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Center Conditions for a Lopsided Quintic Polynomial Vector Field *

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We consider in this article, a lopsided quintic polynomial vector field $X = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} + \sum_{i+j=5}(a_{ij}x^i y^j \frac{\partial}{\partial x})$. We first compute the first non-zero derivative of the return map $r \mapsto L(r, \varepsilon)$. We study then the necessary and sufficient conditions for the existence of a center.

Key Words: Center-focus problem, nonlinear differential equations.

1. INTRODUCTION

Let X be a polynomial vector field of the form

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + \sum_{2\leq i+j\leq d}(a_{ij}x^i y^j \frac{\partial}{\partial x} + b_{ij}x^i y^j \frac{\partial}{\partial y}), \tag{1}$$

where $a_{ij}, b_{ij} \in \mathbf{R}$. Given such a vector field X corresponding to a fixed set of values (a_{ij}, b_{ij}), there exists a neighborhood U of the origin $O \in \mathbf{R}^2$

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on which the flow of X, that is the solution of the differential system

$$\frac{dx}{dt} = -y + \sum_{2\leq i+j\leq d} a_{ij}x^i y^j, \qquad \frac{dy}{dt} = x + \sum_{2\leq i+j\leq d} b_{ij}x^i y^j, \tag{2}$$

exists for all initial values. We can furthermore assume that there is a first return mapping L defined on U. Given an initial point $(r, 0)$, $r > 0$, the solution of (2) with initial data $(r, 0)$ intersects again for the first time the x-axis at some point $(L(r), 0)$, $L(r) > 0$. We denote by $\Sigma = \{(x, 0) \in U\}$ the transversal section, by transversality, the mapping L is analytic and it has a Taylor series

$$L(r) = r + L_2\, r^2 + \cdots + L_k\, r^k + \cdots \tag{3}$$

The coefficients L_k, $k \geq 2$, are called the *Lyapunov-Poincaré* coefficients. The sign of the first non-vanishing Lyapunov-Poincaré coefficient determines the stability of the origin. Let L_{k0} be this coefficient. If $L_{k0} > 0$, then close to the origin, the orbits spirals away and 0 is unstable. On the contrary, if $L_{k0} < 0$, then 0 is stable.

We now assume that (a_{ij}, b_{ij}) varies slightly from a fixed value. In some fixed neighborhood of the origin, the first return mapping L of X relative to Σ still exists and it now depends of the coefficients (a_{ij}, b_{ij}). We write now

$$L(r) = r + L_2(a_{ij}, b_{ij})\, r^2 + \cdots + L_k(a_{ij}, b_{ij})\, r^k + \cdots \tag{4}$$

A direct computation shows in fact that the coefficients L_k are polynomials in the coefficients $(\underline{a}, \underline{b}) = (a_{ij}, b_{ij})$ and thus they are globally defined. The first polynomial $L_{k0}(\underline{a}, \underline{b})$ which is non-zero determines the stability of the origin. The vanishing of the non-zero coefficients $L_k(\underline{a}, \underline{b})$ gives the algebraic expressions which are points of a real algebraic manifold or of a center manifold.

We use the algorithm introduced by J.P. Françoise in [7] for a polynomial vector field

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + \sum_{i+j=d}(a_{ij}x^i y^j \frac{\partial}{\partial x} + b_{ij}x^i y^j \frac{\partial}{\partial y}) \tag{5}$$

in order to compute the first non-zero coefficients for a homogeneous perturbative part. The implementation of this algorithm was developed in [9]. We introduce a parameter ε and consider the 1-parameter family

$$X_\varepsilon = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + \sum_{i+j=d} \varepsilon(a_{ij}x^i y^j \frac{\partial}{\partial x} + b_{ij}x^i y^j \frac{\partial}{\partial y}). \tag{6}$$

The vector fields X_ε has a first return mapping $L(r,\varepsilon)$ of the form

$$L(r,\varepsilon) = r + \varepsilon L_2(r, a_{ij}, b_{ij}) + \cdots + \varepsilon^k L_k(r, a_{ij}, b_{ij}) + \cdots \tag{7}$$

The algorithm allows to compute the first non-zero derivative of $L(r,\varepsilon)$ relatively to ε. This coincides with the coefficients $L_k(a_{ij}, b_{ij})$ in the case of a homogeneous perturbative part. We use the computer algebra methods presented in [9] in order to do our computations for the polynomial vector field (5), where $d = 5$ and $b_{ij} = 0$.

The problem of a *center* for the polynomial vector fields (1) consists in finding all the necessary and sufficient conditions bearing on the coefficients (a_{ij}, b_{ij}), in order that all orbits in a neighborhood of the origin be periodic. For the polynomial vector field (5), these conditions have been found by H. Dulac [5] for $d = 2$ and by Sibirski [16] for $d = 3$. In [8], it was shown that the algorithm introduced in [7] leads to the usual conditions of Dulac for $d = 2$ and of Sibirski for $d = 3$. For $d = 3$ see also the works of N.G. Lloyd and his co-workers [3, 12, 13, 1]. For general references to Hilbert's 16^{th} problem and to the center problem, see [2, 4, 6, 10, 14, 16, 17, 18].

In this work, we determine all the conditions on a polynomial vector field

$$X = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} + \sum_{i+j=5}(a_{ij}x^i y^j \frac{\partial}{\partial x}) \tag{8}$$

and its corresponding differential system

$$\frac{dx}{dt} = y + \sum_{i+j=5} a_{ij}x^i y^j, \qquad \frac{dy}{dt} = -x, \tag{9}$$

such that the origin is a center. We refer to the vector field (8) as the *lopsided quintic vector field* and the system (9) as the *lopsided quintic system.* For a lopsided quartic vector field see [15].This kind of systems have been studied for the first time by Kukles (1944), when $i+j \leq 3$, for example see [11].

2. THE ALGORITHM OF THE MULTIPLICITY OF THE FIRST RETURN MAPPING

We use the algorithm introduced in [7] for the lopsided quintic vector field (8). We include here in general a short description of it. We introduce the complex coordinates $z = \frac{1}{\sqrt{2}}(x + \sqrt{-1}y)$, $\overline{z} = \frac{1}{\sqrt{2}}(x - \sqrt{-1}y)$, and the

1-form $\omega = i_X(dx \wedge dy)$. With this new notation, instead of the polynomial vector field X in (1), we use

$$\omega = dH + \omega_1 = dH + \sum_{2 \le i+j \le d} (A_{ij} z^j \bar{z}^i dz + \bar{A}_{ij} \bar{z}^j z^i d\bar{z}), \tag{10}$$

whose complex coefficients A_{ij} are easily related to the real coefficients (a_{ij}, b_{ij}) of (1). The function H is $H : (z, \bar{z}) \longmapsto H(z, \bar{z}) = z\bar{z}$.
We introduce next a real parameter ε and the 1-parameter family of 1-forms $\omega_\varepsilon = dH + \varepsilon\omega_1$. The corresponding 1-parameter family of vector fields X_ε of (6) such that $i_{X_\varepsilon}(dx \wedge dy) = \omega_\varepsilon$ has a first return mapping $L(r, \varepsilon)$

$$L(r, \varepsilon) = r + \varepsilon L_2(r, A_{ij}, \bar{A}_{ij}) + \cdots + \varepsilon^k L_k(r, A_{ij}, \bar{A}_{ij}) + \cdots \tag{11}$$

We define the *multiplicity* in ε of the first return mapping $L(r, \varepsilon)$ as the first k_0 so that $L_{k_0}(r, A_{ij}, \bar{A}_{ij})$ is not identically (in r) zero. At this point, it is convenient to choose $r = H|_\Sigma$ as the coordinate on the transversal section Σ.

There is a formula due to H. Poincaré which gives

$$L_1(r, A_{ij}, \bar{A}_{ij}) = -\int_{H=r^2} \omega_1 \ .$$

Assume that $L_1(r, A_{ij}, \bar{A}_{ij}) = 0$ (as a function of r) then there is a polynomial g_1 such that

$$\omega_1 = g_1 dH + dR_1$$

and we get

$$L_2(r, A_{ij}, \bar{A}_{ij}) = \int_{H=r^2} g_1\omega_1 \ .$$

One can show inductively that given

$$L_k(r, A_{ij}, \bar{A}_{ij}) = (-1)^k \int_{H=r^2} g_{k-1}\omega_1$$

if $L_k(r, A_{ij}, \bar{A}_{ij}) = 0$ (as a function of r), then there is a polynomial g_k such that

$$g_{k-1}\omega_1 = g_k dH + dR_k \ ,$$

and then

$$L_{k+1}(r, A_{ij}, \bar{A}_{ij}) = (-1)^{k+1} \int\limits_{H=r^2} g_k \omega_1 \ .$$

The consequence is that we can compute the first non-zero coefficient $L_k(r, A_{ij}, \bar{A}_{ij})$ by building the sequence of polynomials $g_1, \dots, g_k, \dots$, at each step k we have the 1-form $g_k\omega_1$. We first compute the differential $d(g_k\omega_1) = F^k(z, \bar{z})\, dz \wedge d\bar{z}$, next we split the polynomial $F^k(z, \bar{z})$ into two parts

$$F^k(z, \bar{z}) = \sum_{i \neq j} F^k_{ij} z^i \bar{z}^j + \sum_{i=j} (F^k_{ij})^k (z\bar{z})^i \ .$$

We find that

$$L_{k+1}(r, A_{ij}, \bar{A}_{ij}) = \sum_l F^k_l r^l \ ,$$

next we compute $d(g_{k+1}\omega_1)$ and we repeat the process.

3. THE LOPSIDED QUINTIC VECTOR FIELD

For the lopsided quintic vector field (8), we have the 1-form

$$\omega = i_X(dx \wedge dy) = \dot{x}dy - \dot{y}dx = (xdx + ydy) - (\sum_{i+j=5} a_{ij} x^i y^j) dy \ .$$

By using the complex coordinates ($z = \frac{1}{\sqrt{2}}(x + \sqrt{-1}y), \bar{z} = \frac{1}{\sqrt{2}}(x - \sqrt{-1}y)$), we get $(xdx + ydy) = d(z\bar{z}) = dH$, and

$$(\sum_{i+j=5} a_{ij} x^i y^j) dy = \sum_{i+j=5} A_{ij} z^i \bar{z}^j d(z - \bar{z}) = (\sum_{i+j=5} A_{ij} z^i \bar{z}^j) dz + (\sum_{i+j=5} \bar{A}_{ij} \bar{z}^i z^j) d\bar{z},$$

where
$A_{50} = -\frac{a_{05}+a_{41}}{8} - i\frac{a_{50}-a_{32}-a_{23}+a_{14}}{8}$,
$A_{41} = \frac{5a_{05}-3a_{41}}{8} + i\frac{-5a_{50}+a_{32}-a_{23}+3a_{14}}{8}$,
$A_{32} = \frac{-2a_{41}-10a_{05}}{8} - i\frac{10a_{50}+2a_{32}+2a_{23}+2a_{14}}{8}$,
$A_{23} = \frac{2a_{41}+10a_{05}}{8} - i\frac{10a_{50}+2a_{32}+2a_{23}+2a_{14}}{8}$,
$A_{14} = \frac{-5a_{05}+3a_{41}}{8} + i\frac{-5a_{50}+a_{32}-a_{23}+3a_{14}}{8}$,
$A_{05} = \frac{a_{05}+a_{41}}{8} - i\frac{a_{50}-a_{32}-a_{23}+a_{14}}{8}$.
So we have

$$\begin{cases} A_{50} &= -\bar{A}_{05} \\ A_{41} &= -\bar{A}_{14} \\ A_{32} &= -\bar{A}_{23}. \end{cases} \tag{12}$$

We refer to (12) as the *lopsided quintic conditions*.
So, after this change of coordinates, for the 1-parameter family

$$X_\varepsilon = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} + \sum_{i+j=5} \varepsilon(a_{ij}x^i y^j \frac{\partial}{\partial x}), \tag{13}$$

we have

$$\omega_\varepsilon = dH + \varepsilon\,(Adz + Bd\bar{z}) \tag{14}$$

with the lopsided quintic conditions (12), where $H = z\bar{z}$, $A = \sum_{i+j=5} A_{ij} z^i \bar{z}^j$, and $B = \sum_{i+j=5} \bar{A}_{ij}\bar{z}^i z^j$.

We have used many times the following theorem [9] in order to do the successive elimination of parameters and to produce center conditions:

Theorem 1. *The first non-zero Lyapunov-Poincaré polynomial $L_i[j]$ is of the form $L_i[j] = P_i[j] - \bar{P}_i[j] = 2i\,\Im(P_i[j])$.*

By using the procedures dvg, swap, and purge [9] on MAPLE with the lopsided quintic conditions (12), we get the Lyapunov-Poincaré polynomials:

Lemma 2. *The non-zero Lyapunov-Poincaré polynomials for the lopsided quintic vector field (8) are $L_i[2i] = 2i\,\Im\,(P_i[2i])$, for $i = 1, 2, 3, 4, 5$, where*

$$P_1[2] = 3\bar{A}_{23},$$

$$P_2[4] = 5\bar{A}_{05}A_{14} + 5A_{23}\bar{A}_{14},$$

$$P_3[6] = \frac{21}{2}\bar{A}_{05}A_{23}^2 - \frac{133}{8}\bar{A}_{05}\bar{A}_{23}A_{14} - \frac{91}{8}\bar{A}_{05}\bar{A}_{23}A_{05} + \frac{175}{24}\bar{A}_{05}A_{23}A_{14} - \frac{7}{3}\bar{A}_{05}\bar{A}_{14}A_{05} - 7A_{14}^2\bar{A}_{05} - 56\bar{A}_{23}A_{23}\bar{A}_{14} - \frac{511}{8}\bar{A}_{23}\bar{A}_{14}A_{14} + \frac{91}{2}A_{23}^2\bar{A}_{14} - 35\bar{A}_{23}^2A_{23},$$

$$P_4[8] = \frac{81}{16}A_{23}^3\bar{A}_{14} - \frac{225}{8}A_{14}^2\bar{A}_{05}^2 + \frac{265}{2}\bar{A}_{05}\bar{A}_{23}A_{05}A_{14} + \frac{1629}{16}A_{23}^3\bar{A}_{05} - \frac{1459}{8}\bar{A}_{23}A_{23}\bar{A}_{05}A_{14} + \frac{1165}{8}\bar{A}_{14}\bar{A}_{05}\bar{A}_{23}A_{05} - \frac{1089}{8}\bar{A}_{14}\bar{A}_{05}A_{23}A_{14} - \frac{423}{4}\bar{A}_{14}^2A_{23}^2 + 54A_{23}\bar{A}_{23}^3 + 162\bar{A}_{14}\bar{A}_{23}^2A_{23} - \frac{1917}{16}A_{14}^2\bar{A}_{14}A_{23} - \frac{675}{16}A_{05}A_{23}^2\bar{A}_{14} + \frac{45}{4}A_{05}^2\bar{A}_{23}\bar{A}_{05} + \frac{2349}{16}\bar{A}_{23}^2A_{05}A_{23} + \frac{3}{2}A_{14}^3\bar{A}_{05} - \frac{1427}{16}\bar{A}_{23}^2A_{05}\bar{A}_{14} + \frac{1053}{8}A_{05}\bar{A}_{14}^2A_{14} + \frac{225}{8}A_{05}^2\bar{A}_{14}\bar{A}_{05} + \frac{903}{16}\bar{A}_{14}^2\bar{A}_{23}A_{05} + \frac{729}{4}A_{14}\bar{A}_{14}\bar{A}_{23}^2 - \frac{21}{16}A_{05}\bar{A}_{14}^2A_{23} + \frac{405}{4}\bar{A}_{23}^2\bar{A}_{05}A_{05} - \frac{3987}{16}\bar{A}_{14}^2A_{23}A_{14} - \frac{3465}{16}A_{23}^2\bar{A}_{14}\bar{A}_{23},$$

$$P_5[10] = -\frac{10659}{8}A_{14}\bar{A}_{23}^4 - \frac{6751349}{5760}A_{05}A_{14}\bar{A}_{14}\bar{A}_{05}\bar{A}_{23} + \frac{36223}{128}\bar{A}_{05}^2\bar{A}_{23}A_{23}A_{05} + \frac{49115}{128}\bar{A}_{23}^2A_{05}A_{23}\bar{A}_{05} + \frac{2008193}{11520}A_{23}\bar{A}_{05}^2A_{05}A_{14} + \frac{10681}{2880}\bar{A}_{05}A_{23}^2A_{05}\bar{A}_{14} - \frac{11341}{30}A_{05}A_{14}\bar{A}_{14}^3 - \frac{5819}{16}\bar{A}_{05}^2A_{23}^2A_{14} - \frac{1175339}{1152}A_{05}\bar{A}_{14}\bar{A}_{23}^3 - \frac{13013}{60}A_{05}^2\bar{A}_{14}^2\bar{A}_{05} -$$

$$\frac{76813}{512}A_{05}^2\bar{A}_{23}\bar{A}_{05}^2+\frac{126731}{512}\bar{A}_{05}^2\bar{A}_{23}A_{14}^2+\frac{17567}{8}\bar{A}_{14}^2A_{23}^2\bar{A}_{23}-\frac{95843}{16}A_{23}^3\bar{A}_{23}\bar{A}_{14}-$$
$$\frac{19635}{128}\bar{A}_{05}A_{14}\bar{A}_{23}^3-\frac{223179}{512}A_{14}^2\bar{A}_{05}^2A_{23}-\frac{334807}{96}A_{14}\bar{A}_{14}^2A_{23}^2+$$
$$\frac{102751}{128}\bar{A}_{14}^2\bar{A}_{23}A_{05}A_{23}-\frac{1188715}{384}\bar{A}_{05}A_{23}^2A_{14}\bar{A}_{23}-\frac{141229}{72}\bar{A}_{05}A_{23}^2A_{14}\bar{A}_{14}+$$
$$\frac{2910721}{1152}\bar{A}_{05}A_{14}\bar{A}_{23}^2A_{23}-\frac{55}{16}A_{05}^3\bar{A}_{05}^2+\frac{274153}{384}A_{14}^2\bar{A}_{23}^2\bar{A}_{05}+\frac{901747}{512}A_{14}^2\bar{A}_{23}\bar{A}_{14}^2-$$
$$\frac{33429}{128}\bar{A}_{05}A_{05}\bar{A}23^3-\frac{309441}{128}\bar{A}_{14}A_{14}\bar{A}_{23}^3-\frac{81059}{96}A_{05}\bar{A}_{14}^2\bar{A}_{23}^2-$$
$$\frac{209}{6}A_{05}\bar{A}_{14}^3A_{23}+\frac{105589}{16}\bar{A}_{14}\bar{A}_{23}^2A_{23}^2+\frac{3245}{24}\bar{A}_{14}^3A_{23}A_{14}-\frac{2035}{12}\bar{A}_{14}^3\bar{A}_{23}A_{05}-$$
$$\frac{21813}{32}A_{14}\bar{A}_{14}^2\bar{A}_{23}^2-\frac{29117}{16}A_{23}^3\bar{A}_{05}\bar{A}_{23}-\frac{33671}{128}\bar{A}_{05}^2A_{23}^2A_{05}-\frac{143}{144}A_{05}^2\bar{A}_{05}^2\bar{A}_{14}-$$
$$\frac{2321}{4}A_{23}^3\bar{A}_{05}\bar{A}_{14}+\frac{9669}{8}\bar{A}_{05}\bar{A}_{23}^2A_{23}^2-1584A_{23}\bar{A}_{14}\bar{A}_{23}^3+\frac{2167}{16}A_{14}^2\bar{A}_{05}\bar{A}_{14}^2+$$
$$\frac{2035}{8}\bar{A}_{05}A_{23}A_{05}\bar{A}_{14}^2-\frac{140789}{3840}A_{14}^2A_{23}\bar{A}_{05}\bar{A}_{14}-\frac{121}{3}A_{05}^2\bar{A}_{14}^3+\frac{26609}{8}A_{23}^2\bar{A}_{a23}^3+$$
$$\frac{13007819}{1152}A_{14}\bar{A}_{23}^2A_{23}\bar{A}_{14}+\frac{69773}{36}\bar{A}_{05}\bar{A}_{23}\bar{A}_{14}A_{23}A_{14}+$$
$$\frac{167497}{192}\bar{A}_{05}\bar{A}_{23}A_{05}A_{23}\bar{A}_{14}+\frac{117997}{24}\bar{A}_{14}^2\bar{A}_{23}A_{14}A_{23}+\frac{215303}{1280}A_{05}\bar{A}_{05}^2\bar{A}_{23}A_{14}+$$
$$\frac{297}{40}\bar{A}_{05}A_{05}\bar{A}_{14}^2A_{14}+\frac{724273}{768}A_{14}^2\bar{A}_{23}\bar{A}_{05}\bar{A}_{14}-\frac{34837}{72}\bar{A}_{05}\bar{A}_{23}^2\bar{A}_{14}A_{05}-$$
$$\frac{13321}{8}\bar{A}_{14}^2A_{23}^3-924\bar{A}_{23}^4A_{23}+\frac{8723}{40}\bar{A}_{14}A_{14}\bar{A}_{05}^2A_{05}+\frac{8723}{16}A_{23}^4\bar{A}_{05}+\frac{11}{16}A_{14}^3\bar{A}_{14}^2.$$

We have computed the other Lyapunov-Poincaré polynomials $L_i[2i]$, for $i = 6, 7, 8, 9, 10$. Their expressions are too lengthy to be reproduced here, the interested reader can get it by http://www-math.unice.fr/˜ salih/lopquint.ps.

Remark 3. Sometimes, the action of the group of rotation allows to assume that $A_{ij} \in \mathbf{R}$ with $i - j + 1 \neq 0$. But, for the lopsided quintic vector field this is not possible, since this vector field is not invariant under the action of the group of rotation.

4. CENTER CONDITIONS FOR A LOPSIDED QUINTIC VECTOR FIELD

In this case, we have $\omega_\varepsilon = dH + \varepsilon\omega$ with the lopsided quintic conditions (12), where $H = z\bar{z}$, and $\omega = \left(\sum_{i+j=5} A_{ij} z^i \bar{z}^j\right) dz + \left(\sum_{i+j=5} \bar{A}_{ij} \bar{z}^i z^j\right) d\bar{z}$.

THEOREM 4. *The origin of $dH + \varepsilon\omega$ is a center if and only if all the coefficients A_{ij} with $i + j = 5$ are real.*

Proof. First, we suppose that the coefficients A_{ij} with $i+j = 5$ are real. The Lyapunov-Poincaré coefficients are polynomials in A_{ij}, so they are real expressions. In this case, by using theorem 1, we get $L_i[2i] = \Im(P_i[2i]) = 0$, for $i \geq 1$. This implies that the origin of $dH + \varepsilon\omega$ is a center.

Now, if the origin of ω_ε is a center, we have to show that all the coefficients A_{ij} with $i + j = 5$ are real. The first non-zero Lyapunov-Poincaré polynomial is

$$L_1[2] = \Im(P_1[2]) = \Im(3\bar{A}_{23}) = -3\Im(A_{23}),$$

the vanishing of $L_1[2]$ gives $A_{23} \in \mathbf{R}$, we suppose that $A_{23} = \alpha$, where $\alpha \in \mathbf{R}$. The second non-zero Lyapunov-Poincaré polynomial is

$$L_2[4] = \Im(P_2[4]) = \Im(5\bar{A}_{05}A_{14} + 5\alpha\bar{A}_{14}) = 5\Im(A_{14}(\bar{A}_{05} - \alpha)).$$

1 If $A_{14} = 0$, the third non-zero Lyapunov-Poincaré polynomial is

$$L_3[6] = -\frac{21}{2}\alpha^2\Im(A_{05}).$$

1.1 If $\alpha = 0$, we get that $L_4[8] = 0$ and

$$L_5[10] = \Im(P_5[10]) = \frac{55}{10}\|A_{05}\|^2\Im(A_{05}).$$

The vanishing of $L_5[10]$ gives that $A_{05} \in \mathbf{R}$, so all the coefficients A_{ij} with $i + j = 5$ are real.
1.2 If $\Im(A_{05}) = 0$ but $\alpha \neq 0$. In this case also we have $A_{05} \in \mathbf{R}$, so all the coefficients A_{ij} with $i + j = 5$ are real.
2 If $A_{14} \neq 0$, in this case $L_2[4] = 5\Im(A_{14}(\bar{A}_{05} - \alpha)) = 0$ gives that $A_{05} = \beta A_{14} + \alpha$, where $\beta \in \mathbf{R}$. Now the computation of $L_3[6]$ gives that

$$L_3[6] = \Im(\psi_1 A_{14}^2 + \psi_2 A_{14}),$$

where $\psi_1 = \alpha(\frac{\beta}{3} - 1)$ and $\psi_2 = \frac{\beta^2\|A_{14}\|^2}{3} - \beta\|A_{14}\|^2 + \frac{\alpha^2}{2} - \frac{3\alpha^2\beta}{2}$. Now we study the following cases:
2.1 If $\psi_1 = 0$ and $\psi_2 \neq 0$, we have $L_3[6] = \psi_2\Im(A_{14})$. So $L_3[6] = 0$ implies that $A_{14} \in \mathbf{R}$, and all the coefficients A_{ij} with $i + j = 5$ are real.
2.2 If $\psi_1 = \psi_2 = 0$, we have the following particular cases:
2.2.1 If $\alpha = \beta = 0$, the computation shows that $L_1[2] = L_2[4] = L_3[6] = L_4[8] = 0$ and

$$L_5[10] = \frac{11}{16}\|A_{14}\|^2\Im(A_{14}).$$

We have $A_{14} \neq 0$, so $L_5[10] = 0$ implies that $A_{14} \in \mathbf{R} \setminus \{0\}$. In this case we have $A_{23} = A_{05} = 0$ and $A_{14} \in \mathbf{R} \setminus \{0\}$.
2.2.2 If $\beta = 3$ and $\alpha = 0$, we have $L_3[6] = 0$. The computation of $L_4[8]$ and $L_5[10]$ gives

$$L_4[8] = \Im(P_4[10]) = \frac{9}{2}\|A_{14}\|^2\Im(A_{14}^2)$$

and

$$L_5[10] = \Im(P_5[10]) = \frac{1182}{5}\|A_{14}\|^2\Im(A_{14}).$$

Since we have $A_{14} \neq 0$, the vanishing of $L_5[10]$ gives $A_{14} \in \mathbf{R} \setminus \{0\}$. therefore in this case we have $A_{23} = 0$, $A_{05} = 3A_{14}$ and $A_{14} \in \mathbf{R} \setminus \{0\}$.
2.3 (general case) If $\psi_1 \neq 0$ and $\psi_2 \neq 0$, when we substitute $\bar{A}_{14}$ in $L_3[6] = P_3[6] - \bar{P}_3[6]$ by $\bar{A}_{14} = \frac{\|A_{14}\|^2}{A_{14}}$, we get

$$L_3[6] = \frac{7}{6A_{14}^2}(A_{14}^2 - \|A_{14}\|^2)(c_{11}A_{14}^2 + c_{12}A_{14} + c_{13}),$$

where $c_{11} = 2\alpha(\beta - 3)$, $c_{12} = 2\beta^2\|A_{14}\|^2 - 6\beta\|A_{14}\|^2 + 3\alpha^2 - 9\beta\alpha^2$ and $c_{13} = 2\alpha\beta\|A_{14}\|^2 - 6\alpha\|A_{14}\|^2$.
Now the computation of $L_4[8], L_5[10]$, and $L_6[12]$ gives:

$$L_4[8] = \frac{3}{4A_{14}^3}(A_{14}^2 - \|A_{14}\|^2)(c_{21}A_{14}^4 + c_{22}A_{14}^3 + c_{23}A_{14}^2 + c_{24}A_{14} + c_{25}),$$

where

$$c_{21} = 2\alpha,$$

$$c_{22} = 30\alpha^2 - 55\alpha^2\beta + 2\beta\|A_{14}\|^2 + 15\alpha^2\beta^2,$$

$$c_{23} = 75\alpha^3\beta - 57\alpha^3 - \alpha\|A_{14}\|^2 + 33\alpha\beta\|A_{14}\|^2 - 55\alpha\beta^2\|A_{14}\|^2 + 15\alpha\beta^3\|A_{14}\|^2,$$

$$c_{24} = \|A_{14}\|^2 c_{22},$$

and

$$c_{25} = \|A_{14}\|^4 c_{21}.$$

$$L_5[10] = -\frac{11}{720A_{14}^3}(A_{14}^2 - \|A_{14}\|^2)(c_{31}A_{14}^4 + c_{32}A_{14}^3 + c_{33}A_{14}^2 + c_{34}A_{14} + c_{35}),$$

where

$$c_{31} = 225\beta^3\alpha^2 - 65\beta^2\alpha^2 + 2454\beta\alpha^2 - 16020\alpha^2,$$

$$c_{32} = 2532\beta^2\|A_{14}\|^2\alpha - 15894\|A_{14}\|^2\alpha + 1755\beta^2\alpha^3 - 130\beta^3\|A_{14}\|^2\alpha - 34856\beta\alpha^3 + 69\alpha^3 + 450\beta^4\|A_{14}\|^2\alpha - 18174\|A_{14}\|^2\beta\alpha,$$

$$c_{33} = -25947\alpha^2\|A_{14}\|^2 + 2205\beta^3\|A_{14}\|^2\alpha^2 - 20811\|A_{14}\|^2\beta^2\alpha^2 - 65\beta^4\|A_{14}\|^4 + 225\|A_{14}\|^4\beta^5 - 21339\beta\|A_{14}\|^2\alpha^2 - 7086\alpha^4 - 45\|A_{14}\|^4 - 2154\beta^2\|A_{14}\|^4 + 78\beta^3\|A_{14}\|^4 - 15879\beta\|A_{14}\|^4 - 2790\beta\alpha^4,$$

$$c_{34} = 2532\beta^2\|A_{14}\|^4\alpha - 15894\alpha\|A_{14}\|^4 + 1755\|A_{14}\|^2\alpha^3\beta^2 - 130\alpha\beta^3\|A_{14}\|^4 - 34856\alpha^3\beta\|A_{14}\|^2 + 69\|A_{14}\|^2\alpha^3 + 450\beta^4\|A_{14}\|^4\alpha - 18174\beta\|A_{14}\|^4\alpha,$$

and

$$c_{35} = -65\alpha^2\beta^2\|A_{14}\|^4 - 16020\|A_{14}\|^4\alpha^2 + 2454\|A_{14}\|^4\beta\alpha^2 + 225\beta^3\|A_{14}\|^4\alpha^2.$$

$$L_6[12] = \frac{13}{4320A_{14}^4}(A_{14}^2 - \|A_{14}\|^2)(\sum_{i=1}^{7} c_{4i}A^{7-i}),$$

where

$$c_{41} = -110160\beta\alpha^2 + 25680\beta^2\alpha^2 + 53100\alpha^2,$$

$$c_{42} = -344835\alpha^3 - 253340\beta^2\alpha^3 + 51360\beta^3\|A_{14}\|^2\alpha - 57816\|A_{14}\|^2\alpha - 16875\beta^3\alpha^3 - 220320\beta^2\|A_{14}\|^2\alpha + 829614\beta\alpha^3 + 130932\|A_{14}\|^2\beta\alpha,$$

$$c_{43} = 727512\|A_{14}\|^2\beta^2\alpha^2 + 602214\alpha^4 - 336816\alpha^2\|A_{14}\|^2 + 77832\beta^2\|A_{14}\|^4 - 602111\beta\alpha^4 + 269739\beta\|A_{14}\|^2\alpha^2 - 57816\beta\|A_{14}\|^4 - 33750\beta^4\|A_{14}\|^2\alpha^2 - 222235\beta^3\|A_{14}\|^2\alpha^2 + 25680\beta^4\|A_{14}\|^4 + 4095\beta^2\alpha^4 - 110160\beta^3\|A_{14}\|^4,$$

$$c_{44} = 213999\alpha^5 - 16875\beta^5\|A_{14}\|^4\alpha + 133254\|A_{14}\|^2\alpha^3\beta^2 - 315180\|A_{14}\|^2\beta^3\alpha^3 + 82206\|A_{14}\|^2\alpha^3 + 31105\beta^4\|A_{14}\|^4\alpha - 317379\beta\|A_{14}\|^4\alpha - 149805\beta\alpha^5 - 76422\alpha\beta^3\|A_{14}\|^4 + 496776\alpha^3\beta\|A_{14}\|^2 + 577314\beta^2\|A_{14}\|^4\alpha - 59967\alpha\|A_{14}\|^4,$$

$$c_{45} = 602214\alpha^4\|A_{14}\|^2 + 4095\beta^2\|A_{14}\|^2\alpha^4 - 336816\|A_{14}\|^4\alpha^2 - 602111\alpha^4\|A_{14}\|^2\beta - 33750\|A_{14}\|^4\beta^4\alpha^2 - 222235\beta^3\|A_{14}\|^4\alpha^2 + 727512\alpha^2\beta^2\|A_{14}\|^4 + 269739\|A_{14}\|^4\beta\alpha^2 + 77832\beta^2\|A_{14}\|^6 + 25680\beta^4\|A_{14}\|^6 - 57816\beta\|A_{14}\|^6 - 110160\beta^3\|A_{14}\|^6,$$

$$c_{46} = -220320\beta^2\|A_{14}\|^6\alpha - 16875\alpha^3\beta^3\|A_{14}\|^4 - 57816\|A_{14}\|^6\alpha + 51360\beta^3\|A_{14}\|^6\alpha + 829614\beta\|A_{14}\|^4\alpha^3 - 253340\beta^2\|A_{14}\|^4\alpha^3 + 130932\alpha\beta\|A_{14}\|^6 - 344835\|A_{14}\|^4\alpha^3,$$

and

$$c_{47} = -110160\|A_{14}\|^6\beta\alpha^2 + 25680\|A_{14}\|^6\beta^2\alpha^2 + 53100\|A_{14}\|^6\alpha^2.$$

Remark 5. We suppose that $\phi_1 = c_{11}A_{14}^2 + c_{12}A_{14} + c_{13}$,
$\phi_2 = c_{21}A_{14}^4 + c_{22}A_{14}^3 + c_{23}A_{14}^2 + c_{24}A_{14} + c_{25}$,
$\phi_3 = c_{31}A_{14}^4 + c_{32}A_{14}^3 + c_{33}A_{14}^2 + c_{34}A_{14} + c_{35}$, and $\phi_4 = \sum_{i=1}^{7} c_{4i}A^{7-i}$.

2.3.1 If $A_{14}^2 - \|A_{14}\|^2 = 0$, we get that $A_{14} = \pm\|A_{14}\| \in \mathbf{R} \setminus \{0\}$, so all coefficients A_{ij} with $i + j = 5$ are real.
2.3.2 If $A_{14}^2 - \|A_{14}\|^2 \neq 0$, in order to show the vanishing of $L_3[6]$ we compute $r(\phi_1, \phi_2, A_{14})$, $r(\phi_1, \phi_3, A_{14})$ and $r(\phi_1, \phi_4, A_{14})$ which are the resultants of the polynomials ϕ_2, ϕ_3 and ϕ_4 with ϕ_1 rapport to A_{14} respectively. We obtain the following

$$r(\phi_1, \phi_2, A_{14}) = 4\alpha^6\|A_{14}\|^4\eta_1^2,$$

$$r(\phi_1, \phi_3, A_{14}) = 9\alpha^4 \|A_{14}\|^4 \eta_2^2,$$

$$r(\phi_1, \phi_4, A_{14}) = 36\alpha^8 \|A_{14}\|^6 \eta_3^2,$$

where

$$\eta_1 = -747\alpha^2 + 585\beta\alpha^2 + 1122\beta^2\alpha^2 - 795\beta^3\alpha^2 + 135\beta^4\alpha^2 - 90\|A_{14}\|^2 + 96\beta\|A_{14}\|^2 + 14\beta^2\|A_{14}\|^2 - 12\beta^3\|A_{14}\|^2,$$

$$\eta_2 = -132678\alpha^4 + 108414\beta\alpha^4 + 244029\beta^2\alpha^4 - 179853\beta^3\alpha^4 + 4725\beta^4\alpha^4 + 6075\beta^5\alpha^4 - 22248\alpha^2\|A_{14}\|^2 + 36540\beta\|A_{14}\|^2\alpha^2 + 1812\|A_{14}\|^2\ beta^2\alpha^2 - 17772\beta^3\|A_{14}\|^2\alpha^2 + 4644\beta^4\|A_{14}\|^2\alpha^2 - 540\|A_{14}\|^4 + 540\beta\|A_{14}\|^4 - 180\beta^2\|A_{14}\|^4 + 20\beta^3\|A_{14}\|^4,$$

and

$$\eta_3 = 15679251\alpha^4 - 39205215\beta\alpha^4 - 2019897\beta^2\alpha^4 + 56007942\beta^3\alpha^4 - 26045787\beta^4\alpha^4 + 2024865\beta^5\alpha^4 + 455625\beta^6\alpha^4 + 18337104\alpha^2\|A_{14}\|^2 - 32313654\beta\|A_{14}\|^2\alpha^2 - 7753950\|A_{14}\|^2\beta^2\alpha^2 + 38735064\beta^3\|A_{14}\|^2\alpha^2 - 24173568\beta^4\|A_{14}\|^2\alpha^2 + 6262446\|A_{14}\|^2\alpha^2\beta^5 - 632610\|A_{14}\|^2\alpha^2\beta^6 + 2003940\|A_{14}\|^4 - 3955824\beta\|A_{14}\|^4 + 2023920\beta^2\|A_{14}\|^4 + 133176\beta^3\|A_{14}\|^4 - 301076\beta^4\|A_{14}\|^4 + 51192\|A_{14}\|^4\beta^5.$$

Since $\psi_1 \neq 0$, α can not be zero. From **2**, $\|A_{14}\|$ also can not be zero. So in order to show the vanishing of $r(\phi_1, \phi_2, A_{14})$, we compute $r(\eta_1, \eta_2, \|A_{14}\|)$ and $r(\eta_1, \eta_3, \|A_{14}\|)$ which are the resultants of the polynomials η_2 and η_3 with η_1 rapport to $\|A_{14}\|$ respectively. We get that

$$r(\eta_1, \eta_2, \|A_{14}\|) = 12\alpha^4(\beta + 1)(\beta - 3)^2\delta_1,$$

$$r(\eta_1, \eta_3, \|A_{14}\|) = -336\alpha^4(\beta + 1)(\beta - 3)^3\delta_2,$$

where
$\delta_1 = 730215\beta^8 - 3916080\beta^7 + 2144202\beta^6 + 15124572\beta^5 - 18090308\beta^4 - 13425096\beta^3 + 28348110\beta^2 - 12214692\beta + 1108485$,
and
$\delta_2 = 273375\beta^9 - 688095\beta^8 - 1358820\beta^7 + 2663514\beta^6 + 5077974\beta^5 - 5349896\beta^4 - 14053692\beta^3 + 22496670\beta^2 - 10408149\beta + 1368495$.
Moreover, the polynomials δ_1 and δ_2 have no common roots, because $r(\delta_1, \delta_2, \beta)$ which is the resultant of the polynomials δ_2 with δ_1 rapport to β can not be zero.
We deduce from $\psi_1 \neq 0$ that neither α can be equal to zero nor β can be equal to three. So the vanishing of $r(\eta_1, \eta_2, \|A_{14}\|)$ and $r(\eta_1, \eta_3, \|A_{14}\|)$

implies that $\beta = -1$. Now if $\beta = -1$, we have $\eta_1 = 720\alpha^2 - 160\|A_{14}\|^2$, then $\eta_1 = 0$ gives that $\|A_{14}\|^2 = \frac{720}{160}\alpha^2$. In this case, we get that

$$L_3[6] = -4\alpha(2A_{14}^2 - 12\alpha A_{14} + 9\alpha^2).$$

Finally, the vanishing of $L_3[6]$ gives that $A_{14} = 3(1 \pm \frac{\sqrt{2}}{2})\alpha \in \mathbf{R} \setminus \{0\}$, so all the coefficients A_{ij} with $i + j = 5$ are real.
▮

Conclusion For the lopsided quintic vector field (8), theorem 4 shows that the origin is a center if and only if all the coefficients A_{ij} with $i+j = 5$ are real, i.e. only if we have symmetry with respect to a line through the origin.

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