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A Polynomial Bound for the Lap Number

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In this note, we show a polynomial bound for the growth of the lap number of a piecewise monotone and piecewise continuous interval map with finitely many periodic points. We use Milnor and Thurston's kneading theory with the coordinates of Baladi and Ruelle, which are useful for extending the theory to the non continuous case.

Key Words: Lap number, Dynamical zeta functions.

1. INTRODUCTION AND MAIN RESULT

We say that $f : [0,1] \to [0,1]$ is piecewise strictly monotone piecewise continuous (PSMC) if there is a finite number of so-called *critical points* $c_1 < \cdots < c_r \in (0,1)$ such that f is continuous and strictly monotone on (c_i, c_{i+1}) and on $(0, c_1)$ and $(c_r, 1)$. The composition of two PSMC maps is also PSMC, usually with a different set of critical points. We denote by $C_f = \{c_i : i = 1 \cdots r\}$ the set of critical points of f (we choose C_f to be minimal in the obvious sense) and by $p_f(n)$ the minimal number of intervals in which f^n $(= f \circ \ldots \circ f\ n$ times) is continuous and strictly monotone. We call $p_f(n)$ the *lap number* of f^n. Clearly, $p_f(n) = \#C_{f^n} + 1$.
We say that $x \in [0,1]$ is *periodic* with period $m = m(x)$ if $f^m(x) = x$. Such an x is *repelling* if there is a neighborhood U of x such that for all $y \in U$, $y \neq x$, there is ℓ with $f^{m\ell}(y) \notin U$.
Set $\epsilon : [0,1] \to \{0, \pm 1\}$ by $\epsilon(x) = 0$ if $x \in C_f$, and $\epsilon(x) = \pm 1$ depending on whether f is increasing or decreasing at $x \notin C_f$. Let $\nu = 0$ if $\epsilon((0, c_1)) \cdot \epsilon((c_r, 1)) = -1$, and 1 otherwise.
In this note, we prove the following Theorem:

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THEOREM 1. *Let $f : [0,1] \to [0,1]$ be a PSMC map, with q periodic points, $q < \infty$. Then, $p_f(n) \le C(s,q) \cdot n^s$, where $s = r + \ell + \nu + 1$, $r = \#C_f$, $\ell = \#\{$repelling periodic orbits of $f\}$ and $C(s,q)$ is a constant depending only on s and q.*

(Compare with [5], Corollary 9.7 page 520.) The error term ν is probably an artifact of the proof and does not seem natural. We will use Milnor and Thurston's kneading theory [5] with the coordinates introduced by Baladi and Ruelle [2]. Since most of the proof can be easily derived from [5] (see [6] for another account), we will pass rather quickly over the details.

We begin by recalling some results and definitions of [2]. Let $f : [0,1] \to [0,1]$ be PSMC, c_i $i = 1 \cdots r$ its critical points. The *address* α of a point $x \in [0,1]$ is a vector in $\{0, \pm 1\}^r$ defined by $(\alpha(x))_i = sgn(x - c_i)$ (where sgn denotes the sign function). If H is a function defined on the interval, we write $H(x+)$ for $\lim_{y \to x, y > x} H(y)$, and similarly for $H(x-)$ (if these limits exists).

DEFINITION 2. Set $\epsilon^{(i)}(x) = \prod_{0<k<i} \epsilon(f^k x)$. The invariant coordinate of $x \in [0,1]$ is the vector of power series

$$\theta_f(x)(t) = \sum_{i \ge 0} t^i \cdot \epsilon^{(i)}(x) \cdot \alpha(f^n x). \tag{1}$$

The kneading matrix $K(t)$ is given by the rows $K_\ell(t) = \frac{1}{2}\big(\theta_f(c_\ell+)(t) - \theta_f(c_\ell-)(t)\big)$ $(\ell = 1 \cdots r)$. The kneading determinant is $\Delta_f(t) = \det(K(t))$.

Denote by $\mathrm{Per}(f)$ the set of *periodic orbits* of f (a periodic orbit of period m is a set $\{x, f(x), \cdots, f^{m-1}\}$ where x is a periodic point of period m). If $\gamma \in \mathrm{Per}(f)$, we write $\pi(\gamma)$ for its period. Baladi and Ruelle proved the following theorem for PSMC maps:

THEOREM 3 (Baladi-Ruelle 1994). *Let f be PSMC. Then,*

$$\frac{1}{\Delta_f(t)} = \frac{\zeta_f^R(t)}{Q(t)}, \tag{2}$$

where $Q(t) = 1 - \frac{1}{2}(\epsilon(0+) + \epsilon(1-)) \cdot t$ and $\zeta_f^R(t)$ is the reduced zeta-function of f, which can be written as

$$\zeta_f^R(t) = \prod_{\gamma \in Per(f)} F(\gamma), \tag{3}$$

$F(\gamma)$ being a polynomial of degree $\pi(\gamma)$ if γ is a non-repelling orbit, and a power series $1 \pm t^{\pi(\gamma)} \pm t^{2\pi(\gamma)} \pm t^{3\pi(\gamma)} \cdots$ if γ is repelling.

(See [2], Theorem 1.1 page 625.) In the continuous case, Milnor and Thurston [5] proved a similar result (with a different definition of the zeta function, which is almost equivalent to the above one). We end these preliminaries with the following remark, which will be useful later:

REMARK 4. *If* $\Omega(t) = \Omega_1(t) \cdot \Omega_2(t) \cdots \Omega_s(t)$ *where* $\Omega_i(t)$ *are power series with coefficients in* $\{0, \pm 1\}$*, then the coefficients of* Ω *cannot grow faster than those of* $\frac{1}{(1-t)^s}$.

We can now begin the proof of Theorem 1.

Proof. Let $f : [0,1] \to [0,1]$ be PSMC, c_i $i = 1 \cdots r$ its critical points. For $1 \leq k \leq r$ and $i \geq 0$, we define (as in [5]) the set:

$$E_k^i = \{x \in [0,1] \,:\, f^i(x) = c_k,\ f^j(x) \notin C_f \text{ for } 0 \leq j < i\},$$

and introduce the power series

$$P(t) = \sum_{n \geq 0} p_f(n) \cdot t^n \quad \text{and} \quad \Gamma_k(t) = \sum_{n \geq 0} \# E_k^n \cdot t^n.$$

Notice that if $x \in E_\ell^i$ then $\alpha(f^j x+) = \alpha^j(f^j x-)$ and $\epsilon^{(j)}(x+) = \epsilon^{(j)}(x-)$ for $0 < j < i$. Therefore,

$$x \in E_\ell^i \Rightarrow \frac{1}{2}[\theta_f(x+)(t) - \theta_f(x-)(t)] = t^i \cdot K_\ell(t) \tag{4}$$

(recall Definition 2). The identity (4) enables us to derive, as in [5] (with slightly different coordinates), that

$$\frac{1}{2}[(\theta_f(1-)(t) - \theta_f(0+)(t))] = \sum_{\ell=1}^{r} \Gamma_\ell(t) \cdot K_\ell(t) \tag{5}$$

and, using (5),

$$\Gamma_k(t) = \sum_{m=1}^{r} \frac{1}{2}[(\theta_f(1-)(t) - \theta_f(0+)(t))]_m \cdot M_{mk}(t), \tag{6}$$

where $[\cdot]_m$ denotes the mth coordinate, and $M(t) = (M_{ij}(t))$ is the inverse matrix of $K(t)$ (by Cramer's rule, $K(t)$ has a formal inverse, which is in fact meromorphic since $K(0)$ is the identity matrix). Detailed computations with our coordinates can be found in [1] (where the notation is slightly different since [1] deals with tree maps). See also [6].

For $k = 1, \cdots, r$, we now define the set

$$R(k) = \{i \in \mathbf{N} : \ f^i(c_k+) = f^i(c_k-) \text{ and } \prod_{j<i} \epsilon(f^j(c_k+)) = \prod_{j<i} \epsilon(f^j(c_k-))\}.$$

We thus have $c_k \in C_{f^n} \Leftrightarrow n \notin R(k)$. Hence, x is a critical point of f^n if and only if there is $0 \le j < n$ and $1 \le k \le r$ such that $f^j(x) = c_k$ and $n - j \notin R(k)$. Therefore,

$$p_n(f) = 1 + \sum_{k=1}^{r} \sum_{j=0}^{n-1} \delta_k^{n-j} \cdot \#E_k^j, \tag{7}$$

where $\delta_k^i = 1$ if $i \notin R(k)$, and $\delta_k^i = 0$ otherwise. Letting $\delta_k(t) = \sum_{j \ge 0} \delta_k^j t^j$, we immediately get from (7) the relation

$$P(t) = \frac{1}{1-t} + t \cdot \sum_{k=1}^{r} \delta_k(t) \Gamma_k(t). \tag{8}$$

Then, from (8) and (6) follows

$$P(t) = \underbrace{\frac{1}{1-t}}_{A} + \sum_{k,m=1}^{r} \underbrace{t \cdot \delta_k(t)}_{B} \cdot \underbrace{\frac{1}{2}[(\theta_f(1-)(t) - \theta_f(0+)(t))]_m}_{C} \cdot \underbrace{M_{mk}(t)}_{D}. \tag{9}$$

If one wants to study the growth of the lap number $p_f(n)$, it is enough to study the terms A, B, C, D of (9). We will use Remark 4 constantly.
A is $(1 + t + t^2 + t^3 + \cdots)$, so its contribution to $p_f(n)$ is just 1. B and C are power series with coefficients in $\{0, \pm 1\}$. D is an entry of the inverse of $K(t)$, hence it is a $r-1$-minor of $K(t)$ divided by $\Delta_f(t) = \det(K(t))$. Since the entries of $K(t)$ have coefficients $0, \pm 1$, a minor is 'at worse' (by Remark 4) $c \cdot (1 + t + t^2 + t^3 + \cdots)^{r-1}$ (with for instance $c = (r-1)!$). Therefore, the terms B, C and D bring at most a contribution of $c \cdot (1 + t + t^2 + t^3 + \cdots)^{r+1} \cdot \frac{1}{\Delta_f(t)}$. Multiplying the constant by r^2, we get rid of the sum.
We now use Theorem 3 to investigate $1/\Delta_f(t)$. Notice that if $\nu = 0$ then $Q(t) = 1$, and if $\nu = 1$, $Q(t) = 1 \pm t$. Therefore, since f has a finite number of periodic orbits (it is the only place where we use this assumption), $\frac{1}{\Delta_f(t)}$ brings at most (by Theorem 3) a contribution of $c' \cdot (1+t+t^2+t^3+\cdots)^{\ell+\nu}$, where ℓ is the number of repelling periodic orbits of f, and $c' - 1$ is the number of non-repelling periodic orbits of f. Theorem 1 follows from the fact that the coefficients a_n of $(1 + t + t^2 + \cdots)^s$ are polynomials of degree s in n. ∎

Unfortunately, this method does not enable us to obtain directly a better upper bound or a lower bound. In particular, we cannot answer (positively or negatively) the following conjecture:

Conjecture (Boshernitzan and Kornfeld [3]). Let f be PSMC, with only fixed points. Then, $p_f(n)$ grows linearly.

Let us finish by pointing out that (9) shows that $P(t)$ is meromorphic in the unit disk, which is the main step toward building a semi-conjugacy between any PSMC map with entropy $h > 0$ and a piecewise linear piecewise continuous map with slope $\pm e^h$. (Recall that $h > 0$ is not compatible with a finite number of periodic orbits.) This has already been done by Milnor and Thurston [5] for continuous maps, and by Preston [6] for PSMC maps. Both proofs use the same argument but use different coordinates to show that $P(t)$ is meromorphic. We have shown here that the proof can also be done with our coordinates. We could then build the semi-conjugacy using $P(t)$, as in Preston's paper [6].
In the case $h = 0$, Buzzi and Hubert [4] obtain similar results using Hofbauer's tower extension's approach. They also get our Theorem 1 (without the term ν) as a corollary, in a slightly restricted case.

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