivalence

$$\Upsilon : \mathcal{V} \longrightarrow \mathbf{R}^3_{(x,y,z)}$$

between $\widetilde{\xi}\mid_{\mathcal{V}}$ and the linear vector field $L = \delta x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}$. Moreover, we have that

$$\begin{aligned}&\Upsilon(\nu(Q)) = 0 \in \mathbf{R}^3.\\ &\Upsilon(\mathcal{V}\cap{\pi_\alpha}^{-1}(0)) = \{x = z = 0\}.\\ &\Upsilon(\Lambda\cap\mathcal{V}) = \{z = 0\}.\\ &\Upsilon(\widetilde{H}\cap\mathcal{V}) = \{x = 0\}\end{aligned}$$

Fix a point $P = (0, 0, z_0)$ and put $P' = \Upsilon^{-1}(P)$. Obviously P' belongs to the trajectory γ_Q of $\widetilde{\xi}$ passing through Q. Let $T'_{P'}$ be a two-dimensional transversal to $\widetilde{\xi}$ at P' such that $T'_{P'}\subset\mathcal{V}$. The flow of $\widetilde{\xi}$ gives an isomorphism

$$\widetilde{\varphi}_1 : V_Q \longrightarrow T'_{P'}$$

where V_Q is a neighborhood of Q in $\widetilde{\Sigma}$. Put

$$P'_n = \widetilde{\varphi}_1(Q_n), \qquad n >> 0$$

(Obviously $\lim_{n\to\infty} P'_n = P'$). Now put $P_n = \Upsilon(P'_n)$, select a point $S = (x_0, 0, 0)$ and a transversal $T_S = \{x = x_o\}$ to the linear vector field L. Denote by S_n the intersection with T_S of the trajectory of L passing through P_n. We have that

$$\lim_{n\to\infty} S_n = S.$$

Put $S' = \Upsilon^{-1}(S)$, $S'_n = \Upsilon^{-1}(S_n)$. The flow of $\widetilde{\xi}$ gives a morphism

$$\widetilde{\varphi}_2 : W_{S'} \longrightarrow \widetilde{\Omega}$$

where $W_{S'}$ is a neighborhood of S'. Moreover, by construction $\widetilde{\varphi}_2(S'_n) = \widetilde{\varphi}(Q_n)$ and $\widetilde{\varphi}_2(S') = \widetilde{Q}$. Hence $\lim_{n\to\infty}\widetilde{\varphi}(Q_n) = \widetilde{Q}$. ▌

Proposition 9. *The foliation $\mathcal{F}_\alpha$ is compatible with the saddle $\mathcal{S}$.*

Proof. Take $\widetilde{\varphi} : \widetilde{\Sigma} \longrightarrow \widetilde{\Omega}$ as in the previous lemma and put $\varphi = \kappa\circ\widetilde{\varphi}\circ\tau^{-1}$. By lemma 7, we get that $\mathcal{F}_\alpha = \mathcal{F}_\varphi$. ▌

6. TOPOLOGICAL EQUIVALENCE FOR SADDLE CONNECTIONS

Let $\mathcal{C}, \mathcal{C}'$ be two adapted saddle connections. Consider $C_0 = {C_\xi}^0(a_0, b_0, \rho_0)$ and $C_1 = {C_\xi}^1(a_1, b_1, \rho_1)$ chimneys associated to the singular points P_0 and P_1 respectively. Let Ω_0, Ω_1 be the tops of C_0 and C_1:

$$\begin{aligned}&\Omega_0 \subset \{x = b_0\}\\ &\Omega_1 \subset \{x = 1 - b_1\}\end{aligned}$$

Assume that $b_0 < \frac{1}{2}, b_1 < \frac{1}{2}$. The situation is analogous for the vector field ξ'. We can also take $\Omega_0' \subset \{x = b_0\}$ and $\Omega_1' \subset \{x = 1-b_1\}$. Let us consider Ω, Ω' transversal discs to Γ, Γ' contained in $x = \frac{1}{2}$ such that putting

$$\begin{aligned} {\Omega_0}^* &= Sat_{\xi,V}(\Omega) \cap \{x = b_0\} \\ {\Omega_1}^* &= Sat_{\xi,V}(\Omega) \cap \{x = 1 - b_1\} \end{aligned}$$

we have ${\Omega_0}^* \subset \Omega_0$ and ${\Omega_1}^* \subset \Omega_1$. Denote by ${C_0}^*$ and ${C_1}^*$ the non-rectified chimneys with tops ${\Omega_0}^*$ and ${\Omega_1}^*$ respectively.

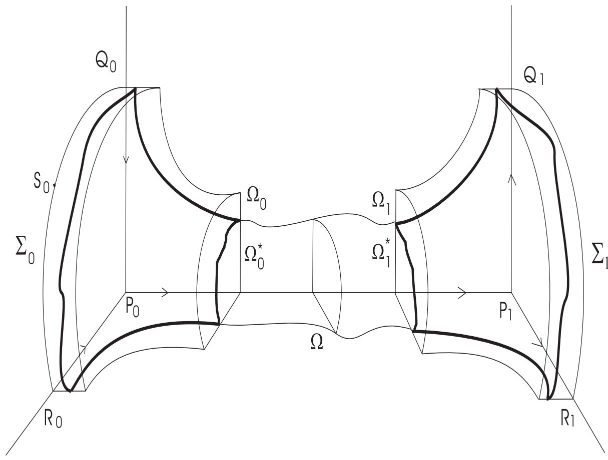

FIG. 7. The neighborhood $\mathcal{U}$.

Let T be the tubular neighborhood

$$T = Sat_{\xi, b_0 \leq x \leq 1-b_1}(\Omega)$$

and put $\mathcal{U} = {C_0}^* \cup T \cup {C_1}^*$. Denote by $R_i = \Delta_i \cap H_i \cap D$ and $Q_i = \Delta_i \cap H_i \cap E$, $i = 0, 1$. (We define $\mathcal{U}', R_i', Q_i'$ in the same way).

Let S_0 be a point in $\Sigma_0 \cap H_0$, $S_0 \notin \{R_0, Q_0\}$. Take a sequence of points $\{S_{0,k}\}$ in Σ_0, such that

$$\lim_{k \to \infty} S_{0,k} = S_0$$

and put $S_{1,k} = \gamma_{S_{0,k}} \cap \Sigma_1$, where $\gamma_{S_{0,k}}$ is the trajectory of ξ passing through $S_{0,k}$.

LEMMA 10. *Let ∇ be the vector field*

$$\nabla = -x(x-1)\frac{\partial}{\partial x} + (2x-1)y\frac{\partial}{\partial y} + ((1+\frac{\beta}{\alpha})x - 1)z\frac{\partial}{\partial z}$$

Consider the adapted saddle connection $\mathcal{D} = (\nabla, \Gamma, V, H_0, H_1, D, E)$ *then*

$$\lim_{k\to\infty} S_{1,k} = \begin{cases} S_1 \notin \{Q_1, R_1\} & \textit{if } \varepsilon(\mathcal{D}) = 1 \\ Q_1 & \textit{if } \varepsilon(\mathcal{D}) < 1 \\ R_1 & \textit{if } \varepsilon(\mathcal{D}) > 1. \end{cases}$$

Proof. Let $\mathcal{R}_m$ be the surface on $0 \leq x \leq 1$, $z \geq 0$, $y \geq 0$ given by

$$f(x, y, z) = y - m(1-x)^{\frac{\beta}{\alpha}-1} z, \qquad m > 0$$

Note that

$$\begin{array}{rl} \{y = mz\} \subset \overline{\mathcal{R}}_m & \text{if } \ \beta = \alpha \\ \{y = 0, x = 1\} \subset \overline{\mathcal{R}}_m & \text{if } \ \beta > \alpha \\ \{z = 0, x = 1\} \subset \overline{\mathcal{R}}_m & \text{if } \ \beta < \alpha. \end{array}$$

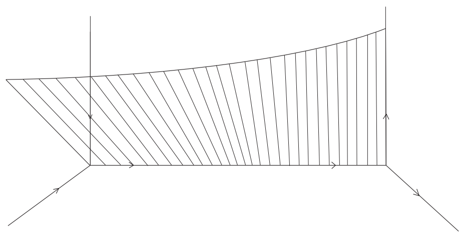

FIG. 8. The surface $\mathcal{R}_m$.

We get that $df(\nabla)\mid_{\{f=0\}} = 0$ and thus ∇ is tangent to the surfaces $\mathcal{R}_m$. Let us consider a sequence $\{S_{1,k}\}$ as in definition above. There exists $k_0 > 0$ and $m > 0$ such that $S_{0,k} \in \mathcal{R}_m$ if $k \geq k_0$ for the corresponding sequence $\{S_{0,k}\}$. By the invariance of $\mathcal{R}_m$ we conclude.

∎

Proof of Theorem 1. *b)* $\Rightarrow$ *a)* Consider the foliations $\mathcal{F}_\alpha, \mathcal{F}_\beta$ in Ω and $\mathcal{F}_{\alpha'}, \mathcal{F}_{\beta'}$ in Ω'. By proposition 2, there exists a homeomorphism $h : \Omega \longrightarrow \Omega'$ such that $h_*\mathcal{F}_\alpha = \mathcal{F}_{\alpha'}$ and $h_*\mathcal{F}_\beta = \mathcal{F}_{\beta'}$. Let us find a topological equivalence $H : \mathcal{U} \longrightarrow \mathcal{U}'$ such that $H\mid_\Omega = h$. First, let us extend h along the tubular neighborhood T. Given a point $P \in T$, we have that $P = \gamma_Q \cap \{x = d\}$, where $Q \in \Omega$ and γ_Q is the trajectory of ξ passing through Q, $d \in [b_0, 1-b_1]$. Put $H_T(P) = \gamma_{h(Q)} \cap \{x = d\}$, we obtain a homeomorphism $H_T : T \longrightarrow T'$. Denote h_0 and h_1 the respective restrictions of H_T to ${\Omega_0}^*$ and ${\Omega_1}^*$.

There are homeomorphisms $f_0 : \Omega \longrightarrow {\Omega_0}^*$ and $f'_0 : \Omega' \longrightarrow {\Omega'_0}^*$ such that

$$h = {f_0}'^{-1} \circ h_0 \circ f_0.$$

Affirmation: The foliations $f_{0*}\mathcal{F}_\alpha$, $f'_{0*}\mathcal{F}_{\alpha'}$ are compatible with the adapted real saddles $\mathcal{S}_0$ and $\mathcal{S}'_0$ respectively (to see this, in proof of proposition 9, consider the homeomorphisms $\widetilde{\varphi} : \widetilde{\Sigma} \longrightarrow \widetilde{\Omega_0}^* = {\pi_\alpha}^{-1}({\Omega_0}^*)$ defined in the same way).

Note that $h_{0*}(f_{0*}\mathcal{F}_\alpha) = (h_0 \circ f_0)_*\mathcal{F}_\alpha = f'_{0*}\mathcal{F}_{\alpha'}$. By proposition 5 we can extend h_0 to a topological equivalence $H_0 : {C_0}^* \longrightarrow {C'_0}^*$. We obtain $H_1 : {C_1}^* \longrightarrow {C'_1}^*$ analogously. We have $H_T \mid_{{\Omega_0}^*} = H_0 \mid_{{\Omega_0}^*}$ and $H_T \mid_{{\Omega_1}^*} = H_1 \mid_{{\Omega_1}^*}$ by gluing H_T, H_0 and H_1 we get the desired topological equivalence $H : \mathcal{U} \longrightarrow \mathcal{U}'$.

$a) \Rightarrow b)$ Assume that b) does not hold. By the implication above if we consider the vector fields

$$\nabla = -x(x-1)\frac{\partial}{\partial x} + (2x-1)y\frac{\partial}{\partial y} + ((1+\frac{\beta}{\alpha})x-1)z\frac{\partial}{\partial z}$$
$$\nabla' = -x(x-1)\frac{\partial}{\partial x} + (2x-1)y\frac{\partial}{\partial y} + ((1+\frac{\beta'}{\alpha'})x-1)z\frac{\partial}{\partial z}$$

it only remains to show that the saddle connections

$$\mathcal{C}^* = (\nabla, \Gamma, V, H_0, H_1, D, E), \qquad \mathcal{C}'^* = (\nabla', \Gamma, V', H_0, H_1, D, E)$$

can not be topologically equivalent. Consider the case $\alpha = \beta$, $\alpha' < \beta'$. If there exists a topological equivalence $H : \mathcal{C}^* \longrightarrow \mathcal{C}'^*$ we have $H(R_i) = R'_i, H(Q_i) = Q'_i$, $i = 0, 1$. By Lemma 10, we know that $\lim_{n\to\infty} {S_1}^k = S_1 \neq \{Q_1, R_1\}$ and we get a contradiction since

$$\lim_{k\to\infty} H(S_{1,k}) = \lim_{k\to\infty} H(\gamma_{S_{0,k}} \cap \Sigma_1) = \lim_{k\to\infty} \gamma'_{H(S_{0,k})} \cap \Sigma'_1 = Q'_1.$$

If $\alpha > \beta$, $\alpha' < \beta'$, we have that $\lim_{k\to\infty} S_{1,k} = R_1$, and $\lim_{k\to\infty} H(S_{1,k}) = Q'_1$. Contradiction.

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