QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **3**, 377–435 (2002)
ARTICLE NO. 46

# Computing Simplest Normal Forms for the Takens–Bogdanov Singularity.

Antonio Algaba

*Departamento de Matemáticas, Universidad de Huelva,*
*21819 La Rábida, Huelva, Spain*
E-mail: algaba@uhu.es

and

Emilio Freire, Estanislao Gamero

*Departamento de Matemática Aplicada II, Universidad de Sevilla,*
*41092 Sevilla, Spain*
E-mail: emilio@ma2.us.es, estanis@ma2.us.es

A computational method to obtain simplest normal forms, which we refer as hypernormal forms, for vector fields having a linear degeneracy corresponding to a double zero eigenvalue with geometric multiplicity one, is presented. The procedure of simplifying the classical normal forms requires some hypothesis on the nonlinear terms of the vector field. Once the hypothesis are established, different hypernormal forms are obtained. Several examples are included in order to show the applicability of our approach in concrete cases.

*Key Words:* Takens–Bogdanov singularity, simplest normal forms.

## 1. INTRODUCTION

The local bifurcation theory of equilibria is based on a set of specific methods that are useful to understand the behavior of a dynamical system or a bifurcation problem. Among these methods, the normal form theory is an important tool, because one can determine restricted classes of vector fields, which can be used to classify the singularity, and also to discuss bifurcations in the corresponding unfolding. The basic idea in this theory is to put the analytic expression of a vector field in a simple and adequate form. This can be done in two ways: by means of coordinate transformations (smooth conjugation) or by changing not only the state variables but also the time (smooth equivalence). Although in this paper we will focus mainly on the use of conjugation, we will present (without proofs) some results showing the improvements in the simplification procedure that can be obtained by using equivalence.

The most cumbersome part in the applications is related to the computation of the normal form coefficients. For this reason, we pay attention to the way this can be done efficiently.

Throughout this paper, we will assume that, after a center manifold reduction if necessary, we deal with a planar autonomous system, with an equilibrium point having a linear degeneracy corresponding to a double zero eigenvalue with geometric multiplicity one. After a translation and a linear transformation, we can assume that the equilibrium point is located at the origin, and the linearization matrix is put in Jordan form.

The underlying idea in the normal form theory is to perform near-identity transformations to remove, in the analytic expression of the vector field, the terms that are unessential in the local dynamical behavior. In the procedure of normalization of the $n$-degree terms, the homological operator (see (2) later) plays an essential role. This linear operator only depends on the linearization matrix, and so, the structure of the normal form is characterized by the linear part of the vector field.

The key in the computation of the $n$-order terms of the normal form is the resolution of the homological equation. In the Takens-Bogdanov singularity, this linear equation has a two-parameter family of solutions. This fact leads to the appearance of pairs of arbitrary constants (parameterizing the kernel of the homological operator) in the expressions of the higher-order normal form coefficients. Selecting these arbitrary constants adequately (for that, we must assume some conditions on the nonlinear terms) one must arrive to simplified normal forms. Our goal is to obtain the simplest ones, which we call hypernormal forms.

A general framework for the approach we follow in this paper (based on Lie transforms) to obtain hypernormal forms, for an arbitrary singularity, can be found in Algaba et al. [1].

The problem of obtaining further simplifications in the Takens-Bogdanov singularity has been addressed by several authors. Ushiki [11] analyze this problem up to fourth-order in the nondegenerate case. In Gamero et al. [7], a higher-order hypernormal form was obtained, including the expressions for the coefficients.

The analysis of the hypernormal form up to an arbitrary order has been carried out by Baider & Sanders [2], in the context of graded Lie algebras. The authors classified three different cases and solved two of them. The remaining case have been addressed by Kokubu et al. [8] and Wang et al. [12]. In this last paper, the simplest normal form is obtained by considering expansions of the vector field in quasi-homogeneous components. In fact, this procedure can be conveniently adapted to the Lie scheme that we adopt in the present paper.

Anyway, as is usually done, we will consider here the expansion of the vector field in homogeneous components, that is, the procedure is done degree by degree.

The afore-mentioned classification of Baider & Sanders [2] into three cases also arises when focusing the problem from the Lie transform perspective. Here, we present results for the three situations. Finally, we remark that we will consider the normal form originally used by Bogdanov, which is the usual one in the study of the Takens-Bogdanov bifurcation (in Baider & Sanders [2], the complementary subspace to the range of the homological operator is the orthogonal subspace with respect to the scalar product defined in Elphick et al. [6]).

It is remarkable that our approach gives different possibilities for the hypernormal form structure (which is of interest when analyzing the topological determinacy of the singularity, and the corresponding bifurcation problems). Moreover, we provide recursive algorithms to obtain the expressions of the coefficients. This fact will be essential to establish the additional simplifications that can be achieved by using $C^\infty$-equivalence.

Another works, which addresses the computational aspects, are Chen & Della Dora [3] that is based in the Carleman linearization procedure, and also Yuan & Yu [13], which uses the conventional method of computing the transformation leading to normal forms.

The present paper is organized as follows. In Section 2, we introduce the notations and state some well-known results concerning with the normal form theory for the Takens-Bogdanov singularity. Also, we outline the procedure we will follow. Some technical results and definitions are presented in Section 3. The main results can be found in Section 4. In the theorems of this section, we assume that some relations hold. Recursive algorithms to verify these relations are presented in Section 5.

In Section 6 we include some particular cases in order to show some concrete hypernormal forms and also to reveal the computational aspect of our approach. Finally, in Section 7, we state without proofs, some results concerning to the additional simplifications when using $C^\infty$-equivalence in two examples included in Section 6. We have also included some appendices where present the proofs and some generalizations of the results reached in Section 4.

## 2. BASIC DEFINITIONS AND PROPERTIES

Let us consider the planar system

$$\begin{aligned} \dot{x} &= y + f(x,y), \\ \dot{y} &= g(x,y), \end{aligned} \tag{1}$$

where $f, g \in C^\infty$ in a neighborhood of the origin in $\mathbb{R}^2$, and they vanish, together their first derivatives, at the origin.

We will denote the vector field of the above system as $v = (y + f(x,y))\partial_x + g(x,y)\partial_y$.

Usually, we deal with Taylor expansions of the vector field. For this reason, we will introduce the following notation: $\mathcal{J}^n v$ denotes the $n$-jet of $v$ (i. e., the $n$-order Taylor polynomial of $v$ at the origin). Likewise, $v_n = \mathcal{J}_n v$ stands for the homogeneous part of degree $n$ of $v$ (that is, $v_n = \mathcal{J}^n v - \mathcal{J}^{n-1} v$). Obviously, $v_1 = y\partial_x$ and $v_n \in \mathcal{H}_n$, the space of planar polynomial homogeneous vector fields of degree $n$, for $n \geq 2$.

The homological operator is determined by the linear part $v_1$, and it is defined by

$$\begin{aligned} L_n \ : \ & \mathcal{H}_n \longmapsto \mathcal{H}_n \\ & U_n \in \mathcal{H}_n \longmapsto L_n(U_n) = [U_n, v_1] \in \mathcal{H}_n, \end{aligned} \tag{2}$$

where $[U, v] = DU \cdot v - Dv \cdot U$ is the Lie product.

Next, we summarize some properties related to the subspaces associated to the homological operator (for details, see Cushman & Sanders [5], Elphick et al. [6]).

- A basis of $\mathcal{R}_n$, the range of $L_n$, is

$$\left\{x^{n-1}y\partial_x, \ldots, y^n\partial_x, -x^n\partial_x + nx^{n-1}y\partial_y, x^{n-2}y^2\partial_y, \ldots, y^n\partial_y\right\}$$

- A basis of $\mathcal{K}_n = \operatorname{Ker} L_n$ is $\left\{f_{n,0} = y^n\partial_x, f_{n,1} = xy^{n-1}\partial_x + y^n\partial_y\right\}$.
- A complementary subspace to $\mathcal{R}_n$, that we denote by $\mathcal{C}_n$, is spanned by $\left\{x^n\partial_y, x^{n-1}y\partial_y\right\}$.

It is well known that, by means of a near-identity transformation $(x, y) \to (x, y) + U_n(x, y)$, with $U_n \in \mathcal{H}_n$ $(n \geq 2)$, it is possible to simplify $v_n$ (annihilating the part belonging to the range of the homological operator), leaving unaltered the terms of order less than $n$ (this fact is the basis of the Normal Form Theorem). To be precise, splitting $\mathcal{H}_n = \mathcal{R}_n \oplus \mathcal{C}_n$, we can write $v_n = v_n^r + v_n^c$ with $v_n^r \in \mathcal{R}_n$ and $v_n^c \in \mathcal{C}_n$. It is enough to choose $U_n$ satisfying the homological equation

$$L_n(U_n) = v_n^r, \tag{3}$$

to achieve that the $n$-order terms in the transformed vector field become $v_n^* = v_n^c \in \mathcal{C}_n$. The solution set of the linear equation (3) is parameterized by arbitrary elements of the kernel of the homological operator. For each $n \geq 2$, this yields the appearance of two arbitrary constants in the expressions of the higher order normal form coefficients. These degrees of freedom can be used, under adequate hypothesis in the nonlinear terms, to annihilate some terms in the normal form, and so we can arrive to simpler normal forms. Our aim is to obtain simplest normal forms, called hypernormal forms.

In our analysis, we will assume, without loss of generality, that we have performed a $C^\infty$–conjugation, leading system (1) to normal form, and then, our hypothesis will be based on the normal form coefficients. In this way, we deal with the vector field

$$v = y\partial_x + \sum_{n\geq 2}(a_n x^n + b_n x^{n-1} y)\partial_y, \tag{4}$$

and consequently, $v_n = (a_n x^n + b_n x^{n-1} y)\partial_y$ for $n \geq 2$.

Next, we describe the basic ideas of the approach we follow in this paper to obtain the hypernormal forms for the vector field (4). The key is to perform the transformations in a way well adapted to computations. Let us consider a near-identity transformation:

$$(x, y) = \varphi(\tilde{x}, \tilde{y}), \tag{5}$$

corresponding to a generator $U$. This means that the change is the time-one flow of the autonomous system corresponding to the vector field $U$. That is, $\varphi(\tilde{x}, \tilde{y}) = u(\tilde{x}, \tilde{y}, 1)$; where $u(\tilde{x}, \tilde{y}, \varepsilon)$ is the unique solution of the following initial value problem:

$$\frac{\partial}{\partial \varepsilon} u(\tilde{x}, \tilde{y}, \varepsilon) = U(u(\tilde{x}, \tilde{y}, \varepsilon)), \qquad u(\tilde{x}, \tilde{y}, 0) = (\tilde{x}, \tilde{y}). \tag{6}$$

There is a correspondence between changes and generators, so that any change has a generator associated and vice-versa. For details, we refer to Algaba et al. [1].

The transformation (5) carries over the vector field (4) into

$$v^* = v + [v, U] + \tfrac{1}{2!}[[v, U], U] + \tfrac{1}{3!}[[[v, U], U], U] + \cdots = v + \sum_{n\geq 1}\frac{1}{n!}T_U^n(v),$$

where $T_U(v) = [v, U]$ and $T_U^n(v) = T_U \circ \cdots \circ T_U(v)$.

The above expressions show how the generator $U$ can be used to accomplish a change of variables, without using the expressions of this change. Also, it is possible to use a recursive procedure that provides the change from the generator (see Chow & Hale [4]), but usually one seek for an adequate canonical expression for the original system and do not worry about the change leading to such canonical expression.

There are some properties of the Lie product which can be easily derived. So, taking $v_k \in \mathcal{H}_k$ and $U_j \in \mathcal{H}_j$ we have $[v_k, U_j] \in \mathcal{H}_{k+j-1}$.

We will consider a transformation whose generator $U = \sum_{n\geq 1} U_n$ satisfies:

$$\mathrm{Proj}_{\mathcal{R}_n}\left(\sum_{j\geq 1}\frac{1}{j!}T_U^j(v)\right) = 0, \text{ for all } n \geq 2. \tag{7}$$

In this way, the transformed vector field $v^*$ is also put in normal form. Moreover, such a generator $U$ will be selected so that $v^*$ has the maximum number of vanishing coefficients.

The structure of $U$, degree by degree, is as follows.

*(1)* First, we impose that the linear parts of $v^*$ and $v$ agree. This condition leads to the relation $[v_1, U_1] = 0$, and then $U_1(x,y) = A_{1,0}f_{1,0} + A_{1,1}f_{1,1} \in \mathcal{K}_1$, where $A_{1,1}, A_{1,0} \in \mathbb{R}$ are arbitrary. In Lemmas 20, 21 we will show that these arbitrary constants do not permit to annihilate terms in $v^*$. So, we will take the generator with zero linear part: $U_1 = 0$.

*(2)* Taking $n = 2$ in (7), we get that the second-order terms of $U$ satisfies $[v_1, U_2] = 0$. Hence, $U_2(x,y) = A_{2,0}f_{2,0} + A_{2,1}f_{2,1} \in \mathcal{K}_2$.

*(3)* For $n = 3$, the third-order terms of $U$ will be taken such that

$$[v_1, U_3] + \mathrm{Proj}_{\mathcal{R}_3}([v_2, U_2]) = 0 \iff L_3(U_3) = \mathrm{Proj}_{\mathcal{R}_3}([v_2, U_2]).$$

Then,

$$U_3 = A_{3,0}f_{3,0} + A_{3,1}f_{3,1} + U_3^L,$$

where $U_3^L$ depends linearly on $A_{2,0}$, $A_{2,1}$.

*(4)* In the remaining cases $n \geq 4$, the $n$-degree of $U$ satisfies

$$[v_1, U_n] + \mathrm{Proj}_{\mathcal{R}_n}\left([v_2, U_{n-1}] + \cdots + [v_{n-1}, U_2]\frac{1}{2!}[[v, U], U] + \cdots\right) = 0.$$

Then,

$$U_n = A_{n,0}f_{n,0} + A_{n,1}f_{n,1} + U_n^L + U_n^{NL},$$

where $U_n^L$ depends linearly on $A_{2,0}, A_{2,1}, \ldots, A_{n-1,0}, A_{n-1,1}$ and $U_n^{NL}$ depends nonlinearly on these constants.

The above definition can be extended to $U_2$ and $U_3$, by taking $U_2^L = U_2^{NL} = U_3^{NL} = 0$. For $n \geq 3$, $U_n^L$ and $U_n^{NL}$ will be selected as follows:

- $U_n^L$ is a particular solution of

$$L_n\left(U_n^L\right) = \mathrm{Proj}_{\mathcal{R}_n}\left([v_2, U_{n-1}^L] + \cdots + [v_{n-1}, U_2^L]\right). \tag{8}$$

- $U_n^{NL}$ is a particular solution of

$$L_n\left(U_n^{NL}\right) = \mathrm{Proj}_{\mathcal{R}_n}\left([v_2, U_{n-1}^{NL}] + \cdots + [v_{n-1}, U_2^{NL}]\sum_{j\geq 2}\frac{1}{j!}T_U^j(v)\right). \tag{9}$$

Once we have obtained the generator $U$ from (7), as a function of $A_{n,0}$, $A_{n,1}$ ($n \geq 2$), we must analyze how these arbitrary constants appear in $v^*$ and describe how much this can be simplified.

We will proceed recursively performing transformations, each of one depending on one arbitrary constant. In this way, we will see how each arbitrary constant appear $v^*$, and determine which term can be annihilated selecting adequately the arbitrary constant.

We fix $h \geq 2$, $i = 0, 1$, and consider the generator satisfying (7) that depends only on $A_{h,i}$ (i. e., we annihilate the remaining arbitrary constants). This generator is $U = U_h + U_{h+1} + \cdots$, where $U_h = A_{h,i} f_{h,i}$ and $U_n = U_n^L + U_n^{NL}$ for all $n \geq h+1$.

As $\mathcal{J}^h[v, U] = 0$, there exists $\lambda \geq h$ such that

$$\begin{aligned}&\mathrm{Proj}_{\mathcal{C}_k}[v, U] = [v, U]_k = 0 \quad \text{for } k = 2, \ldots, \lambda, \quad \text{and}\\ &[v, U]_{\lambda+1} = \mathrm{Proj}_{\mathcal{C}_{\lambda+1}}[v, U] \neq 0.\end{aligned}$$

Moreover, it can be proved that $\mathcal{J}^{\lambda+(j-1)(h-1)+1} T_U^j(v) = 0$ for $j \geq 3$. Then, we get $U_h = U_h^L, \ldots, U_{h+\lambda-1} = U_{h+\lambda-1}^L$. Moreover, there exists $\delta \geq h + \lambda$ such that

$$[v_2, U_{k-1}^{NL}] + \cdots + [v_{k-h}, U_{h+1}^{NL}] + \mathcal{J}_k \left\{ \sum_{j \geq 2} \frac{1}{j!} T_U^j(v) \right\} = 0,$$

for $k = h+1, \ldots, \delta - 1$.

This means that the arbitrary constant appears linearly in the terms of order less than $\delta$ of the generator. Also, there exists $\mu \geq \delta$ such that

$$\mathrm{Proj}_{\mathcal{C}_k} \left\{ [v_2, U_{k-1}^{NL}] + \cdots + [v_{k-h}, U_{h+1}^{NL}] + \sum_{j \geq 2} \frac{1}{j!} T_U^j(v) \right\} = 0,$$

for $k = h+1, \ldots, \mu$. Then:

$$v^* = v_1 + \cdots + v_\lambda + (v_{\lambda+1} + \widetilde{v}_{\lambda+1}) + \cdots + (v_\mu + \widetilde{v}_\mu) + v_{\mu+1}^* + v_{\mu+2}^* + \cdots,$$

where

$$\widetilde{v}_k = \mathrm{Proj}_{\mathcal{C}_k} \left\{ [v_2, U_{k-1}^L] + \cdots + [v_{k-h+1}, U_h^L] \right\}, \tag{10}$$

for $k = \lambda + 1, \ldots, \mu$. Moreover, the following relations must be fulfilled

$$\mathrm{Proj}_{\mathcal{R}_k} \left\{ [v_1, U_k^L] + [v_2, U_{k-1}^L] + \cdots + [v_{k-h+1}, U_h^L] \right\} = 0, \tag{11}$$

for $k = h+1, \ldots, \mu$. Moreover, $A_{h,i}$ can appear nonlinearly in the terms of $v^*$ of order greater than $\mu + 1$. Obviously, in our analysis, we will focus on the linear appearance of these arbitrary constants.

## 3. TECHNICAL RESULTS

In this section we include some definitions and results we will use in the next section to derive the hypernormal forms.

Recall that our approach is the following: for each $A_{h,i} \in \mathbb{R}$, ($h \geq 2$, $i = 0, 1$), we must determine $\lambda$, $\delta$, $\mu$; solve the equation (11) and then compute $\widetilde{v}_k$ from (10).

As we will need to manage normal form coefficients, we define the projection operators $\Pi_1,\ \Pi_2\ :\ \mathcal{C}_n \longmapsto \mathbb{R}$, by:

$$\Pi_1\ \left((Ax^n + Bx^{n-1}y)\partial_y\right) = A, \quad \Pi_2\ \left((Ax^n + Bx^{n-1}y)\partial_y\right) = B,$$

that allows to extract each normal form coefficient.

To solve (11), we must deal with the Lie product for vector fields $v_n = (a_n x^n + b_n x^{n-1} y)\partial_y$. For that, we define the vector subspaces $\mathcal{H}_{k;j}$ ($k \in \mathbb{N}$, $j \in \mathbb{Z}$) and the operators $L(n,j)$, $N(n,j)$ ($n \in \mathbb{N}$, $j \in \mathbb{Z}$) (see Lemma 4):

- $\mathcal{H}_{k;j} = \left\{Ax^{k-j}y^j\partial_x + Bx^{k-j-1}y^{j+1}\partial_y : A, B \in \mathbb{R}\right\}$ for $j = 0, \ldots, k-1$.
- $\mathcal{H}_{k;k} = \left\{Ay^k\partial_x : A \in \mathbb{R}\right\}$.
- $\mathcal{H}_{k;-1} = \left\{Ax^k\partial_y : A \in \mathbb{R}\right\}$.
- $\mathcal{H}_{k;j} = \{0\}$, for $j < -1$, $j \geq k+1$.

Observe that $\mathcal{H}_k = \bigoplus_{j=-1}^{k} \mathcal{H}_{k;j} = \bigoplus_{j=-\infty}^{\infty} \mathcal{H}_{k,j}$. Also:

- $\mathcal{H}_{k;j} \subset \mathcal{R}_k$, for $j = 1, \ldots, k$;
- $\mathcal{H}_{k;-1} \subset \mathcal{C}_k$,
- $\mathcal{K}_k \subset \mathcal{H}_{k;k-1} \bigoplus \mathcal{H}_{k;k}$.

Usually, we will write the elements of $\mathcal{H}_{k;j}$ as $f_{k;k-j}$. Alternatively, we will use the notation $f_{k;j}$ for the elements belonging to $\mathcal{H}_{k;k-j}$.

Taking canonical basis, we can represent the subspaces $\mathcal{H}_{k;j}$ as:

$$\mathcal{H}_{k;j} = \left\{\begin{pmatrix} A \\ B \end{pmatrix} : A, B \in \mathbb{R}\right\}, \quad \text{for} \quad j = 0, \ldots, k-1.$$
$$\mathcal{H}_{k;k} = \left\{\begin{pmatrix} A \\ 0 \end{pmatrix} : A \in \mathbb{R}\right\}.$$
$$\mathcal{H}_{k;-1} = \left\{\begin{pmatrix} 0 \\ A \end{pmatrix} : A \in \mathbb{R}\right\}.$$
$$\mathcal{H}_{k;j} = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}, \quad \text{for} \quad j \leq -2, \quad \text{or} \quad j \geq k+1.$$

For instance, the elements of the basis of $\mathcal{K}_k = \operatorname{Ker} L_k$ are $f_{k,0} = y^k\partial_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathcal{H}_{k;k}$ and also $f_{k,1} = xy^{k-1}\partial_x + y^k\partial_y = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathcal{H}_{k;k-1}$.

The linear operators $L$, $N$ arise when manipulating Lie products involving normal form terms. They depend upon the normal form coefficients, and are defined by:

• $L(n,j) : \mathcal{H}_{k;j} \longmapsto \mathcal{H}_{k+n;j}$

$$f_{k;k-j} = \begin{pmatrix} A \\ B \end{pmatrix} \longmapsto b_{n+1} \begin{pmatrix} -j & 0 \\ n & -j \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

• $N(n,j) : \mathcal{H}_{k;j} \longmapsto \mathcal{H}_{k+n;j-1}$

$$f_{k;k-j} = \begin{pmatrix} A \\ B \end{pmatrix} \longmapsto a_{n+1} \begin{pmatrix} -j & 0 \\ n+1 & -j-1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

Next, we define another operator $M(k-j)$, we will use to solve the homological equation:

• If $k-j \neq 0, -1$, then $M(k-j) : \mathcal{H}_{k;j} \bigcap \mathcal{R}_k \longmapsto \mathcal{H}_{k;j-1}$ is defined by

$$f_{k;k-j} = \begin{pmatrix} A \\ B \end{pmatrix} \longmapsto \frac{1}{(k-j)(k-j+1)} \begin{pmatrix} k-j & 1 \\ 0 & k-j+1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

• If $k-j=0$, then $M(0) : \mathcal{H}_{k;k} \longmapsto \mathcal{H}_{k;k-1}$ is defined by $\begin{pmatrix} A \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ -A \end{pmatrix}$.

• If $k-j=-1$, then the definition for $M(-1)$ is obvious, because the domain is the zero subspace.

It is easy to prove that the above operators are well defined.

Definition 1. Let us consider $l > 0$. A $l$-index is an element $\mathbf{n} = (n_1, \dots, n_l) \in \mathbb{N}^l$ (here, $\mathbb{N}$ is the set of natural numbers not including zero. Through this paper, $\mathbb{N}_0$ will denote the set of natural numbers including zero). The set of $l$-indices is denoted by $\mathcal{I}_l$. The modulus of a $l$-index is $|\mathbf{n}| = n_1 + \cdots + n_l$.

Given a $l$-index $\mathbf{n} \in \mathcal{I}_l$, we say that $\mathbf{n}' \in \mathcal{I}_l$ is associated to $\mathbf{n}$, and we write $\mathbf{n}' \sim \mathbf{n}$, if $n'_j - n_j = 1$, or $n'_j - n_j = 2$, for all $j = 1, \dots, l$.

The elements appearing in the next definitions are useful to solve explicitly the equation (11) (see Theorem 12). Also, we use them to obtain $\widetilde{v}_k$, described in (10) (see Theorem 13). These elements, which depend linearly on $A_{h,i}$, determine the terms in $v^*$ which can be annihilated by selecting the arbitrary constant $A_{h,i} \in \mathbb{R}$ adequately.

Definition 2. Let us consider $f_{h;i} \in \mathcal{H}_{h;h-i}$ and $\mathbf{n}, \mathbf{n}' \in \mathcal{I}_l$ with $\mathbf{n}' \sim \mathbf{n}$. We define $f_{h,\mathbf{n};i,\mathbf{n}'} \in \mathcal{H}_{h+|\mathbf{n}|;h-i+|\mathbf{n}|-|\mathbf{n}'|}$ by induction in $l$:

For $l=1$,

$$f_{h,n_1;i,n'_1} = 0, \text{ if } h-i+n_1-n'_1 \leq -2,$$
$$f_{h,n_1;i,n'_1} = M(i+n'_1-1) \circ \mathrm{Proj}_{\mathcal{R}_{h+n_1}}(H_{n_1} \circ f_{h;i}), \text{ if } h-i+n_1-n'_1 \geq -1,$$

where

$$H_{n_1} = \begin{cases} L(n_1, h-i), & \text{whenever } n_1' - n_1 = 1, \\ N(n_1, h-i), & \text{whenever } n_1' - n_1 = 2. \end{cases} \tag{12}$$

Assume that we have already defined $f_{h,\mathbf{n};i,\mathbf{n}'}$ for any $\mathbf{n} = (n_1, \ldots, n_l) \in \mathcal{I}_l$, $\mathbf{n}' = (n_1', \ldots, n_l') \in \mathcal{I}_l$, with $\mathbf{n}' \sim \mathbf{n}$. Consider $\tilde{\mathbf{n}} = (\mathbf{n}, n_{l+1}) \in \mathcal{I}_{l+1}$, $\tilde{\mathbf{n}}' = (\mathbf{n}', n_{l+1}') \in \mathcal{I}_{l+1}$ such that $\tilde{\mathbf{n}}' \sim \tilde{\mathbf{n}}$. We define

$f_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}'} = 0$, if $h - i + |\tilde{\mathbf{n}}| - |\tilde{\mathbf{n}}'| \leq -2$,

$f_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}'} = M(i + |\tilde{\mathbf{n}}'| - 1) \circ \mathrm{Proj}_{\mathcal{R}_{h+|\tilde{\mathbf{n}}|}}(H_{n_{l+1}} \circ f_{h,\mathbf{n};i,\mathbf{n}'})$, if $h - i + |\tilde{\mathbf{n}}| - |\tilde{\mathbf{n}}'| \geq -1$,

where

$$H_{n_{l+1}} = \begin{cases} L(n_{l+1}, h-i+|\mathbf{n}|-|\mathbf{n}'|), & \text{whenever } n_{l+1}' - n_{l+1} = 1, \\ N(n_{l+1}, h-i+|\mathbf{n}|-|\mathbf{n}'|), & \text{whenever } n_{l+1}' - n_{l+1} = 2. \end{cases} \tag{13}$$

Definition 3. Let us consider $f_{h;i} \in \mathcal{H}_{h;h-i}$ and $\mathbf{n}, \mathbf{n}' \in \mathcal{I}_l$ with $\mathbf{n}' \sim \mathbf{n}$. We define $\widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} \in \mathcal{H}_{h+|\mathbf{n}|;h-i+|\mathbf{n}|-|\mathbf{n}'|+1}$ by induction in $l$ as follows:

For $l = 1$:

$\widetilde{f}_{h,n_1;i,n_1'} = 0$, if $h - i + n_1 - n_1' \leq -3$,

$\widetilde{f}_{h,n_1;i,n_1'} = H_{n_1} \circ f_{h;i}$, if $h - i + n_1 - n_1' \geq -2$, where $H_{n_1}$ is given in (12).

Assume that we have already defined $\widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'}$ for any $\mathbf{n} = (n_1, \ldots, n_l) \in \mathcal{I}_l$, $\mathbf{n}' = (n_1', \ldots, n_l') \in \mathcal{I}_l$, with $\mathbf{n}' \sim \mathbf{n}$. Consider $\tilde{\mathbf{n}} = (\mathbf{n}, n_{l+1}) \in \mathcal{I}_{l+1}$, $\tilde{\mathbf{n}}' = (\mathbf{n}', n_{l+1}') \in \mathcal{I}_{l+1}$ such that $\tilde{\mathbf{n}}' \sim \tilde{\mathbf{n}}$. We define

$\widetilde{f}_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}'} = 0$, if $h - i + |\tilde{\mathbf{n}}| - |\tilde{\mathbf{n}}'| \leq -3$,

$\widetilde{f}_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}'} = H_{n_{l+1}} \circ M(i + |\mathbf{n}'| - 1) \circ \mathrm{Proj}_{\mathcal{R}_{h+|\mathbf{n}|}} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'}$, if $h - i + |\tilde{\mathbf{n}}| - |\tilde{\mathbf{n}}'| \geq -2$, where $H_{n_{l+1}}$ is given in (13).

It is easy to show that the elements $f_{h,\mathbf{n};i,\mathbf{n}'}$, $\widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'}$ are well defined. The following lemmas provide some properties of subspaces $\mathcal{H}_{k;j}$ and the above-defined operators.

Lemma 4.

*(1)*$[\mathcal{H}_{m;r}, \mathcal{H}_{k;j}] \subseteq \mathcal{H}_{m+k-1;r+j}$.

*(2)Consider* $f_{k;j} \in \mathcal{H}_{k;k-j}$*. Then:*

$$[v_n, f_{k;j}] = \widetilde{f}_{k,n-1;j,n} + \widetilde{f}_{k,n-1;j,n+1} \in \mathcal{H}_{n+k-1;k-j} \bigoplus \mathcal{H}_{n+k-1;k-j-1}.$$

*(3) (a)Consider* $f_{k;k-j} \in \mathcal{H}_{k;j} \subset \mathcal{R}_k$ *(with* $j \neq 0, -1$*). Then,* $U_k = M(k-j) \circ f_{k;k-j}$ *is a solution of* $L_k(U_k) = f_{k;k-j}$*.*

*(b)Consider* $f_{k;k} = Ax^k\partial_x + Bx^{k-1}y\partial_y \in \mathcal{H}_{k;0} \bigcap \mathcal{R}_k$ *(i. e.,* $B = -kA$*). Then,* $U_k = M(k) \circ f_{k;k}$ *is a solution of* $L_k(U_k) = f_{k;k}$*.*

*(4)Consider* $f_{k;k} = Ax^k\partial_x + Bx^{k-1}y\partial_y \in \mathcal{H}_{k;0}$*. Then,* $\mathrm{Proj}_{\mathcal{C}_k} f_{k;k} = (B + kA)x^{k-1}y\partial_y$*.*

*(5)Consider* $\mathbf{n}, \mathbf{n}' \in \mathcal{I}_l$ *with* $\mathbf{n}' \sim \mathbf{n}$*.*

*(a)Let* $\tilde{\mathbf{n}} = (\mathbf{n}, m) \in \mathcal{I}_{l+1}$*,* $\tilde{\mathbf{n}}' = (\mathbf{n}', m+1) \sim \tilde{\mathbf{n}}$*. Then:* $\widetilde{f}_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}'} = L(m, h-i+|\mathbf{n}|-|\mathbf{n}'|) \circ f_{h,\mathbf{n};i,\mathbf{n}'}$*.*

*(b)Let* $\tilde{\mathbf{n}} = (\mathbf{n}, m) \in \mathcal{I}_{l+1}$*,* $\tilde{\mathbf{n}}' = (\mathbf{n}', m+2) \sim \tilde{\mathbf{n}}$*. Then:* $\widetilde{f}_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}'} = N(m, h-i+|\mathbf{n}|-|\mathbf{n}'|) \circ f_{h,\mathbf{n};i,\mathbf{n}'}$*.*

*(c)*$f_{h,\mathbf{n};i,\mathbf{n}'} = M(i+|\mathbf{n}'|-1) \circ \mathrm{Proj}_{\mathcal{R}_{h+|\mathbf{n}|}} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'}$*.*

*Proof.* **(1)** We will consider here the case $0 \leq r+j < m+k-1$.

Denote $f_{m;m-r} = Ax^{m-r}y^r\partial_x + Bx^{m-r-1}y^{m+1}\partial_y \in \mathcal{H}_{m;r}$, $f_{k;k-j} = Cx^{k-j}y^j\partial_x + Dx^{k-j-1}y^{j+1}\partial_y \in \mathcal{H}_{k;j}$. After some computations, we get

$$\begin{aligned}[f_{m;m-r}, f_{k;k-j}] = &((m-r-k+j)AC + rAD - jBC)\, x^{m+k-r-j-1}y^{r+j}\partial_x \\ &+ ((m-r-1)BC + (r+j)BD - (k-j-1)AD)\, x^{m+k-r-j-2}y^{r+j+1}\partial_y,\end{aligned} \tag{14}$$

which belongs to $\mathcal{H}_{m+k-1;r+j}$. The remaining cases: $r+j < -1$, $r+j = -1$, $r+j = m+k-1$, $r+j > m+k-1$ are analogous.

**(2)** As in the above item, we will consider here only one case: $0 < k-j-1 < n+k-1$ (the remaining ones are analogous).

Let $f_{k;j} = Ax^jy^{k-j}\partial_x + Bx^{j-1}y^{k-j+1}\partial_y \in \mathcal{H}_{k;k-j}$. Then:

$$\begin{aligned}&[v_n, f_{k;j}] = -(k-j)a_nAx^{n+j}y^{k-j-1}\partial_x \\ &+ (na_nA - (k-j+1)a_nB)\, x^{n+j-1}y^{k-j}\partial_y - \\ &-(k-j)b_nAx^{n+j-1}y^{k-j}\partial_x + ((n-1)b_nA - (k-j)b_nB)\, x^{n+j-2}y^{k-j+1}\partial_y,\end{aligned}$$

which belongs to $\mathcal{H}_{n+k-1;k-j-1} \bigoplus \mathcal{H}_{n+k-1;k-j}$. Using coordinates in $\mathcal{H}_{k;k-j}$, $\mathcal{H}_{n+k-1;k-j-1}$, $\mathcal{H}_{n+k-1;k-j}$, and writing

$$\begin{aligned}&\begin{pmatrix} -(k-j)a_nA \\ na_nA - (k-j-1)a_nB \end{pmatrix} = a_n \begin{pmatrix} -(k-j) & 0 \\ n & -(k-j+1) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \\ &= N(n-1, k-j) \circ f_{k;j} = \widetilde{f}_{k,n-1;j,n+1},\end{aligned}$$

$$\begin{pmatrix} -(k-j)b_nA \\ (n-1)b_nA-(k-j)b_nB \end{pmatrix} = b_n \begin{pmatrix} -(k-j) & 0 \\ n-1 & -(k-j) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \\ = L(n-1,k-j)\circ f_{k;j} = \widetilde{f}_{k,n-1;j,n},$$

we obtain the result.

**(3) (a)** It is enough to see that $L_k(M(k-j)\circ f_{k;k-j}) = f_{k;k-j}$. We will consider here the case $1 \le j < k$. The remaining situations ($j < -1, j = k, j > k$) are similar. Let $f_{k;k-j} = \begin{pmatrix} A \\ B \end{pmatrix} \in \mathcal{H}_{k;j} \subset \mathcal{R}_k$. Then:

$$M(k-j)\circ f_{k;k-j} = \begin{pmatrix} \frac{1}{k-j+1}\left(A+\frac{1}{k-j}B\right) \\ \frac{1}{k-j}B \end{pmatrix} \in \mathcal{H}_{k;j-1}.$$

It is a straightforward computation to obtain

$$L_k(M(k-j)\circ f_{k;k-j}) = \\ = \left[\frac{1}{k-j+1}\left(A+\frac{1}{k-j}B\right)x^{k-j+1}y^{j-1}\partial_x + \frac{1}{k-j}Bx^{k-j}y^j\partial_y, y\partial_x\right] = \\ = Ax^jy^{k-j}\partial_x + Bx^{j-1}y^{k-j+1}\partial_y = f_{k;k-j}.$$

The proof of item (b) is analogous.

**(4)** It is enough to use

$$\mathrm{Proj}_{\mathcal{C}_k}(\left(A_1x^k + A_2x^{k-1}y + \cdots + A_{k+1}y^k\right)\partial_x + \\ + \left(B_1x^k + B_2x^{k-1}y + \cdots + B_{k+1}y^k\right)\partial_y) = \\ \left(B_1x^k + (B_2+kA_1)x^{k-1}y\right)\partial_y.$$

**(5)** This item follows directly from Definitions 2, 3. ▮

The proofs of the next lemmas are based on the use of induction and some manipulations, and they are omitted for the sake of brevity.

Lemma 5. *Let* $\mathbf{n} = (n_1,\ldots,n_l) \in \mathcal{I}_l$ *with* $l \ge 2$. *Consider* $j$ *such that* $1 \le j < l$, *and define* $\tilde{\mathbf{n}} = (n_1,\ldots,n_j) \in \mathcal{I}_j$, $\tilde{\mathbf{n}}' = (n_1+2,\ldots,n_j+2) \sim \tilde{\mathbf{n}}$. *Then:*

$$f_{2l-1,\tilde{\mathbf{n}};0,\tilde{\mathbf{n}}'} = c_j \begin{pmatrix} 2(j-l) \\ |\tilde{\mathbf{n}}|+2j \end{pmatrix} \in \mathcal{H}_{2l-1+|\tilde{\mathbf{n}}|;2l-2j-1},$$

*where* $f_{2l-1;0} = y^{2l-1}\partial_x \in \mathcal{H}_{2l-1;2l-1}$, $c_1 = -\frac{a_{n_1+1}}{n_1+2}$, $c_j = -\frac{2(j-l-1)a_{n_j+1}}{|\tilde{\mathbf{n}}|+2j}c_{j-1}$ *for* $j \ge 2$.

Using this lemma with $j = l-1$, we can prove:

Lemma 6. *Let* $\mathbf{n} \in \mathcal{I}_l$ *with* $l \geq 1$, *and define* $\mathbf{n}' = (n_1 + 2, \ldots, n_l + 2) \sim \mathbf{n}$. *Then:*

$$\widetilde{f}_{2l-1,\mathbf{n};0,\mathbf{n}'} = c_{l-1} a_{n_l+1} \begin{pmatrix} 2 \\ -2(|\mathbf{n}| + 2l - 1) \end{pmatrix} \in \mathcal{H}_{2l-1+|\mathbf{n}|;0},$$

*where* $f_{2l-1;0} = y^{2l-1}\partial_x \in \mathcal{H}_{2l-1;2l-1}$.

*In particular,* $\mathrm{Proj}_{\mathcal{C}_{2l-1+|\mathbf{n}|}} \widetilde{f}_{2l-1,\mathbf{n};0,\mathbf{n}'} = 0$.

Lemma 7. *Let* $\mathbf{n} = (n_1, \ldots, n_l) \in \mathcal{I}_l$ *with* $l \geq 2$. *Consider* $j$ *such that* $1 \leq j < l$, *and define* $\tilde{\mathbf{n}} = (n_1, \ldots, n_j) \in \mathcal{I}_j$, $\tilde{\mathbf{n}}' = (n_1 + 1, \ldots, n_j + 1) \sim \tilde{\mathbf{n}}$. *Then:*

$$f_{l-1,\tilde{\mathbf{n}};0,\tilde{\mathbf{n}}'} = d_j \begin{pmatrix} j - l + 1 \\ |\tilde{\mathbf{n}}| + j \end{pmatrix} \in \mathcal{H}_{l+|\tilde{\mathbf{n}}|-1;l-j-1},$$

*where* $f_{l-1;0} = y^{l-1}\partial_x \in \mathcal{H}_{l-1;l-1}$, $d_1 = -\frac{b_{n_1+1}}{n_1+1}$ *and* $d_j = -\frac{(j-l)b_{n_j+1}}{|\tilde{\mathbf{n}}|+j} d_{j-1}$ *for* $j \geq 2$.

Using the above lemma with $j = l - 1$, we obtain:

Lemma 8. *Let* $\mathbf{n} \in \mathcal{I}_l$ *with* $l \geq 2$, *and define* $\mathbf{n}' = (n_1 + 1, \ldots, n_l + 1) \sim \mathbf{n}$. *Then:*

$$\widetilde{f}_{l-1,\mathbf{n};0,\mathbf{n}'} = 0 \in \mathcal{H}_{l+|\mathbf{n}|-1;0},$$

*where* $f_{l-1;0} = y^{l-1}\partial_x \in \mathcal{H}_{l-1;l-1}$.

Lemma 9. *Let* $\mathbf{n} \in \mathcal{I}_l$ *with* $l \geq 2$. *Define the following pairs of* $l$*-indices:*

$$\begin{aligned}
&\mathbf{n}^{(1)} = (n_1, n_2, \ldots, n_{l-1}, n_l) = \mathbf{n}, \\
&\mathbf{n}^{(1)'} = (n_1 + 2, n_2 + 1, \ldots, n_{l-1} + 1, n_l + 1) \sim \mathbf{n}^{(1)}, \\
&\mathbf{n}^{(2)} = (n_2, n_1, \ldots, n_{l-1}, n_l), \\
&\mathbf{n}^{(2)'} = (n_2 + 1, n_1 + 2, \ldots, n_{l-1} + 1, n_l + 1) \sim \mathbf{n}^{(2)}, \\
&\vdots \\
&\mathbf{n}^{(l-1)} = (n_2, n_3, \ldots, n_1, n_l), \\
&\mathbf{n}^{(l-1)'} = (n_2 + 1, n_3 + 1, \ldots, n_1 + 2, n_l + 1) \sim \mathbf{n}^{(l-1)}, \\
&\mathbf{n}^{(l)} = (n_2, n_3, \ldots, n_l, n_1), \\
&\mathbf{n}^{(l)'} = (n_2 + 1, n_3 + 1, \ldots, n_l + 1, n_1 + 2) \sim \mathbf{n}^{(l)}.
\end{aligned}$$

*Then*

$$\sum_{j=1}^{l} \widetilde{f}_{l-1,\mathbf{n}^{(j)};0,\mathbf{n}^{(j)'}} = 0 \in \mathcal{H}_{l+|\mathbf{n}|-1;-1}.$$

Using this lemma, we can prove the next one:

LEMMA 10. *Let* $\mathbf{n} \in \mathcal{I}_l$ *with* $l \geq 2$, *and define* $\mathbf{n}' = (n_1{+}2, n_2{+}1, \ldots, n_l{+}1) \sim \mathbf{n}$. *Then:*

$$\sum{}'_{\sigma \in \varsigma\{1,\ldots,l\}} \widetilde{f}_{l-1,\sigma(\mathbf{n});0,\sigma(\mathbf{n}')} = 0 \in \mathcal{H}_{l+|\mathbf{n}|-1;-1},$$

*where* $\varsigma\{1,\ldots,l\}$ *is the permutation group of elements* $\{1,\ldots,l\}$; *and we have denoted* $\sigma(\mathbf{n}) = \left(n_{\sigma(1)}, \ldots, n_{\sigma(l)}\right) \in \mathcal{I}_l$; *and* $f_{l-1;0} = y^{l-1}\partial_x \in \mathcal{H}_{l-1;l-1}$. *The symbol* $\sum'$ *denotes that only the summands* $\widetilde{f}_{l-1,\sigma(\mathbf{n});0,\sigma(\mathbf{n}')}$ *that are different must be considered (observe that two different permutations should define the same element).*

Another property, we will use later, is the following:

LEMMA 11. *Let* $\mathbf{n} \in \mathcal{I}_l$ *with* $l \geq 2$, *and define* $\mathbf{n}' = (n'_1, \ldots, n'_{l-2}, n_{l-1}+1, n_l+1) \sim \mathbf{n}$ *(i. e.,* $n'_j - n_j = 1, 2$ *for* $j = 1, \ldots, l-2$*). Consider* $h \in \mathbb{N}$, $i \in \mathbb{Z}$ *such that* $h - i + |\mathbf{n}| - |\mathbf{n}'| = -2$. *Then:*

$$\widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} = 0 \in \mathcal{H}_{h+|\mathbf{n}|;-1}.$$

The next question we address in this section is the computation of the generators in $\bigoplus_{j=1}^{n} \mathcal{H}_j$ which leave the vector field in normal form up to order $n$. The relations $\mathrm{Proj}_{\mathcal{R}_j} \mathcal{J}^n[v,U] = 0$ $(j = 1,\ldots,n)$ must hold. Theorem 12 provides the solution of these equations. Later, in Theorem 13, we will compute $\mathrm{Proj}_{\mathcal{C}_j} \mathcal{J}^n[v,U]$ $(j = 1,\ldots,n)$, in order to analyze how is altered the normal form up to order $n$.

THEOREM 12. *Let us consider the vector field* $v$ *given in (4). Then, the general solution of the PDE*

$$\mathrm{Proj}_{\bigoplus_{k=1}^{n} \mathcal{R}_k}([v,U]) = 0, \tag{15}$$

*(see (11)) is* $U = U_1 + U_2 + \cdots + U_n$, *where*

$$\begin{aligned} U_k \;=\; & A_{k,0} f_{k;0} + A_{k,1} f_{k;1} + \sum_{\substack{h=1,\ldots,k-1 \\ i=0,1}} A_{h,i} \left\{ \sum_{\substack{l>0,\mathbf{n}\in\mathcal{I}_l,\mathbf{n}'\sim\mathbf{n} \\ h+|\mathbf{n}|=k,\, i+|\mathbf{n}'|<k+1}} f_{h,\mathbf{n};i,\mathbf{n}'} + \right. \\ & \left. + \sum_{\substack{l>0,\mathbf{n}\in\mathcal{I}_l,\mathbf{n}'\sim\mathbf{n} \\ h+|\mathbf{n}|=k,\, i+|\mathbf{n}'|=k+1}} M(k) \circ \mathrm{Proj}_{\mathcal{R}_k} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} \right\}, \end{aligned} \tag{16}$$

*for* $k = 1,\ldots,n$ *where* $f_{h;1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathcal{H}_{h;h-1} \subset \mathcal{K}_h$, $f_{h;0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathcal{H}_{h;h} \subset \mathcal{K}_h$, *and* $A_{h,i} \in \mathbb{R}$, *for* $h = 1,\ldots,k$, $i = 0, 1$.

*Proof.* Equation (15) is equivalent to

$$\begin{aligned}
&[v_1, U_1] = 0,\\
&[v_1, U_2] + \mathrm{Proj}_{\mathcal{R}_2}[v_2, U_1] = 0, \qquad (17)\\
&\quad \cdots\\
&[v_1, U_n] + \mathrm{Proj}_{\mathcal{R}_n}\left([v_2, U_{n-1}] + \cdots + [v_n, U_1]\right) = 0.
\end{aligned}$$

We proceed by induction on $n$.

• $n = 1$. In this case, there is only one equation: $[v_1, U_1] = 0$. Its general solution is

$$U_1 = A_{1,0} f_{1;0} + A_{1,1} f_{1;1} \in \mathcal{K}_1,$$

where $f_{1;0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathcal{H}_{1;1}$, $f_{1;1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathcal{H}_{1;0}$, and $A_{1,0}, A_{1,1} \in \mathbb{R}$. The above expression agrees with (16) for $k = 1$ (for this value of $k$, any index $\mathbf{n} \in \mathcal{I}_l$ satisfies $h + |\mathbf{n}| > k$ and the sums inside the braces in (16) are null).

• Assume that the solution of the first $n-1$ equations in (17) is $U = U_1 + U_2 + \cdots + U_{n-1}$, where $U_k$ are given in (16) for $k = 1, \ldots, n-1$.

To prove the result for $n$ equations, we must solve in addition the $n$th equation of (17):

$$[v_1, U_n] + \mathrm{Proj}_{\mathcal{R}_n}\left([v_2, U_{n-1}] + \cdots + [v_n, U_1]\right) = 0,$$

or equivalently

$$L_n(U_n) = \mathrm{Proj}_{\mathcal{R}_n}\left(\sum_{k=1}^{n-1}[v_{n-k+1}, U_k]\right).$$

For $k = 1, \ldots, n-1$, we have

$$\begin{aligned}
[v_{n-k+1}, U_k] &= A_{k,0}[v_{n-k+1}, f_{k;0}] + A_{k,1}[v_{n-k+1}, f_{k;1}] + \qquad (18)\\
&+ \sum_{\substack{h=1,\ldots,k-1\\ i=0,1}} A_{h,i} \left\{ \sum_{\substack{l>0, \mathbf{n}\in\mathcal{I}_l, \mathbf{n}'\sim\mathbf{n}\\ h+|\mathbf{n}|=k, i+|\mathbf{n}'|<k+1}} [v_{n-k+1}, f_{h,\mathbf{n};i,\mathbf{n}'}] + \right.\\
&+ \left. \sum_{\substack{l>0, \mathbf{n}\in\mathcal{I}_l, \mathbf{n}'\sim\mathbf{n}\\ h+|\mathbf{n}|=k, i+|\mathbf{n}'|=k+1}} \left[v_{n-k+1}, M(k)\circ \mathrm{Proj}_{\mathcal{R}_k} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'}\right] \right\}.
\end{aligned}$$

For each $h = 1, \ldots, n-1$, $i = 0, 1$, we deal separately with the double sums in the second and third line of (18). With respect to the sum in the third line, using that

$M(k) \circ \operatorname{Proj}_{\mathcal{R}_k} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} \in \mathcal{H}_{k;-1} \subseteq \mathcal{C}_k$, we find $\left[v_{n-k+1}, M(k) \circ \operatorname{Proj}_{\mathcal{R}_k} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'}\right] \in \mathcal{H}_{n;-1} \subseteq \mathcal{C}_n$, and then its projection onto $\mathcal{R}_n$ vanishes.

With respect to the elements in the first line and the double sum in the second line of (18), applying item (2) of Lemma 4, we have

$$\begin{aligned} [v_{n-k+1}, f_{k;i}] &= \widetilde{f}_{k,n-k;i,n-k+1} + \widetilde{f}_{k,n-k;i,n-k+2}, \text{ and} \\ [v_{n-k+1}, f_{h,\mathbf{n};i,\mathbf{n}'}] &= \widetilde{f}_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}'} + \widetilde{f}_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}''}, \end{aligned} \tag{19}$$

where $\tilde{\mathbf{n}} = (\mathbf{n}, n-k) \in \mathcal{I}_{l+1}$, $\tilde{\mathbf{n}}' = (\mathbf{n}', n-k+1)$ and $\tilde{\mathbf{n}}'' = (\mathbf{n}', n-k+2)$. Note that both, $\tilde{\mathbf{n}}' \sim \tilde{\mathbf{n}}$ and $\tilde{\mathbf{n}}'' \sim \tilde{\mathbf{n}}$. Also, $h + |\tilde{\mathbf{n}}| = n$, $i + |\tilde{\mathbf{n}}'| < n+2$ and $i + |\tilde{\mathbf{n}}''| < n+3$. Moving $k$ throughout the values $1, \ldots, n-1$, $\tilde{\mathbf{n}}$ runs the set $\mathcal{I}_{l+1}$ and $\tilde{\mathbf{n}}'$, $\tilde{\mathbf{n}}''$ are all the $(l+1)$-indices associated to $\tilde{\mathbf{n}}$. So,

$$\begin{aligned} &\operatorname{Proj}_{\mathcal{R}_n}\left(\sum_{k=1}^{n-1}[v_{n-k+1}, U_k]\right) = \\ &= \operatorname{Proj}_{\mathcal{R}_n} \sum_{\substack{h=1,\ldots,n-1 \\ i=0,1}} A_{h,i} \left\{ \sum_{\substack{l\geq 0, \tilde{\mathbf{n}}\in\mathcal{I}_{l+1}, \tilde{\mathbf{n}}'\sim\tilde{\mathbf{n}} \\ h+|\tilde{\mathbf{n}}|=n, i+|\tilde{\mathbf{n}}'|<n+3}} \widetilde{f}_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}'} \right\} \\ &= \operatorname{Proj}_{\mathcal{R}_n} \sum_{\substack{h=1,\ldots,n-1 \\ i=0,1}} A_{h,i} \left\{ \sum_{\substack{l> 0, \mathbf{n}\in\mathcal{I}_{l}, \mathbf{n}'\sim\mathbf{n} \\ h+|\mathbf{n}|=n, i+|\mathbf{n}'|<n+3}} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} \right\}. \end{aligned}$$

In the last equality, we have displaced the summation index $l+1 \longrightarrow l$.

In the last sum, we must consider $\mathbf{n}' \sim \mathbf{n}$ with $i + |\mathbf{n}'| < n+3$. There are three possibilities:

*(a)* $i + |\mathbf{n}'| < n+1$. Here, $\widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} \in \mathcal{R}_n$, and $\operatorname{Proj}_{\mathcal{R}_n} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} = \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'}$.

*(b)* $i + |\mathbf{n}'| = n+1$, where $\widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} \in \mathcal{H}_{n;0}$.

*(c)* $i + |\mathbf{n}'| = n+2$. Now, $\widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} \in \mathcal{H}_{n;-1} \subseteq \mathcal{C}_n$, and then $\operatorname{Proj}_{\mathcal{R}_n} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} = 0$.

So, we can write

$$\operatorname{Proj}_{\mathcal{R}_n}\left(\sum_{k=1}^{n-1}[v_{n-k+1}, U_k]\right) = \sum_{\substack{h=1,\ldots,n-1 \\ i=0,1}} A_{h,i} \left\{ \sum_{\substack{l> 0, \mathbf{n}\in\mathcal{I}_{l}, \mathbf{n}'\sim\mathbf{n} \\ h+|\mathbf{n}|=n, i+|\mathbf{n}'|<n+1}} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} + \right.$$

$$+ \sum_{\substack{l>0,\mathbf{n}\in\mathcal{I}_l,\mathbf{n}'\sim\mathbf{n}\\ h+|\mathbf{n}|=n,i+|\mathbf{n}'|=n+1}} \mathrm{Proj}_{\mathcal{R}_n} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} \Bigg\}.$$

Using now item (3) of Lemma 4, the general solution of the linear equation $L_n(U_n) = \mathrm{Proj}_{\mathcal{R}_n} \sum_{k=1}^{n-1}[v_{n-k+1}, U_k]$ can be expressed as

$$\begin{aligned} U_n &= A_{n,0} f_{n;0} + A_{n,1} f_{n;1} + \\ &+ \sum_{\substack{h=1,\dots,n-1\\ i=0,1}} A_{h,i} \Bigg\{ \sum_{\substack{l>0,\mathbf{n}\in\mathcal{I}_l,\mathbf{n}'\sim\mathbf{n}\\ h+|\mathbf{n}|=n,i+|\mathbf{n}'|<n+1}} M(i+|\mathbf{n}'|-1)\circ \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} \\ &+ \sum_{\substack{l>0,\mathbf{n}\in\mathcal{I}_l,\mathbf{n}'\sim\mathbf{n}\\ h+|\mathbf{n}|=n,i+|\mathbf{n}'|=n+1}} M(i+|\mathbf{n}'|-1)\circ \mathrm{Proj}_{\mathcal{R}_n} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} \Bigg\}. \end{aligned}$$

To fulfill the proof, it is enough to use item (5)(c) of Lemma 4. ∎

THEOREM 13. *Let us consider the general solution* $U = \sum_{k=1}^{n} U_k \in \bigoplus_{k=1}^{n} \mathcal{H}_k$ *of equation (15), obtained in the above theorem. Then,*

$$\begin{aligned} \widetilde{v}_k = \mathrm{Proj}_{\mathcal{C}_k}([v,U]) = &\sum_{\substack{h=1,\dots,k-1\\ i=0,1}} A_{h,i} \Bigg\{ \sum_{\substack{l>0,\mathbf{n}\in\mathcal{I}_l,\mathbf{n}'\sim\mathbf{n}\\ h+|\mathbf{n}|=k,i+|\mathbf{n}'|=k+2}} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} \\ &+ \sum_{\substack{l>0,\mathbf{n}\in\mathcal{I}_l,\mathbf{n}'\sim\mathbf{n}\\ h+|\mathbf{n}|=k,i+|\mathbf{n}'|=k+1}} \mathrm{Proj}_{\mathcal{C}_k} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} \Bigg\}, \end{aligned} \tag{20}$$

*for* $k = 2, \dots, n$.

*Proof.* As $[v_1, U_n] \in \mathcal{R}_n$, we get $\widetilde{v}_n = \mathrm{Proj}_{\mathcal{C}_n}([v_1,U_n] + [v_2,U_{n-1}] + \cdots + [v_n,U_1]) = \mathrm{Proj}_{\mathcal{C}_n}([v_2,U_{n-1}] + \cdots + [v_n,U_1])$.

We focus into obtaining $\mathrm{Proj}_{\mathcal{C}_n}[v_{n-k+1}, U_k]$ for $k = 1, \dots, n-1$. The expression of $[v_{n-k+1}, U_k]$ appears in (18). As in the proof of the above theorem, we start analyzing the double sum in the third line of (18). We have seen that its elements belongs to $\mathcal{C}_n$. Using item (5)(c) of Lemma 4, we can write its summands as

$$\Big[v_{n-k+1}, M(k) \circ \mathrm{Proj}_{\mathcal{R}_k} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'}\Big] = [v_{n-k+1}, f_{h,\mathbf{n};i,\mathbf{n}'}].$$

Denote $\tilde{\mathbf{n}} = (\mathbf{n}, n-k) \in \mathcal{I}_{l+1}$, $\tilde{\mathbf{n}}' = (\mathbf{n}', n-k+1) \sim \tilde{\mathbf{n}}$ and $\tilde{\mathbf{n}}'' = (\mathbf{n}', n-k+2) \sim \tilde{\mathbf{n}}$. Observe that $h + |\tilde{\mathbf{n}}| = n$, $i + |\tilde{\mathbf{n}}'| = n+2$ and $i + |\tilde{\mathbf{n}}''| = n+3$. Applying item (2) of Lemma 4, we obtain

$$[v_{n-k+1}, f_{h,\mathbf{n};i,\mathbf{n}'}] = \widetilde{f}_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}'} + \widetilde{f}_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}''} = \widetilde{f}_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}'},$$

because $h + |\tilde{\mathbf{n}}| - i - |\tilde{\mathbf{n}}''| = -3$, and then $\widetilde{f}_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}''} = 0$.

In summary

$$\Big[v_{n-k+1}, M(k) \circ \mathrm{Proj}_{\mathcal{R}_k} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'}\Big] = \widetilde{f}_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}'}.$$

The remaining elements of (18) (the two summands in the first line and the double sum in the second line) appear in (19). In the double sum in the second line of (18), we must consider indices $\mathbf{n} \in \mathcal{I}_l, \mathbf{n}' \sim \mathbf{n}$ with $h + |\mathbf{n}| = k$, $i + |\mathbf{n}'| < k+1$. As $\widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} \in \mathcal{R}_{h+|\mathbf{n}|}$, whenever $h + |\mathbf{n}| - i - |\mathbf{n}'| \geq 0$, when projecting onto $\mathcal{C}_n$ we must consider only the indices such that $i + |\mathbf{n}'| = k$ (the remaining have zero projection).

Now, we move $k$ throughout the values $1, \ldots, n-1$, so that $\tilde{\mathbf{n}}$ runs the set $\mathcal{I}_{l+1}$ and $\tilde{\mathbf{n}}'$, $\tilde{\mathbf{n}}''$ run the set of $(l+1)$-indices associated to $\tilde{\mathbf{n}}$. Consequently:

$$\widetilde{v}_n = \mathrm{Proj}_{\mathcal{C}_n} \sum_{k=1}^{n-1} [v_{n-k+1}, U_k] = \sum_{\substack{h=1,\ldots,n-1 \\ i=0,1}} A_{h,i} \left\{ \sum_{\substack{l\geq 0, \tilde{\mathbf{n}} \in \mathcal{I}_{l+1}, \tilde{\mathbf{n}}' \sim \tilde{\mathbf{n}} \\ h+|\tilde{\mathbf{n}}|=n, i+|\tilde{\mathbf{n}}'|=n+2}} \widetilde{f}_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}'} \right.$$
$$\left. + \sum_{\substack{l\geq 0, \tilde{\mathbf{n}} \in \mathcal{I}_{l+1}, \tilde{\mathbf{n}}' \sim \tilde{\mathbf{n}} \\ h+|\tilde{\mathbf{n}}|=n, i+|\tilde{\mathbf{n}}'|=n+1}} \mathrm{Proj}_{\mathcal{C}_n} \widetilde{f}_{h,\tilde{\mathbf{n}};i,\tilde{\mathbf{n}}'} \right\}.$$

To fulfill the proof, it is enough to displace the summation index $l+1 \longrightarrow l$. ∎

We are interested in the expressions (16), (20) but depending only on one arbitrary constant $A_{h,i}$ (with $h$, $i$ fixed) and annihilating the remaining ones. The quoted arbitrary constant appears in the $h$-order terms and upper (see Theorem 12). The structure of the coefficient of such arbitrary constant suggest the following definitions:

Definition 14. We will denote $\mathcal{H}^*_{k;j} = \mathcal{H}_{k;j} \bigoplus \mathcal{H}_{k;j+1} \bigoplus \cdots \bigoplus \mathcal{H}_{k;k}$. Consider $f \in \mathcal{H}^*_{k;j}$, $f \neq 0$. We define the order and the difference of $f$ by:

$$\begin{aligned} \operatorname{ord}(f) &= k, \\ \operatorname{dif}(f) &= \max\left\{m \in \mathbb{Z} : \ f \in \mathcal{H}^*_{k;m}\right\}. \end{aligned}$$

The next properties follow directly from the above definition:

Lemma 15.

*(1)Consider* $f \in \mathcal{H}_{k;j}$, $f \neq 0$. *Then* $\operatorname{ord}(f) = k$, $\operatorname{dif}(f) = j$.
*(2)Consider* $f \in \mathcal{H}^*_{k;j}$, *with* $j \leq -2$. *Then* $\operatorname{dif}(f) \geq -1$.
*(3)Consider* $f \in \mathcal{H}^*_{k;j}$, $f \neq 0$. *Then* $-1 \leq \operatorname{dif}(f) \leq k$.
*(4)Consider* $f \in \mathcal{H}^*_{k;j}$, $f \neq 0$.

*If* $\operatorname{dif}(f) \geq 1$, *then* $f \in \mathcal{R}_k$.

*If* $\operatorname{dif}(f) = -1$, *then* $f \in \mathcal{C}_k$.

*If* $\operatorname{dif}(f) = 0$, *then* $f$ *may have nonzero projection onto* $\mathcal{R}_k$ *and* $\mathcal{C}_k$.

*In particular, the elements in* $\mathcal{H}^*_{k;j}$ *with nonzero projection onto* $\mathcal{C}_k$ *have difference negative or zero.*

Next, we extend the operators $N$, $L$, $M$ to $\mathcal{H}^*_{k;j}$.

Definition 16. Consider $f = \sum_{m \geq j} f_{k;k-m} \in \mathcal{H}^*_{k;j}$, where $f_{k;k-m} \in \mathcal{H}_{k;m}$ for $m \geq j$. We define

$$\begin{aligned} N_n f &= \sum_{m \geq j} N(n,m) \circ f_{k;k-m} \in \mathcal{H}^*_{k+n;j-1}, \\ L_n f &= \sum_{m \geq j} L(n,m) \circ f_{k;k-m} \in \mathcal{H}^*_{k+n;j}, \\ M f &= \sum_{m \geq j} M(k-m) \circ \operatorname{Proj}_{\mathcal{R}_k} f_{k;k-m} \in \mathcal{H}^*_{k;j-1}. \end{aligned}$$

Lemma 4 can be generalized as follows:

Lemma 17.

*(1)*$\left[\mathcal{H}^*_{m;r}, \mathcal{H}^*_{k;j}\right] \subseteq \mathcal{H}^*_{m+k-1;r+j}$.
*(2)Consider* $f \in \mathcal{H}^*_{k;j}$. *Then* $[v_n, f] = N_{n-1} f + L_{n-1} f \in \mathcal{H}^*_{n+k-1;j-1}$.
*(3)Consider* $f \in \mathcal{H}^*_{k;j}$. *Then* $U_k = Mf$ *is a solution of the homological equation* $L_k(U_k) = \operatorname{Proj}_{\mathcal{R}_k} f$.

*(4)Consider* $f \in \mathcal{H}^*_{k;j}$ *with* $j > 0$. *Then* $\mathrm{Proj}_{\mathcal{C}_k} f = 0$.
*(5)Consider* $f_{k;j} \in \mathcal{H}_{k;k-j}$. *Then:*

$$f_{k,n;j,n+1} = ML_n f_{k;j} \in \mathcal{H}_{k+n;k-j-1}, \quad f_{k,n;j,n+2} = MN_n f_{k;j} \in \mathcal{H}_{k+n;k-j-2},$$
$$\widetilde{f}_{k,n;j,n+1} = L_n f_{k;j} \in \mathcal{H}_{k+n;k-j}, \quad \widetilde{f}_{k,n;j,n+2} = N_n f_{k;j} \in \mathcal{H}_{k+n;k-j-1}.$$

*Also, consider* $\mathbf{n} \in \mathcal{I}_l$ *and* $\mathbf{n}' \sim \mathbf{n}$. *Then:*

$$\begin{aligned}\widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} &= H_{n_l} M H_{n_{l-1}} \cdots M H_{n_1} f_{h;i} \in \mathcal{H}_{h+|\mathbf{n}|;h+|\mathbf{n}|-i-|\mathbf{n}'|+1},\\ f_{h,\mathbf{n};i,\mathbf{n}'} &= M H_{n_l} \cdots M H_{n_1} f_{h;i} \in \mathcal{H}_{h+|\mathbf{n}|;h+|\mathbf{n}|-i-|\mathbf{n}'|},\end{aligned}$$

*where* $H_{n_k}$ *is given in (12), (13).*

Definition 18. We will denote

$$\mathcal{P}\left(H_{n_l} M H_{n_{l-1}} \cdots M H_{n_1} f_{h;i}\right) = \sum_{\sigma \in \varsigma\{1,\cdots,l\}}{}' H_{n_{\sigma(1)}} M H_{n_{\sigma(2)}} \cdots M H_{n_{\sigma(l)}} f_{h;i},$$

(recall that $\varsigma\{1,\ldots,l\}$ is the permutation group of elements $\{1,\ldots,l\}$). Also, $\mathcal{P}\left(MH_{n_l} \cdots MH_{n_1} f_{h;i}\right) = \sum'_{\sigma\in\varsigma\{1,\cdots,l\}} MH_{n_{\sigma(1)}} \cdots MH_{n_{\sigma(l)}} f_{h;i}$, where, as before, $\sum'$ denotes that only the summands that are different must be considered.

Next, we analyze the effect of $N_n$, $L_n$, $M$ on the difference of an element.

Lemma 19. *Consider* $f \in \mathcal{H}^*_{k;j}$, $f \neq 0$, *with* $-1 \leq j \leq k$.

*(1)* *If* $\mathrm{dif}(f) = 0$ *and* $\mathrm{Proj}_{\mathcal{H}_{k;0}} f = \begin{pmatrix} 0 \\ B \end{pmatrix}$, *then* $\mathrm{dif}(L_n f) > \mathrm{dif}(f)$.

*If* $\mathrm{dif}(f) \neq 0$ *or* $\mathrm{Proj}_{\mathcal{H}_{k;0}} f \neq \begin{pmatrix} 0 \\ B \end{pmatrix}$, *then* $\mathrm{dif}(L_n f) = \mathrm{dif}(f)$.

*(2)* *If* $\mathrm{dif}(f) = -1$, *then* $\mathrm{dif}(N_n f) \geq \mathrm{dif}(f) - 1$.

*If* $\mathrm{dif}(f) = 0$ *and* $\mathrm{Proj}_{\mathcal{H}_{k;0}} f = \begin{pmatrix} A \\ (n+1)A \end{pmatrix}$, *then* $\mathrm{dif}(N_n f) > \mathrm{dif}(f) - 1$.

*In any other case,* $\mathrm{dif}(N_n f) = \mathrm{dif}(f) - 1$.

*(3)*$\mathrm{dif}(Mf) = \mathrm{dif}(f) - 1$ *whenever* $0 \leq j \leq k$.

*Proof.*

*(1)* It is enough to use $\mathrm{Ker}\, N(n,j) = \{0\}$ for $j = 1,\ldots,k$; $\mathrm{Ker}\, N(n,-1) = \mathcal{H}_{k;-1}$; and $\mathrm{Ker}\, N(n,0) = \left\{\begin{pmatrix} A \\ (n+1)A \end{pmatrix} : A \in \mathbb{R}\right\} \subset \mathcal{H}_{k;0}$.

*(2)* The relations in the statement hold because $\operatorname{Ker} L(n,j) = \{0\}$ for $j \neq 0$; and $\operatorname{Ker} L(n,0) = \left\{ \begin{pmatrix} 0 \\ B \end{pmatrix} : B \in \mathbb{R} \right\} \subset \mathcal{H}_{k;0}$.

*(3)* In this case, $\operatorname{Ker} M(k-j) = \{0\}$ for $j = 0, \ldots, k$.

∎

*Remark:* If $\operatorname{dif}(f) > 0$, then $\operatorname{dif}(N_n f) = \operatorname{dif}(f) - 1$, $\operatorname{dif}(L_n f) = \operatorname{dif}(f)$. The same equalities hold in the case $\operatorname{dif}(f) = 0$ and $\operatorname{Proj}_{\mathcal{H}_{k;0}} f = \begin{pmatrix} -A \\ kA \end{pmatrix}$, with $A \neq 0$ (i.e., for $\operatorname{Proj}_{\mathcal{C}_k} f = 0$).

## 4. MAIN RESULTS

In this section we present the different possibilities of annihilating terms in the normal form:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \sum_{n \geq 2} (a_n x^n + b_n x^{n-1} y). \end{aligned} \tag{21}$$

The results achieved are based in the relative position of the indices corresponding to the first non-vanishing coefficients in the above normal form. This indices will be denoted $r$, $s$. So, $a_2 = \cdots = a_{r-1} = 0$, $a_r \neq 0$, and $b_2 = \cdots = b_{s-1} = 0$, $b_s \neq 0$.

To analyze how much we can simplify the above normal form by using transformations in the state variables, we will transform the above normal form into

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \sum_{n \geq 2} (a_n^* x^n + b_n^* x^{n-1} y), \end{aligned} \tag{22}$$

where the coefficients $a_n^*, b_n^*$ depend on the arbitrary constants parameterizing $\bigoplus_{k \geq 1} \mathcal{K}_k$. Later, these arbitrary constants will be selected in order to annihilate the maximum number of coefficients $a_n^*, b_n^*$.

Under the above hypothesis, we have

$$\begin{aligned} &N_1 = \cdots = N_{r-2} = 0, \ N_{r-1} \neq 0, \ \text{and} \\ &L_1 = \cdots = L_{s-2} = 0, \ L_{s-1} \neq 0. \end{aligned}$$

From Theorem 13, one arbitrary constant $A_{h,i}$ appears in the $k$-order terms of the transformed vector field multiplied by:

$$\sum_{\substack{l>0,\mathbf{n}\in\mathcal{I}_l,\mathbf{n}'\sim\mathbf{n}\\ h+|\mathbf{n}|=k,i+|\mathbf{n}'|=k+2}} \widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} + \sum_{\substack{l>0,\mathbf{n}\in\mathcal{I}_l,\mathbf{n}'\sim\mathbf{n}\\ h+|\mathbf{n}|=k,i+|\mathbf{n}'|=k+1}} \mathrm{Proj}_{\mathcal{C}_k}\widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'}$$

Using Lemma 17, the summands of the above sum can be expressed as

$$\widetilde{f}_{h,\mathbf{n};i,\mathbf{n}'} = H_{n_l}MH_{n_{l-1}}\cdots MH_{n_1}f_{h;i},$$

where $f_{h;i}\in\mathcal{H}_{h;h-i}$. As $h\geq 2$, $i=0,1$, we find $\mathrm{dif}\,(f_{h;i})>0$. We seek for those summands with zero or negative difference (in this case, the projection onto the co-range is non-vanishing). Moreover, we are interested in the lower order where the arbitrary constant $A_{h,i}$ appears. This will depend on the expression of $H_{n_j}$ (see (12), (13)).

When we apply $MN_{r-1}$ to an element, its difference decreases two unities and its order increases $r-1$ unities. That is, $\mathrm{dif}(MN_{r-1}f)=\mathrm{dif}(f)-2$ and $\mathrm{ord}(MN_{r-1}f)=\mathrm{ord}(f)+r-1$, whenever the projection of $f$ into the co-range vanishes (see Remark after Lemma 19). On the other hand, when applying $ML_{s-1}$, the difference decreases one unity and the order increases $s-1$. Consequently, looking at the decreasing of the difference, the operators $MN_{r-1}$, $ML_{s-1}ML_{s-1}$ have the same effect.

To determine the lower order where $A_{h,i}$ appears, we must distinguish three cases:

*(1)* $r<2s-1$. In this case, we must pay attention to the appearance of $MN_{r-1}$ instead $ML_{s-1}ML_{s-1}$, because the first operation increases less the order than the second one.

*(2)* $r=2s-1$. Now, the effect on the increasing of the order of $MN_{r-1}$ and $ML_{s-1}ML_{s-1}$ is the same and we must deal with both.

*(3)* $r>2s-1$. Here, we focus in the summands where $ML_{s-1}ML_{s-1}$ (instead $MN_{r-1}$) appear.

Before dealing with each one of these cases, we will see that the arbitrary constants $A_{1,0}$, $A_{1,1}$ (parameterizing $\mathcal{K}_1$), can not be used to annihilate terms in the normal form. Namely, $A_{1,0}$ do not appear and $A_{1,1}$ can only be used to rescaling.

Lemma 20. *The arbitrary constant $A_{1,0}$ do not appear in the normal form (22).*

*Proof.* The constant $A_{1,0}$ appear in the normal form (22) in the following terms:

- $\mathrm{Proj}_{\mathcal{C}_{k+1}}N_kf_{1;0}$, in the $k+1$ order terms,
- $\mathcal{P}(L_kMN_mf_{1;0})$, and $\mathrm{Proj}_{\mathcal{C}_{k+m+1}}\mathcal{P}(L_kML_mf_{1;0})$ in the $k+m+1$ order terms,

- $\mathcal{P}(L_n M L_k M L_m f_{1;0})$, in the $k+m+n+1$ order terms, for any $k, m, n \in \mathbb{N}$,

where $f_{1;0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathcal{H}_{1,1}$.

The above mentioned elements vanish, and consequently the quoted arbitrary constant do not appear in the transformed vector field. ▮

LEMMA 21. *The arbitrary constant $A_{1,1}$ can be used in the normal form (22) only to rescaling.*

*Proof.* The change of variables corresponding to a generator

$$U = A_{1,1} f_{1;1} = A_{1,1} \left( x\partial_x + y\partial_y \right),$$

is given by

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = e^{A_{1,1}} \begin{pmatrix} x \\ y \end{pmatrix},$$

and so $A_{1,1}$ can be used only to rescaling some term in the normal form. ▮

Next, we analyze the different cases for the hypernormal form of system (21).

### 4.1. Case I: $r < 2s - 1$.

In this case, there are different possibilities in the simplifications. We will consider two subcases:

#### *4.1.1. Subcase I.1: $s < r < 2s - 1$.*

To analyze how each arbitrary constant $A_{h,i}$ appear in the transformed vector field $v^*$, we deal with the cases $h = 2n$ even and $h = 2n + 1$ odd separately.

The main results we have obtained appear detailed in Appendix A.1. Here, we present one of the cases, that corresponds to the first appearance of each arbitrary constant in $v^*$.

(A) Role of $A_{2n,1}$, $n \geq 1$.

LEMMA 22. *Let $\lambda = n(r+1) - 1$. Then, the vector field (21) is $C^\infty$–conjugate to (22), where*

$$\begin{aligned}
a_k^* &= a_k, \ \text{for } k = 2, \ldots, \lambda + s - 1, \\
b_k^* &= b_k, \ \text{for } k = 2, \ldots, \lambda, \\
a_{\lambda+s}^* &= a_{\lambda+s} + A_{2n,1} a_1^{2n,1}, \\
b_{\lambda+1}^* &= b_{\lambda+1} + A_{2n,1} a_2^{2n,1},
\end{aligned}$$

*where*

$$a_1^{2n,1} = \Pi_1\mathcal{P}(L_{s-1}MN_{r-1} \overset{n}{\cdots} MN_{r-1}f_{2n,1}), \quad (23)$$

$$a_2^{2n,1} = \Pi_2\text{Proj}_{\mathcal{C}_{\lambda+1}}N_{r-1}MN_{r-1} \overset{n-1}{\cdots} MN_{r-1}f_{2n,1}. \quad (24)$$

Moreover, $A_{2n,1}$ appear linearly in the normal form coefficients up to order $\mu = (2n-1)(r+1)+s-1$. In the Appendix A.1 we will show that we can know how it appears and then different possibilities to annihilating normal form coefficients are obtained.

<u>(B) Role of $A_{2n,0} \in \mathbb{R}$, $n \geq 1$.</u>

LEMMA 23. *Let $\lambda = n(r+1)+s-2$. Then, the vector field (21) is $C^\infty$–conjugate to (22), with*

$$\begin{aligned} a_k^* &= a_k, \ \textit{for } k = 2,\ldots,\lambda+r-s, \\ b_k^* &= b_k, \ \textit{for } k = 2,\ldots,\lambda, \\ a_{\lambda+r-s+1}^* &= a_{\lambda+r-s+1} + A_{2n,0}a_1^{2n,0}, \\ b_{\lambda+1}^* &= b_{\lambda+1} + A_{2n,0}a_2^{2n,0}, \ \textit{for } k = \lambda+1,\ldots,\mu, \end{aligned}$$

*where*

$$a_1^{2n,0} = = \Pi_1\left(N_{r-1}MN_{r-1} \overset{n}{\cdots} MN_{r-1}f_{2n,0}\right), \quad (25)$$

$$a_2^{2n,0} = \Pi_2\text{Proj}_{\mathcal{C}_{\lambda+1}}\mathcal{P}(L_{s-1}MN_{r-1} \overset{n}{\cdots} MN_{r-1}f_{2n,0}). \quad (26)$$

In Appendix, we will carry out a deeper study, and we will show that $A_{2n,0}$ appear linearly in the normal form coefficients up to order $\mu = 2n(r+1)+s-3$. This allow different possibilities to select the normal form coefficients to be annihilated.

<u>(C) Role of $A_{2n+1,1} \in \mathbb{R}$, $n \geq 1$.</u>

LEMMA 24. *Let $\lambda = n(r+1)+s-1$. Then, the vector field (21) is $C^\infty$–conjugate to (22), where*

$$\begin{aligned} a_k^* &= a_k, \ \textit{for } k = 2,\ldots,\lambda+r-s, \\ b_k^* &= b_k, \ \textit{for } k = 2,\ldots,\lambda, \\ a_{\lambda+r-s+1}^* &= a_{\lambda+r-s+1} + A_{2n+1,1}a_1^{2n+1,1}, \\ b_{\lambda+1}^* &= b_{\lambda+1} + A_{2n+1,1}a_2^{2n+1,1}, \end{aligned}$$

*where*

$$a_1^{2n+1,1} = \Pi_1\left(N_{r-1}MN_{r-1} \overset{n}{\cdots} MN_{r-1}f_{2n+1,1}\right), \tag{27}$$

$$a_2^{2n+1,1} = \Pi_2\mathrm{Proj}_{\mathcal{C}_{\lambda+1}}\mathcal{P}(L_{s-1}MN_{r-1} \overset{n}{\cdots} MN_{r-1}f_{2n+1,1}). \tag{28}$$

As in the above cases, we will later show that $A_{2n+1,1}$ appears linearly in the normal form up to order $\mu = 2n(r+1)+s-1$.

(D) Role of $A_{2n-1,0} \in \mathbb{R}$, $n \geq 2$.

LEMMA 25. *Let $\lambda = (n-1)(r+1)+2s-2$. Then, the vector field (21) is* $C^\infty$*–conjugate to (22), with*

$$\begin{aligned} a_k^* &= a_k, \ \text{for } k = 2,\ldots,\lambda+r-s, \\ b_k^* &= b_k, \ \text{for } k = 2,\ldots,\lambda, \\ a_{\lambda+r-s+1}^* &= a_{\lambda+r-s+1} + A_{2n-1,0}a_1^{2n-1,0}, \\ b_{\lambda+1}^* &= b_{\lambda+1} + A_{2n-1,0}a_2^{2n-1,0}, \end{aligned}$$

*where*

$$a_1^{2n-1,0} = \Pi_1\mathcal{P}(L_{s-1}MN_{r-1} \overset{n}{\cdots} MN_{r-1}f_{2n-1,0}), \tag{29}$$

$$a_2^{2n-1,0} = \Pi_2\mathrm{Proj}_{\mathcal{C}_{\lambda+1}}\mathcal{P}(L_{s-1}ML_{s-1}MN_{r-1} \overset{n-1}{\cdots} MN_{r-1}f_{2n-1,0}). \tag{30}$$

In this case, $A_{2n-1,0}$ appears linearly up to order $\mu = 2(n-1)(r+1)+r+s-2$.

(E) Statement of the main result.

The information achieved in the four items above provides the following result.

THEOREM 26. *Let us consider the vector field (21), with $a_2 = \cdots = a_{r-1} = 0, a_r \neq 0$, $b_2 = \cdots = b_{s-1} = 0, b_s \neq 0$ where $r < 2s-1$. Then, the quoted vector field is* $C^\infty$*–conjugate to (22), where*

*(i)* • $a_k^* = a_k$ *for* $k = 2,\ldots,r+s-1$.

• $b_k^* = b_k$ *for* $k = 2,\ldots,r$.

*(ii)* • $a_{r+s}^* = 0$ *or* $b_{r+1}^* = 0$, *whenever* $r \neq s+1$. *If* $r = s+1$, *there is only one possibility:* $b_{r+1}^* = 0$.

• $a_{2r}^* = 0$ *or* $b_{r+s}^* = 0$.

*•$a^*_{2r+1} = 0$ or $b^*_{r+s+1} = 0$.*

*(iii)For each $n \geq 2$, we can choose a case of simplifications of item (a), and another of item (b), listed below:*

*(a-1) $b^*_{n(r+1)} = 0$ whenever $a_2^{2n,1} \neq 0$; and $b^*_{2s-r-2+n(r+1)} = 0$ whenever $a_2^{2n-1,0} \neq 0$.*

*(a-2) $b^*_{n(r+1)} = 0$ whenever $a_2^{2n,1} \neq 0$; and $a^*_{s-2+n(r+1)} = 0$ whenever $a_1^{2n-1,0} \neq 0$.*

*(a-3) $a^*_{s-2+n(r+1)} = 0$ whenever $a_1^{2n-1,0} \neq 0$; and $a^*_{s-1+n(r+1)} = 0$ whenever $a_1^{2n,1} \neq 0$.*

*(b-1) $b^*_{s-1+n(r+1)} = 0$ whenever $a_2^{2n,0} \neq 0$; and $b^*_{s+n(r+1)} = 0$ whenever $a_2^{2n+1,1} \neq 0$.*

*(b-2) $a^*_{r-1+n(r+1)} = 0$ whenever $a_1^{2n,0} \neq 0$; and $b^*_{s+n(r+1)} = 0$ whenever $a_2^{2n+1,1} \neq 0$.*

*(b-3) $b^*_{s-1+n(r+1)} = 0$ whenever $a_2^{2n,0} \neq 0$; and $a^*_{r+n(r+1)} = 0$ whenever $a_1^{2n+1,1} \neq 0$.*

*(b-4) $a^*_{r-1+n(r+1)} = 0$ whenever $a_1^{2n,0} \neq 0$; and $a^*_{r+n(r+1)} = 0$ whenever $a_1^{2n+1,1} \neq 0$.*

*The constants $a_j^{h,i}$ are given in (23), (24), (25), (26), (27), (28), (29), (30).*

*Proof.* Item (i) is straightforward. To prove item (ii), we firstly perform some calculations and obtain:

$$\begin{aligned}
a_1^{2,1} &= 2(1-r+s)a_r b_s/(s(s+1)),\\
a_2^{2,1} &= -3a_r,\\
a_1^{2,0} &= -(r+s+1)a_r^2/(r+1),\\
a_2^{2,0} &= -(r+s)a_r b_s/(r+1),\\
a_1^{3,1} &= -2(r-1)(r+3)a_r^2/((r+1)(r+2)),\\
a_2^{3,1} &= (-rs^2-r^2s-5s^2-2rs+9r^2+3s+27r+18)a_r b_s/(s(r+1)(r+2)).
\end{aligned}$$

In the hypothesis of item (ii), the above constants do not vanish, and it is enough to select $A_{2,0}$, $A_{2,1}$, $A_{3,1}$ to fulfill the proof.

We focus in the proof of item (iii). First, we transform the vector field $v$ as indicated in Lemma 22. We obtain the transformed vector field $v^*$ given in (22), where

$$a^*_k = a_k, \quad \text{for} \quad k = 2, \ldots s-2+n(r+1),$$

$$\begin{aligned} a^*_{s-1+n(r+1)} &= a_{s-1+n(r+1)} + A_{2n,1}a_1^{2n,1}, \\ b^*_k &= b_k, \quad \text{for} \quad k = 2, \ldots, n(r+1)-1, \\ b^*_{n(r+1)} &= b_{n(r+1)} + A_{2n,1}a_2^{2n,1}, \end{aligned}$$

with $a_1^{2n,1}$, $a_2^{2n,1}$ given in (23), (24), respectively. Next, we transform $v^*$ as in Lemma 25. We obtain now $v^{**}$, which is also in normal form, and whose coefficients are

$$\begin{aligned} a^{**}_k &= a^*_k, \quad \text{for} \quad k = 2, \ldots, s-3+n(r+1), \\ a^{**}_{s-2+n(r+1)} &= a^*_{s-2+n(r+1)} + A_{2n-1,0}\tilde{a}_1^{2n-1,0}, \\ b^{**}_k &= b^*_k, \quad \text{for} \quad k = 2, \ldots, n(r+1)+r-3, \\ b^{**}_{2s-r-2+n(r+1)} &= b^*_{2s-r-2+n(r+1)} + A_{2n-1,0}\tilde{a}_2^{2n-1,0}. \end{aligned}$$

As $a^*_r = a_r$, $b^*_s = b_s$, we deduce $\tilde{a}_1^{2n-1,0} = a_1^{2n-1,0}$, $\tilde{a}_2^{2n-1,0} = a_2^{2n-1,0}$. Moreover, these constants are given in (29), (30). The possibilities of annihilating terms in $v^{**}$ are summarized as follows:

- In the case $a_2^{2n,1} \neq 0$, we can select $A_{2n,1} = -b_{n(r+1)}/a_2^{2n,1}$, and then $b^{**}_{n(r+1)} = b^*_{n(r+1)} = 0$ for $r \neq 2s-2$. If $a_2^{2n-1,0} \neq 0$, then taking $A_{2n-1,0}$ adequately, we can annihilate $b^{**}_{2s-r-2+n(r+1)}$. For $r = 2s-2$ (where the indices $n(r+1)$, $2s-r-2+n(r+1)$ agree), we can do $b^*_{n(r+1)} = 0$ assuming $a_2^{2n,1} \neq 0$, and the second transformation is not necessary.
- Assuming $a_2^{2n,1} \neq 0$, we can achieve $b^{**}_{n(r+1)} = b^*_{n(r+1)} = 0$ for $r \neq 2s-2$. Analogously, if $a_1^{2n-1,0} \neq 0$, we can vanish $a^{**}_{s-2+n(r+1)}$.

Item (a-3), and the remaining case of (a-2), are obtained by applying consecutively Lemmas 25, 22; and then selecting $A_{2n-1,0}$, $A_{2n,1}$ in order to achieve the corresponding simplifications. The reasoning to prove item (b) is the same, but applying now Lemmas 23, 24 and taking $A_{2n,0}$, $A_{2n+1,1}$ in a convenient way. Similar comments hold for items (b-3), (b-4).

Finally, it is necessary to remark that each step in the procedure do not affect to the terms we have annihilated in previous steps. ▮

#### 4.1.2. *Subcase I.2:* $\boldsymbol{r \leq s}$.

The analysis of the remaining cases can be accomplished following a parallel scheme to the above one. The results presented below can be greatly proved analogously to the above ones, so that we only include in some cases the main differences with the case I.1.

In the case I.2, we consider that $a_2 = \cdots = a_{r-1} = 0$, $a_r \neq 0$, $b_2 = \cdots = b_{s-1} = 0$, $b_s \neq 0$, with $r \leq s$.

The analysis of how the arbitrary constants $A_{h,i}$ affect the transformed vector field is carried out in next items. Their proofs are similar to the above subcase. For this reason, we do not include them.

(A) Role of $A_{2n,1}$, $n \geq 1$.

Lemma 27. *Let $n \in \mathbb{N}$. Then, the vector field (21) is* C$^\infty$*–conjugate to (22), where*

*(a)If $nr+n \leq s$, then*

$$\begin{aligned} a_k^* &= a_k, \quad \textit{for} \quad k = 2, \dots, 2n+2nr-2, \\ b_k^* &= b_k, \quad \textit{for} \quad k = 2, \dots, nr+n-1, \\ b_{nr+n}^* &= b_{nr+n} + A_{2n,1} a_2^{2n,1}. \end{aligned}$$

*In this case, the arbitrary constant $A_{2n,1}$ appears for the first time in the expression of $a_{2n+2nr-1}^*$ in a nonlinear way.*

*(b)If $nr+n \geq s+1$, then*

$$\begin{aligned} a_k^* &= a_k, \quad \textit{for} \quad k = 2, \dots, n+nr+s-2, \\ a_{n+nr+s-1}^* &= a_{n+nr+s-1} + A_{2n,1} a_1^{2n,1}, \\ b_k^* &= b_k, \quad \textit{for} \quad k = 2, \dots, nr+n-1, \\ b_{nr+n}^* &= b_{nr+n} + A_{2n,1} a_2^{2n,1}. \end{aligned}$$

*The constants $a_1^{2n,1}$, $a_2^{2n,1}$ are given in (23), (24), respectively.*

(B) Role of $A_{2n,0} \in \mathbb{R}$, $n \geq 1$.

Lemma 28. *Let $n \in \mathbb{N}$. Then, the vector field (21) is* C$^\infty$*–conjugate to (22), where*

$$\begin{aligned} a_k^* &= a_k, \quad \textit{for} \quad k = 2, \dots, n+nr+r-2, \\ a_{n+nr+r-1}^* &= a_{n+nr+r-1} + A_{2n,0} a_1^{2n,0}, \\ b_k^* &= b_k, \quad \textit{for} \quad k = 2, \dots, n+nr+s-2, \\ b_{n+nr+s-1}^* &= b_{n+nr+s-1} + A_{2n,0} a_2^{2n,0}, \end{aligned}$$

*where the expressions of $a_1^{2n,0}$, $a_2^{2n,0}$ are given in (25), (26).*

(C) Role of $A_{2n+1,1} \in \mathbb{R}$, $n \geq 1$.

Lemma 29. *Let $n \in \mathbb{N}$. Then, the vector field (21) is $\mathrm{C}^\infty$–conjugate to (22), where*

$$\begin{aligned} a_k^* &= a_k, \quad \textit{for} \quad k = 2, \ldots, n + nr + r - 1, \\ a_{n+nr+r}^* &= a_{n+nr+r} + A_{2n+1,1} a_1^{2n+1,1}, \\ b_k^* &= b_k, \quad \textit{for} \quad k = 2, \ldots, n + nr + s - 1, \\ b_{n+nr+s}^* &= b_{n+nr+s} + A_{2n+1,1} a_2^{2n+1,1}, \end{aligned}$$

*where $a_1^{2n+1,1}$, $a_2^{2n+1,1}$ appear in (27), (28).*

(D) Role of $A_{2n-1,0} \in \mathbb{R}$, $n \geq 2$.

Lemma 30. *Let $n \in \mathbb{N}$. Then, the vector field (21) is $\mathrm{C}^\infty$–conjugate to (22), where*

*(a)If $nr + n - 1 \leq s$, then*

$$\begin{aligned} a_k^* &= a_k, \quad \textit{for} \quad k = 2, \ldots, n + nr + s - 3, \\ a_{n+nr+s-2}^* &= a_{n+nr+s-2} + A_{2n-1,0} a_1^{2n-1,0}, \\ b_k^* &= b_k, \quad \textit{for} \quad k = 2, \ldots, 2n + 2nr - r + s - 4. \end{aligned}$$

*Here, the arbitrary constant $A_{2n-1,0}$ appears for the first time in $b_{2n+2nr-r+s-3}^*$, but in a nonlinear way.*

*(b)If $nr + n - 1 \geq s + 1$, then*

$$\begin{aligned} a_k^* &= a_k, \quad \textit{for} \quad k = 2, \ldots, n + nr + s - 3, \\ a_{n+nr+s-2}^* &= a_{n+nr+s-2} + A_{2n-1,0} a_1^{2n-1,0}, \\ b_k^* &= b_k, \quad \textit{for} \quad k = 2, \ldots, n + nr - r + 2s - 3, \\ b_{n+nr-r+2s-2}^* &= b_{n+nr-r+2s-2} + A_{2n-1,0} a_2^{2n-1,0}. \end{aligned}$$

*The constants $a_1^{2n-1,0}$, $a_2^{2n-1,0}$ are given in (29), (30).*

(E) Statement of the main results.

To obtain the results by applying these lemmas, it is necessary to distinguish different cases, depending on the relation between $r$ and $s$.

THEOREM 31. *Assume that* $s - r \not\equiv 1, 2 \ (mod\ r+1)$.
*Let* $m = \min\{j \in \mathbb{N} : j(r+1) \geq s\}$. *Then, the vector field (21) is* $C^\infty$*–conjugate to (22), where*

*(i)* • $a_k^* = a_k$, *for* $k = 2, \dots, 2r-1$,

•$b_k^* = b_k$, *for* $k = 2, \dots, m(r+1)-1$.

*(ii)For* $n \geq m$, *we can achieve* $b_{n(r+1)}^* = 0$ *whenever* $a_2^{2n,1} \neq 0$.

*(iii)For* $n \geq 1$, *we can get* $a_{n(r+1)+r-1}^* = 0$ *whenever* $a_1^{2n,0} \neq 0$, *or* $b_{n(r+1)+s-1}^* = 0$ *whenever* $a_2^{2n,0} \neq 0$.

*(iv)For* $n \geq 1$, *we can obtain* $a_{n(r+1)+r}^* = 0$ *whenever* $a_1^{2n+1,1} \neq 0$, *or* $b_{n(r+1)+s}^* = 0$ *whenever* $a_2^{2n+1,1} \neq 0$.

*(v)For* $n \geq 2$, *we can obtain* $a_{n(r+1)+s-2}^* = 0$ *whenever* $a_1^{2n-1,0} \neq 0$.

THEOREM 32. *Assume that* $s - r \equiv 1 (mod\ r+1)$, *that is,* $s = r+1+j(r+1)$. *Then, the vector field (21) is* $C^\infty$*–conjugate to (22), where*

*(i)* • $a_k^* = a_k$ *for* $k = 2, \dots, 2r-1$,

•$b_k^* = b_k$ *for* $k = 2, \dots, (j+1)(r+1)-2$.

*(ii)For* $n \geq j+1$, *we can achieve* $b_{n(r+1)}^* = 0$ *whenever* $a_2^{2n,1} \neq 0$.

*(iii)For* $n \geq 1$, *we can get* $a_{n(r+1)+r}^* = 0$ *whenever* $a_1^{2n+1,1} \neq 0$, *or* $b_{(n+j+1)(r+1)}^* = 0$ *whenever* $a_2^{2n+1,1} \neq 0$.

*(iv)For* $n = 1, \dots, 2j+1$, *we can obtain* $a_{n(r+1)+r-1}^* = 0$ *if* $a_1^{2n,0} \neq 0$, *or* $b_{(n+j+1)(r+1)-1}^* = 0$ *whenever* $a_2^{2n,0} \neq 0$.

*(v)For* $n \geq j+2$, *we can obtain* $a_{(n+j)(r+1)+r-1}^* = b_{(n+2j)(r+1)+r}^* = 0$, *whenever* $a_1^{2(n+j),0} a_2^{2n-1,0} - a_2^{2(n+j),0} a_1^{2n-1,0} \neq 0$. *If this expression vanishes, there are two possibilities:* $a_{(n+j)(r+1)+r-1}^* = 0$ *whenever* $a_1^{2(n+j),0} \neq 0$ *or* $a_1^{2n-1,0} \neq 0$; *or* $b_{(n+j)(r+1)+s-1}^* = 0$ *whenever* $a_2^{2(n+j),0} \neq 0$.

THEOREM 33. *Assume that* $s - r \equiv 2 (mod\ r+1)$, *that is* $s = r+2+j(r+1)$. *Then, the vector field (21) is* $C^\infty$*–conjugate to (22), where*

*(i)* • $a_k^* = a_k$, *for* $k = 2, \dots, 2r-1$,

•$b_k^* = b_k$, *for* $k = 2, \dots, (j+2)(r+1)-2$.

*(ii)For $n \geq j+2$, we can achieve $b^*_{n(r+1)} = 0$ whenever $a_2^{2n,1} \neq 0$.*

*(iii)For $n \geq 1$, we can get $a^*_{n(r+1)+r-1} = 0$, whenever $a_1^{2n,0} \neq 0$, or $b^*_{(n+j+1)(r+1)} = 0$ whenever $a_2^{2n,0} \neq 0$.*

*(iv)For $n = 1,\ldots,2j+1$, we can obtain $a^*_{n(r+1)+r} = 0$ if $a_1^{2n+1,1} \neq 0$, or $b^*_{(n+j+1)(r+1)+1} = 0$ whenever $a_2^{2n+1,1} \neq 0$.*

*(v)For $n \geq j+2$, we can obtain $a^*_{(n+j)(r+1)+r} = b^*_{(n+2j+1)(r+1)+1} = 0$, whenever $a_1^{2(n+j)+1,1} a_2^{2n-1,0} - a_2^{2(n+j)+1,1} a_1^{2n-1,0} \neq 0$. If this expression vanishes, there are two possibilities: $a^*_{(n+j)(r+1)+r} = 0$ whenever $a_1^{2(n+j)+1,1} \neq 0$ or $a_1^{2n-1,0} \neq 0$; or $b^*_{(n+j)(r+1)+s} = 0$ whenever $a_2^{2(n+j)+1,1} \neq 0$.*

### 4.2. Case II: $r = 2s - 1$.

In the situation we analyze now, the first nonzero normal form coefficients are $a_{2s-1}$, $b_s$. Next lemmas show how the arbitrary constants appear in the normal form. An outline of its proofs appears in Appendix A.2.

(A) Role of $A_{2n,1}$, $n \geq 1$.

LEMMA 34. *Let $n \in \mathbb{N}$. Then, the vector field (21) is $C^\infty$–conjugate to (22), where*

$$\begin{aligned} a^*_k &= a_k, \quad \textit{for} \ \ k = 2,\ldots,2ns+s-2, \\ a^*_{2ns+s-1} &= a_{2ns+s-1} + A_{2n,1} a_1^{2n,1}, \\ b^*_k &= b_k, \quad \textit{for} \ \ k = 2,\ldots,2ns-1, \\ b^*_{2ns} &= b_{2ns} + A_{2n,1} a_2^{2n,1}, \end{aligned}$$

*where*

$$a_1^{2n,1} = \Pi_1 \sum_{\substack{\mathbf{n}\in\mathcal{I}_l,\mathbf{n}'\sim\mathbf{n} \\ n_j=s-1,n'_j=s;n_j=2s-2;n'_j=2s \\ 2n+|\mathbf{n}|=2ns+s-1,1+|\mathbf{n}'|=2ns+s+1}} \widetilde{f}_{2n,\mathbf{n};1,\mathbf{n}'}, \tag{31}$$

$$a_2^{2n,1} = \Pi_2 \sum_{\substack{\mathbf{n}\in\mathcal{I}_l,\mathbf{n}'\sim\mathbf{n} \\ n_j=s-1,n'_j=s;n_j=2s-2;n'_j=2s \\ 2n+|\mathbf{n}|=2ns,1+|\mathbf{n}'|=2ns+1}} \mathrm{Proj}_{\mathcal{C}_{2ns}} \widetilde{f}_{2n,\mathbf{n};1,\mathbf{n}'}. \tag{32}$$

(B) Role of $A_{2n,0} \in \mathbb{R}$, $n \geq 1$.

Lemma 35. *Let $n \in \mathbb{N}$. Then, the vector field (21) is $C^\infty$–conjugate to (22), where*

$$\begin{aligned} a_k^* &= a_k, \quad \textit{for} \quad k = 2, \ldots, 2ns+2s-3, \\ a_{2ns+2s-2}^* &= a_{2ns+2s-2} + A_{2n,0} a_1^{2n,0}, \\ b_k^* &= b_k, \quad \textit{for} \ \ k = 2, \ldots, 2ns+s-2, \\ b_{2ns+s-1}^* &= b_{2ns+s-1} + A_{2n,0} a_2^{2n,0}, \end{aligned}$$

*where*

$$a_1^{2n,0} = \Pi_1 \sum_{\substack{\mathbf{n}\in\mathcal{I}_l, \mathbf{n}'\sim\mathbf{n} \\ n_j=s-1, n_j'=s; n_j=2s-2; n_j'=2s \\ 2n+|\mathbf{n}|=2ns+2s-2, |\mathbf{n}'|=2ns+2s}} \widetilde{f}_{2n,\mathbf{n};0,\mathbf{n}'}, \tag{33}$$

$$a_2^{2n,0} = \Pi_2 \sum_{\substack{\mathbf{n}\in\mathcal{I}_l, \mathbf{n}'\sim\mathbf{n} \\ n_j=s-1, n_j'=s; n_j=2s-2; n_j'=2s \\ 2n+|\mathbf{n}|=2ns+s-1, |\mathbf{n}'|=2ns+s}} \mathrm{Proj}_{\mathcal{C}_{2ns+s-1}} \widetilde{f}_{2n,\mathbf{n};0,\mathbf{n}'}. \tag{34}$$

(C) Role of $A_{2n+1,1} \in \mathbb{R}$, $n \geq 1$.

Lemma 36. *Let $n \in \mathbb{N}$. Then, the vector field (21) is $C^\infty$–conjugate to (22), where*

$$\begin{aligned} a_k^* &= a_k, \quad \textit{for} \ \ k = 2, \ldots, 2ns+2s-2, \\ a_{2ns+2s-1}^* &= a_{2ns+2s-1} + A_{2n+1,1} a_1^{2n+1,1}, \\ b_k^* &= b_k, \quad \textit{for} \ \ k = 2, \ldots, 2ns+s-1, \\ b_{2ns+s}^* &= b_{2ns+s} + A_{2n+1,1} a_2^{2n+1,1}, \end{aligned}$$

*where*

$$a_1^{2n+1,1} = \Pi_1 \sum_{\substack{\mathbf{n}\in\mathcal{I}_l, \mathbf{n}'\sim\mathbf{n} \\ n_j=s-1, n_j'=s; n_j=2s-2; n_j'=2s \\ 2n+1+|\mathbf{n}|=2ns+2s-1, 1+|\mathbf{n}'|=2ns+2s+1}} \widetilde{f}_{2n+1,\mathbf{n};1,\mathbf{n}'}, \tag{35}$$

$$a_2^{2n+1,1} = \Pi_2 \sum_{\substack{\mathbf{n}\in\mathcal{I}_l, \mathbf{n}'\sim\mathbf{n} \\ n_j=s-1, n_j'=s; n_j=2s-2; n_j'=2s \\ 2n+1+|\mathbf{n}|=2ns+s, 1+|\mathbf{n}'|=2ns+s+1}} \mathrm{Proj}_{\mathcal{C}_{2ns+s}} \widetilde{f}_{2n+1,\mathbf{n};1,\mathbf{n}'}. \tag{36}$$

<u>(D) Role of $A_{2n-1,0} \in \mathbb{R}$, $n \geq 2$.</u>

LEMMA 37. *Let $n \in \mathbb{N}$. Then, the vector field (21) is $\mathrm{C}^\infty$–conjugate to (22), where*

$$\begin{aligned} a_k^* &= a_k, \quad \textit{for} \quad k = 2, \ldots, 2(n+1)s + s - 3, \\ a_{2(n+1)s+s-2}^* &= a_{2(n+1)s+s-2} + A_{2n+1,0} a_1^{2n+1,0}, \\ b_k^* &= b_k, \quad \textit{for} \quad k = 2, \ldots, 2(n+1)s - 2, \\ b_{2(n+1)s-1}^* &= b_{2(n+1)s-1} + A_{2n+1,0} a_2^{2n+1,0}, \end{aligned}$$

*where*

$$a_1^{2n+1,0} = \Pi_1 \sum_{\substack{\mathbf{n} \in \mathcal{I}_l, \mathbf{n}' \sim \mathbf{n} \\ n_j = s-1, n'_j = s; n_j = 2s-2; n'_j = 2s \\ 2n+1+|\mathbf{n}| = 2(n+1)s+s-2, |\mathbf{n}'| = 2(n+1)s+s}} \widetilde{f}_{2n+1,\mathbf{n};0,\mathbf{n}'}, \tag{37}$$

$$a_2^{2n+1,0} = \Pi_2 \sum_{\substack{\mathbf{n} \in \mathcal{I}_l, \mathbf{n}' \sim \mathbf{n} \\ n_j = s-1, n'_j = s; n_j = 2s-2; n'_j = 2s \\ 2n+1+|\mathbf{n}| = 2(n+1)s-1, |\mathbf{n}'| = 2(n+1)s}} \mathrm{Proj}_{\mathcal{C}_{2(n+1)s-1}} \widetilde{f}_{2n+1,\mathbf{n};0,\mathbf{n}'}. \tag{38}$$

<u>(E) Statement of the main result.</u>

The above information yields the following result.

THEOREM 38. *Let us consider the vector field (21), with $a_2 = \cdots = a_{2s-2} = 0$, $a_{2s-1} \neq 0$, $b_2 = \cdots = b_{s-1} = 0$, $b_s \neq 0$. Then, the quoted vector field is $\mathrm{C}^\infty$–conjugate to (22), where*

*(i)* • $a_k^* = a_k$ *for* $k = 2, \ldots, 3s-2$.

•$b_k^* = b_k$ *for* $k = 2, \ldots, 2s-1$.

*(ii)For $n \geq 1$, we can achieve*

•$b_{2ns}^* = 0$ *whenever* $a_2^{2n,1} \neq 0$, *or* $a_{2ns+s-1}^* = 0$ *whenever* $a_1^{2n,1} \neq 0$.

•$b_{2ns+s-1}^* = 0$ *whenever* $a_2^{2n,0} \neq 0$, *or* $a_{2ns+2s-2}^* = 0$ *whenever* $a_1^{2n,0} \neq 0$.

•$b_{2ns+s}^* = 0$ *whenever* $a_2^{2n+1,1} \neq 0$, *or* $a_{2ns+2s-1}^* = 0$ *whenever* $a_1^{2n+1,1} \neq 0$.

•$b_{2ns+2s-1}^* = 0$ *whenever* $a_2^{2n+1,0} \neq 0$, *or* $a_{2ns+3s-2}^* = 0$ *whenever* $a_1^{2n+1,0} \neq 0$.

*The constants* $a_j^{h,i}$ *are defined in (31), (32), (33), (34), (35), (36), (37), (38).*

*Proof.* It is enough to apply successively Lemmas 34, 35, 36, 37, for each $n \geq 1$. ▮

### 4.3. Case III: $r > 2s - 1$.

The last case corresponds to the situation when the first nonzero normal form coefficients are $a_r$, $b_s$, with $r > 2s - 1$. Next, we show how an arbitrary constant $A_{h,0}$ appears in the transformed vector field. The proofs appears in Appendix A.3.

(A) Role of $A_{h,0}$, $h \geq 1$.

LEMMA 39. *Let $h \in \mathbb{N}$, $h \geq 2$. Then, the vector field (21) is* $C^\infty$*–conjugate to (22), where*

$$\begin{aligned} a_k^* &= a_k, \quad \text{for } \ k = 2, \ldots, hs + 2r - 2s - 1, \\ a_{hs+2r-2s}^* &= a_{hs+2r-2s} + A_{h,0} a_1^{h,0}, \\ b_k^* &= b_k, \quad \text{for } \ k = 2, \ldots, hs + r - s - 1, \\ b_{hs+r-s}^* &= b_{hs+r-s} + A_{h,0} a_2^{h,0}, \end{aligned}$$

*where*

$$a_1^{h,0} = \Pi_1 \mathcal{P}(N_{r-1} M N_{r-1} M L_{s-1} \overset{h-2}{\cdots} M L_{s-1} f_{h,0}), \tag{39}$$

$$a_2^{h,0} = \Pi_2 \mathrm{Proj}_{\mathcal{C}_{hs+r-s}} \mathcal{P}(N_{r-1} M L_{s-1} \overset{h-1}{\cdots} M L_{s-1} f_{h,0}). \tag{40}$$

(B) Role of $A_{h,1}$, $h \geq 1$.

LEMMA 40.
*Let $h \in \mathbb{N}$, $h \geq 2$. Then, the vector field (21) is* $C^\infty$*–conjugate to (22), where*

$$\begin{aligned} a_k^* &= a_k, \quad \text{for } \ k = 2, \ldots, hs + r - s - 1, \\ a_{hs+r-s}^* &= a_{hs+r-s} + A_{h,1} a_1^{h,1}, \\ b_k^* &= b_k, \quad \text{for } \ k = 2, \ldots, hs - 1, \\ b_{hs}^* &= b_{hs} + A_{h,1} a_2^{h,1}, \end{aligned}$$

*where*

$$a_1^{h,1} = \Pi_1 \mathcal{P}(N_{r-1} M L_{s-1} \overset{h-1}{\cdots} M L_{s-1} f_{h,1}), \quad (41)$$

$$a_2^{h,1} = \Pi_2 \mathrm{Proj}_{\mathcal{C}_{hs}} \left( L_{s-1} M L_{s-1} \overset{h-1}{\cdots} M L_{s-1} f_{h,1} \right). \quad (42)$$

(E) Statement of the main results.

In this case, we must distinguish two cases, depending on the $r$ and $s$.

THEOREM 41. *Assume that $r-s$ is not a multiple of $s$ ($r-s \neq ms$, for any $m \geq 1$). Then, the vector field (21) is* $C^\infty$*–conjugate to (22), where*

*(i)* • $a_k^* = a_k$, *for* $k = 2, \ldots, r+s-1$.

•$b_k^* = b_k$, *for* $k = 2, \ldots, 2s-1$.

*(ii)For each $h \geq 2$, we can obtain*

•$a_{hs+r-s}^* = 0$ *whenever* $a_1^{h,1} \neq 0$, *or* $b_{hs}^* = 0$ *whenever* $a_2^{h,1} \neq 0$,

•$a_{hs+2r-2s}^* = 0$ *whenever* $a_1^{h,0} \neq 0$, *or* $b_{hs+r-s}^* = 0$ *whenever* $a_2^{h,0} \neq 0$.

*The constants $a_j^{h,i}$ are defined in (39), (40), (41), (42).*

*Proof.* We can write $r = s + ms + \tilde{m}$, with $m \geq 1$ and $0 < \tilde{m} < s$. We apply successively Lemma 40 for $h = 2, \ldots, m+1$. Each time we apply the lemma, we take $A_{h,1}$ in order to annihilate $a_{hs+r-s}^*$ if $a_1^{h,1} \neq 0$, or $b_{hs}^*$ if $a_2^{h,1} \neq 0$. This procedure does not alter the terms annihilated in previous steps.

Next, we apply also Lemma 40 with $h = m+2$ and choose $A_{m+2,1}$ so that $a_{(m+2)s+r-s}^* = 0$ or $b_{(j+2)s}^* = 0$. Using now Lemma 39 with $h = 2$, we can annihilate $a_{2r}^*$ or $b_{r+s}^*$ selecting adequately $A_{2,0}$.

Later, we apply Lemma 40 with $h = m+3$ and Lemma 39 with $h = 3$, and so on. In this way, we fulfill the proof. ▌

The remaining case, considered in the next theorem, can be proved analogously.

THEOREM 42. *Assume that $r-s$ is a multiple of $s$: $r = s + ms$, for some $m \geq 1$. Then, the vector field (21) is* $C^\infty$*–conjugate to (22), where*

*(i)* • $a_k^* = a_k$, *for* $k = 2, \ldots, (2+m)s-1$.

•$b_k^* = b_k$, *for* $k = 2, \ldots, 2s-1$.

*(ii)For each $h = 2, \ldots, m+1$, we can obtain $a_{hs+r-s}^* = 0$ whenever $a_1^{h,1} \neq 0$, or $b_{hs}^* = 0$ whenever $a_2^{h,1} \neq 0$.*

*(iii)For each $h \geq 2$, we can achieve $b^*_{(h+m)s} = a^*_{(h+2m)s} = 0$ whenever $a_1^{h,0} a_2^{h+m,1} - a_2^{h,0} a_1^{h+m,1} \neq 0$. If this expression vanishes, we have two possibilities: $a^*_{(h+2m)s} = 0$ whenever $a_1^{h+m,1} \neq 0$ or $a_1^{h,0} \neq 0$; or $b^*_{(h+m)s} = 0$ whenever $a_2^{h+m,1} \neq 0$ or $a_2^{h,0} \neq 0$. The constants $a_j^{h,i}$ are defined in (39), (40), (41), (42).*

## 5. RECURSIVE COMPUTATION OF THE CONSTANTS $A_J^{H,I}$

The constants $a_j^{h,i}$ that appear in the hypothesis of the theorems in the previous section can be explicitly computed using recursive procedures.

Up to the order we have been able to compute, we have found $a_1^{h,i} \neq 0$ or $a_2^{h,i} \neq 0$ (although we have not proved it, we conjecture that this is always true). So, we have used each arbitrary constant $A_{h,i}$ to annihilate one normal form coefficient, and we can assure that the normal forms obtained are the simplest ones.

To determine the structure of the hypernormal form, it is not necessary to compute such constants, but only to know if they vanish or not. Nevertheless, for the computation of the hypernormal form for a given vector field, we require to know the values of these constants. For details, see the next section where we present some examples.

We will focus in the computation of some constants in the hypothesis of theorems of the above section. The remaining cases can be handled analogously.

LEMMA 43. *Let $a_2^{2n,1}(p) = \Pi_2 \left( \mathrm{Proj}_{\mathcal{C}_{2n+np}} N_p M N_p \overset{n-1}{\cdots} M N_p f_{2n,1}. \right)$, with $n \geq 1$. Then:*

$$a_2^{2n,1}(p) = a_{p+1} \left( (-2n - (n-1)p + 1) A_n - 2B_n \right),$$

*where $A_n$, $B_n$ satisfy the recursive algorithm:*

$$\begin{aligned} A_1 &= 1, \quad B_1 = 1, \\ A_{k+1} &= \alpha_k \left( \left( (pk+2k)(-2n+2k-1) + (p+1) \right) A_k + (-2n+2k-2) B_k \right), \\ B_{k+1} &= \alpha_k \left( (pk+2k+1)(p+1) A_k + (pk+2k+1)(-2n+2k-2) B_k \right), \end{aligned}$$

*and* $\alpha_k = \dfrac{a_{p+1}}{(pk+2k)(pk+2k+1)}$ *for* $k = 1, \ldots, n-1$.

*Proof.* For $k = 0, \ldots, n-1$, let $u_{k+1} = M N_p \overset{k}{\cdots} M N_p f_{2n,1} = \begin{pmatrix} A_{k+1} \\ B_{k+1} \end{pmatrix} \in \mathcal{H}_{2n+kp;2n-1-2k}$. As

$$\begin{aligned} a_2^{2n,1}(p) &= \Pi_2 \left( \mathrm{Proj}_{\mathcal{C}_{2n+np}} N(p,1) \circ u_n \right) = \\ &= a_{p+1} \Pi_2 \left( \mathrm{Proj}_{\mathcal{C}_{2n+np}} \begin{pmatrix} -A_n \\ (p+1)A_n - 2B_n \end{pmatrix} \right), \end{aligned}$$

we get $a_2^{2n,1}(p) = a_{p+1}\left((-2n-(n-1)p+1)A_n - 2B_n\right)$.

It is enough to note that $u_n$ can be computed recursively by:

$$u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad u_{k+1} = M(k(p+2)) \circ N(p, 2n-1-2(k-1)) \circ u_k,$$

to derive the algorithm that appears in the statement of the lemma. ▮

*Remark:* The constant $a_2^{2n,1}$ that appears in the statement of Theorems 26, Theorems 31, 32, 33, is just $a_2^{2n,1}(r-1)$ computed in the above lemma. Also, the constant $a_2^{2n,1}$ of Theorem 38 agrees with $a_2^{2n,1}(2s-2)$.

Next lemma can be proved as the above one:

LEMMA 44. *Let* $a_1^{2n+1,1}(p) = \Pi_1\left(N_p M N_p \overset{n}{\cdots} M N_p f_{2n+1,1}\right)$, *with* $n \geq 1$. *Then:*

$$a_1^{2n+1,1}(p) = a_{p+1}((p+1)A_{n+1} - B_{n+1}),$$

*where* $A_{n+1}$, $B_{n+1}$ *are given by the recursive algorithm:*

$$\begin{aligned} A_1 &= 1, \ B_1 = 1, \\ A_{k+1} &= \alpha_k\left(((pk+2k)(-2n+2k-2)+(p+1))A_k + (-2n+2k-3)B_k\right), \\ B_{k+1} &= \alpha_k\left((pk+2k+1)(p+1)A_k + (pk+2k+1)(-2n+2k-3)B_k\right), \end{aligned}$$

*being* $\alpha_k = \dfrac{a_{p+1}}{(pk+2k)(pk+2k+1)}$ *for* $k = 1, \ldots, n$.

*Remark:* In this case, the constant $a_1^{2n+1,1}$ of the statement of Theorems 26, 31, 32, 33, corresponds with $a_1^{2n+1,1}(r-1)$. Moreover, $a_1^{2n+1,1}(2s-2)$ agrees with the constant $a_1^{2n+1,1}$ of Theorem 38.

LEMMA 45. *Let* $a_1^{2n+1,0}(p,q) = \Pi_1 \mathcal{P}(L_q M N_p \overset{n+1}{\cdots} M N_p f_{2n+1,0})$, *with* $n \geq 1$. *Then:*

$$a_1^{2n+1;0}(p,q) = A_0 + A_1 + \cdots + A_n + A_{n+1},$$

*with:*

*(a)*$A_{n+1} = -b_{q+1} A_{n+1}^{(1)}$, *where* $A_{n+1}^{(1)}$ *is given by the recursive algorithm:*

$$\begin{aligned} A_1^{(1)} &= -(2n+1)a_{p+1}, \ B_1^{(1)} = (p+1)a_{p+1}, \qquad (43) \\ A_{k+1}^{(1)} &= \alpha_k^{(1)}\left((pk+2k-1)(-2n+2k-1)A_k^{(1)} + (-2n+2k-1)B_k^{(1)}\right), \\ B_{k+1}^{(1)} &= \alpha_k^{(1)}((pk+2k-1)(p+1)A_k^{(1)} + \\ &\quad +(p+1+(pk+2k)(-2n+2k-2))B_k^{(1)}), \end{aligned}$$

*being* $\alpha_k^{(1)} = \dfrac{a_{p+1}}{(pk+2k-1)(pk+2k)}$ *for* $k = 1, \ldots, n$.

*(b)For each* $j = 0, \ldots, n$, *we have* $A_j = B_{n+1}^{(2)}$, *where* $B_{n+1}^{(2)}$ *is obtained from the recursive algorithm:*

$$\begin{aligned} A_j^{(2)} &= b_{q+1}(-2n+2j-1)A_j^{(3)}, \quad B_j^{(2)} = b_{q+1}\left(pA_j^{(3)} + (-2n+2j-1)B_j^{(3)}\right), \\ A_{k+1}^{(2)} &= \alpha_k^{(2)}\left((pk+2k+q)(-2n+2k)A_k^{(2)} + (-2n+2k)B_k^{(2)}\right), \qquad (44) \\ B_{k+1}^{(2)} &= \alpha_k^{(2)}((pk+2k+q)(p+1)A_k^{(2)} + \\ &\quad +(p+1+(pk+2k+q+1)(-2n+2k-1))B_k^{(2)}), \end{aligned}$$

*with* $\alpha_k^{(2)} = \dfrac{a_{p+1}}{(pk+2k+q+1)(pk+2k+q)}$ *for* $k = j, \ldots, n$; *and* $A_j^{(3)}$, $B_j^{(3)}$ *satisfy:*

$$\begin{aligned} A_0^{(3)} &= 1, \quad B_0^{(3)} = 0, \qquad (45) \\ A_{k+1}^{(3)} &= \alpha_k^{(3)}(((p(k+1)+2k+1)(-2n+2k-1)+p+1)A_k^{(3)} + \\ &\quad +(-2n+2k-2)B_k^{(3)}), \\ B_{k+1}^{(3)} &= \alpha_k^{(3)}((p(k+1)+2k+2)(p+1)A_k^{(3)} + \\ &\quad +(p(k+1)+2k+2)(-2n+2k-2)B_k^{(3)}), \end{aligned}$$

*where* $\alpha_k^{(3)} = \dfrac{a_{p+1}}{(p(k+1)+2k+1)(p(k+1)+2k+2)}$ *for* $k = 0, \ldots, j-1$.

*Proof.* Observe that

$$\mathcal{P}(L_qMN_p \overset{n+1}{\cdots} MN_pf_{2n+1,0}) = \sum_{j=0}^{n+1} N_pM \overset{n-j+1}{\cdots} N_pML_qMN_p \overset{j}{\cdots} MN_pf_{2n+1,0}.$$

So, taking $A_j = \Pi_1(N_pM \overset{n-j+1}{\cdots} N_pML_qMN_p \overset{j}{\cdots} MN_pf_{2n+1,0})$ for $j = 0, \ldots, n+1$, we can write $a_1^{2n+1,0}(p,q) = A_0 + \cdots + A_{n+1}$.

To compute $A_j$, we deal separately with the cases $j = n+1$ and $j \neq n+1$ (see item (3) of Lemma 4).

*(a)* If $j = n+1$, then

$$A_{n+1} = \Pi_1\left(L_qM\mathrm{Proj}_{\mathcal{R}_{2n+1+(n+1)p}}N_pMN_p \overset{n}{\cdots} MN_pf_{2n+1,0}\right).$$

Denote

$$u_k^{(1)} = N_pMN_p \overset{k-1}{\cdots} MN_pf_{2n+1,0} = \begin{pmatrix} A_k^{(1)} \\ B_k^{(1)} \end{pmatrix} \in \mathcal{H}_{2n+1+kp;2n-2(k-1)},$$

for $k = 1, \ldots, n+1$. Then,

$$u_1^{(1)} = a_{p+1} \begin{pmatrix} -(2n+1) \\ p+1 \end{pmatrix}, \tag{46}$$
$$u_{k+1}^{(1)} = N(p, 2n+1-2k) \circ M(k(p+2)-1) \circ u_k^{(1)}.$$

To derive the algorithm (43), it is enough to observe that

$$\begin{aligned} A_{n+1} &= \Pi_1 \Big( L(q,-1) \circ M(2n+1+(n+1)p) \circ \mathrm{Proj}_{\mathcal{R}_{2n+1+(n+1)p}} u_{n+1}^{(1)} \Big) = \\ &= -b_{q+1} A_{n+1}^{(1)}, \end{aligned}$$

where $A_{n+1}^{(1)}$ are obtained from (46).

*(b)* To obtain $A_j$ $(j = 0, \ldots, n)$, we define

$$u_k^{(3)} = MN_p \overset{k}{\cdots} MN_p f_{2n+1,0} = \begin{pmatrix} A_k^{(3)} \\ B_k^{(3)} \end{pmatrix} \in \mathcal{H}_{2n+1+kp;2n+1-2k},$$

for $k = 0, \ldots, j$. Once we have $u_j^{(3)}$, we build $u_j^{(2)} = L_q u_j^{(3)}$, and then

$$u_k^{(2)} = N_p M \overset{k-j}{\cdots} N_p M u_j^{(2)} = \begin{pmatrix} A_k^{(2)} \\ B_k^{(2)} \end{pmatrix} \in \mathcal{H}_{2n+1+kp+q;2n+1-2k},$$

for $k = j+1, \ldots, n+1$. Notice that $A_j = \Pi_1 u_{n+1}^{(2)} = B_{n+1}^{(2)}$. Moreover:

*(1)* $u_0^{(3)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $u_{k+1}^{(3)} = M((k+1)(p+2)-1) \circ N(2n+1-2k) \circ u_k^{(3)}$, for $k = 0, \ldots, j-1$. These relations provide the algorithm (45).

*(2)* $u_j^{(2)} = b_{q+1} \begin{pmatrix} (-2n+2j-1) A_j^{(3)} \\ p A_j^{(3)} + (-2n+2j-1) B_j^{(3)} \end{pmatrix}$, and $u_{k+1}^{(2)} = N(p, 2n-2k) \circ M(k(p+2)+q) \circ u_k^{(2)}$, for $k = j, \ldots, n$. The algorithm (44) is derived from these relations.

∎

*Remark:* Taking $p = r-1$ and $q = s-1$ in the above lemma, we obtain the constant $a_1^{2n+1,0}$ of Theorems 26, 31, 32, 33. Also, with $p = 2s-2$ and $q = s-1$ we get the constant $a_1^{2n+1,0}$ of Theorem 38.

In the same way, we can obtain recursive algorithms to compute the constants that appear in the hypothesis of the remaining theorems of the above section.

## 6. EXAMPLES

In this last section, we will obtain some hypernormal forms. The reason for including this section is not only to describe the structure of the hypernormal form in some specific cases. The main objective is to show how our approach is useful to compute the expressions for the hypernormal form coefficients.

We will assume that the system (1) has been put previously in normal form. Then, we deal with the vector field (4), and our conditions will be given on the normal form coefficients $a_n$, $b_n$. The recursive computation of these coefficients is addressed in Gamero et al. [7].

**Example 1:** Case I.1.

Consider the vector field (4), with $a_2 = a_3 = a_4 = a_5 = 0$, $a_6 \neq 0$, $b_2 = b_3 = 0$, $b_4 \neq 0$. Among the possibilities presented in Theorem 26 we will consider here one obtained by using items (a-2), (b-1) Different hypernormal forms can be obtained using other items.

We have computed the constants $a_j^{h,i}$ (which appear in the items (a-2), (b-1) of Theorem 26) for $h = 1, \ldots, 20$, and they are nonzero. Then, a hypernormal form up to order 146 is

$$v^* = y\partial_x + \left( a_9^* x^9 + \sum_{\substack{n=6 \\ n\neq 7k+2}}^{146} a_n^* x^n + \sum_{\substack{n=4 \\ n\neq 7k, n\neq 7k+3, n\neq 7k+4}}^{146} b_n^* x^{n-1} y \right) \partial_y. \quad (47)$$

Moreover, the expressions for the coefficients up to order 11, are:

$$\begin{aligned}
b_4^* &= b_4, \quad b_5^* = b_5, \quad b_6^* = b_6, \quad a_6^* = a_6, \quad a_7^* = a_7, \\
a_8^* &= a_8, \quad b_8^* = b_8 - \left(10a_7b_7 + b_4^2 b_7\right) / \left(10a_6\right), \\
a_9^* &= a_9, \quad b_9^* = b_9 - \left(5a_8b_7 + b_4b_5b_7\right) / 5a_6, \\
a_{10}^* &= a_{10} - \left(b_4b_7\right) / 30, \quad a_{11}^* = a_{11} - \left(a_7b_4b_7\right) / \left(15a_6\right),
\end{aligned}$$

The expressions for the coefficients of order greater than 11 are omitted for the sake of brevity.

We summarize below the procedure used in this example (see Section 4):

First, we compute the generator which depends only on $A_{2,1}$. We denote $U_k^{[j]}$ the part of the $k$-degree homogeneous terms of this generator depending on the power $j$ of $A_{2,1}$. Obviously, $U_k^{NL} = \sum_{j\geq 2} U_k^{[j]}$.

The expressions for $U_k^L$ (see (16) with $A_{h,i} = 0$, $h \neq 2$, $i \neq 1$) are:

$$U_1^L = 0, \quad U_2^L = A_{2,1} f_{2,1}, \quad U_3^L = U_4^L = 0.$$

In the following, we will denote with $f_{2;1} = A_{2,1} f_{2,1} \in \mathcal{H}_{2;1}$. With this notation,

$$U_5^L = ML_3 f_{2;1}, \quad U_6^L = ML_4 f_{2;1}, \quad U_7^L = (MN_5 + ML_5) f_{2;1}.$$

Taking $U_k^{NL} = 0$ for $k = 1, \ldots, 7$, we have $\mathcal{J}^7 v^* = \mathcal{J}^7 v + \widetilde{v}_7$, where

$$\widetilde{v}_7 = \mathrm{Proj}_{\mathcal{C}_7} N_5 f_{2;1} = -3a_6 A_{2,1} x^6 y \partial_y.$$

In the sequel, we denote $\widetilde{L}$, $\widetilde{N}$ the operators associated to the vector field $\widetilde{v}$.

One can see that, if dif $([[v_j, U_l], U_k]) \leq 0$ (i. e., its projection onto the co-range may be nonzero), then its order is at least 11. Consequently, $A_{2,1}^2$ appears in $b_{11}^*$ and in the higher order terms of $v^*$. In the same way, looking at the triple Lie products, we can show that $A_{2,1}^3$ appears in $b_{14}^*$ and in the higher order terms of $v^*$. Also, $A_{2,1}^4$ appears in the terms of $v^*$ of order greater than 18, and so on.

As we are interested in the transformed vector field up to order 11, we neglect the terms of order greater than 2 in $A_{2,1}$. So, we do not consider the Lie products involving more than 2 functions, neither $U_k^{[j]}$ with $j \geq 2$ in the generator.

Taking $U_8 = U_8^L + U_8^{[2]}$, with $U_8^{[2]} = \dfrac{1}{2} M\widetilde{L}_6 f_{2;1}$, we find $v_8^* = v_8 + \widetilde{v}_8$, where $\widetilde{v}_8 = \mathrm{Proj}_{\mathcal{C}_8}(N_6 + L_3 M L_3) f_{2;1} = (-3a_7 - \frac{3}{10} b_4^2) A_{2,1} x^7 y \partial_y$.

The terms of order 9 in the generator are $U_9 = U_9^L + U_9^{[2]} + \mathcal{O}(3)$, where $U_9^{[2]} = \frac{1}{2} M\widetilde{L}_7 f_{2;1}$. So, we have $v_9^* = v_9 + \widetilde{v}_9$, with $\widetilde{v}_9 = \mathrm{Proj}_{\mathcal{C}_9}\left(N_7 + \mathcal{P}(L_3 M L_4)\right) f_{2;1}$, and then $v_9^* = v_9 + (-3a_8 - \frac{3}{5} b_4 b_5) A_{2,1} x^8 y \partial_y$.

The terms of order 10 in the generator are $U_{10} = U_{10}^L + U_{10}^{[2]} + \mathcal{O}(3)$, with $U_{10}^{[2]} = \frac{1}{2} M\widetilde{L}_8 f_{2;1}$. Then, $v_{10}^* = v_{10} + \widetilde{v}_{10}$, where

$$\widetilde{v}_{10} = \mathcal{P}(N_5 M L_3) f_{2;1} + \mathrm{Proj}_{\mathcal{C}_{10}}\left(N_8 + \mathcal{P}(L_3 M L_5) + L_4 M L_4\right) f_{2;1},$$

and consequently

$$v_{10}^* = v_{10} + \left(-\tfrac{1}{10} a_6 b_4 A_{2,1} x^{10} + (-3a_9 - \tfrac{9}{14} b_4 b_6 - \tfrac{4}{15} b_5^2) A_{2,1} x^9 y\right) \partial_y.$$

Reasoning analogously, we get $U_{11} = U_{11}^L + U_{11}^{[2]} + \mathcal{O}(3)$, and $v_{11}^* = v_{11} + \widetilde{v}_{11} + \widetilde{v}_{11}^{[2]}$ ($\widetilde{v}_k^{[2]}$ denotes the part of $\widetilde{v}_k$ depending on the square of$A_{2,1}$), with

$$\begin{aligned}
\widetilde{v}_{11} &= \mathcal{P}(N_6 M L_3) f_{2;1} + \mathrm{Proj}_{\mathcal{C}_{11}}\left(N_9 + \mathcal{P}(L_3 M L_6) + \mathcal{P}(L_4 M L_5)\right) f_{2;1}, \\
\widetilde{v}_{11}^{[2]} &= \frac{1}{2} \mathrm{Proj}_{\mathcal{C}_{11}}\left(\widetilde{N}_9 + \mathcal{P}(L_3 M \widetilde{L}_6)\right) f_{2;1}.
\end{aligned}$$

After some computations, we get

$$\begin{aligned}
v_{11}^* &= v_{11} + ((-\tfrac{1}{5} a_7 b_4 A_{2,1}) x^{11} + \\
&\quad + ((-3a_{10} - \tfrac{11}{21} b_5 b_6 - \tfrac{99}{140} b_4 b_7) A_{2,1} + \tfrac{339}{280} a_6 b_4 A_{2,1}^2) x^{10} y) \partial_y.
\end{aligned}$$

In summary, the transformed vector field up to order 11 is

$$v^* = y\partial_x + \sum_{n=2}^{11} (a_n^* x^n + b_n^* x^{n-1} y)\partial_y, \tag{48}$$

with $a_n^* = a_n$ for $n = 2, \ldots, 9$, $b_n^* = b_n$ for $n = 2, \ldots, 6$, and

$$\begin{array}{l} b_7^* = b_7 - 3a_6 A_{2,1}, \quad b_8^* = b_8 - (3a_7 - \frac{3}{10} b_4^2) A_{2,1}, \\ b_9^* = b_9 + (-3a_8 - \frac{3}{5} b_4 b_5) A_{2,1}, \quad a_{10}^* = a_{10} - \frac{1}{10} a_6 b_4 A_{2,1}, \\ b_{10}^* = b_{10} + (-3a_9 - \frac{9}{14} b_4 b_6 - \frac{4}{15} b_5^2) A_{2,1}, \quad a_{11}^* = a_{11} - \frac{1}{5} a_7 b_4 A_{2,1}, \\ b_{11}^* = b_{11} + (-3a_{10} - \frac{11}{21} b_5 b_6 - \frac{99}{140} b_4 b_7) A_{2,1} + \frac{339}{280} a_6 b_4 A_{2,1}^2. \end{array}$$

Now, we consider a generator depending only on $A_{2,0}$. As before, we obtain another transformed vector field $v^{**}$, which agrees with $v^*$ up to order 9, and

$$\begin{array}{rcl} v_{10}^{**} &=& v_{10}^* + \mathrm{Proj}_{\mathcal{C}_{10}} \mathcal{P}(N_5 M L_3) f_{2;0}, \\ v_{11}^{**} &=& v_{11}^* + \mathrm{Proj}_{\mathcal{C}_{11}} \left( \mathcal{P}(N_5 M L_4) + \mathcal{P}(N_6 M L_3) \right) f_{2;0}, \end{array}$$

where $f_{2;0} = A_{2,0} f_{2,0}$. The coefficients of $v^{**}$ satisfy $a_n^{**} = a_n^*$ for $n = 2, \ldots, 11$, $b_n^{**} = b_n^*$ for $n = 2, \ldots, 9$, and

$$\begin{array}{rcl} b_{10}^{**} &=& b_{10}^* - \frac{10}{7} a_6^* b_4^* A_{2,0}, \\ b_{11}^{**} &=& b_{11}^* + (-\frac{11}{8} a_7^* b_4^* - \frac{11}{7} a_6^* b_5^*) A_{2,0}. \end{array}$$

Finally, we consider a generator depending only on $A_{3,1}$. The transformed vector field $v^{***}$ agrees with $v^{**}$ up to order 10, and $v_{11}^{***} = v_{11}^{**} + \mathrm{Proj}_{\mathcal{C}_{11}} \mathcal{P}(N_5 M L_3) f_{3;1}$, where $f_{3;1} = A_{3,1} f_{3,1}$. So, its coefficients satisfy $a_n^{***} = a_n^{**}$ for $n = 2, \ldots, 11$, $b_n^{***} = b_n^{**}$ for $n = 2, \ldots, 10$, and

$$b_{11}^{***} = b_{11}^{**} + \frac{37}{112} a_6^{**} b_4^{**} A_{3,1}.$$

Taking $A_{2,1}$, $A_{2,0}$, $A_{3,1}$ adequately, we obtain the hypernormal form (47) up to order 11.

In the remaining cases considered in this last section, one can proceed analogously, but for the sake of brevity we only present the hypernormal form as well as the expressions for the first coefficients.

**Example 2:** Case I.2.

We will assume now that the second order normal form coefficients are nonzero ($a_2, b_2 \neq 0$).

We plain to apply Theorem 31. We have computed the constants $a_1^{h,i}$, $a_2^{h,i}$ for $h = 2, \ldots, 60$, and they are nonzero. Then, the vector field (21) is $\mathcal{C}^\infty$–conjugate, up to order 93, to

$$y\partial_x + \left(a_2^* x^2 + b_2^* xy + a_3^* x^3 + \sum_{j=1}^{30} (b_{3j+1}^* x^{3j} y + b_{3j+2}^* x^{3j+1} y)\right) \partial_y.$$

where the first coefficients are:

$$\begin{aligned} a_2^* &= a_2, \quad b_2^* = b_2, \quad a_3^* = a_3, \\ b_4^* &= b_4 - \left(b_2^2 b_3 + 4a_4 b_2 + 5a_3 b_3\right) / \left(5a_2\right), \\ b_5^* &= b_5 - \left(10a_2 a_4 b_3 + 81a_3 a_4 b_2 + 9a_3 b_2^2 b_3 - 35a_2 a_5 b_2\right) / \left(10a_2^2\right). \end{aligned}$$

Also, we have computed the expressions for the coefficients up to order 14, but they are not included here for the sake of brevity.

**Example 3:** Case II.

We will consider now that the coefficients of the vector field (4) satisfy $a_2 = 0$, $a_3 \neq 0$, $b_2 \neq 0$. In the present case, we can show that the constants that appear in the hypothesis of Theorem 38 have an special structure (they are polynomials in $a_3$, $b_2$):

*(1)* $a_1^{2n,1} = b_2(\alpha_{1,n} a_3^n + \alpha_{2,n} a_3^{n-1} b_2^2 + \cdots + \alpha_{n,n} a_3 b_2^{2n-2})$. The first ones are

$$\begin{aligned} a_1^{2,1} &= 0, \\ a_1^{4,1} &= b_2(\tfrac{19}{21} a_3^2 + \tfrac{2}{7} b_2^2), \\ a_1^{6,1} &= b_2(-\tfrac{237}{220} a_3^3 - \tfrac{13}{220} a_3^2 b_2^2 + \tfrac{3}{22} a_3 b_2^4), \\ a_1^{8,1} &= b_2(\tfrac{11317}{12320} a_3^4 - \tfrac{12517}{22176} a_3^3 b_2^2 - \tfrac{5287}{18480} a_3^2 b_2^4 + \tfrac{1}{20} a_3 b_2^6). \end{aligned}$$

*(2)* $a_1^{2n-1,0} = b_2(\alpha_{1,n} a_3^n + \alpha_{2,n} a_3^{n-1} b_2^2 + \cdots + \alpha_{n-1,n} a_3^2 b_2^{2n-4})$.
*(3)* $a_2^{2n-1,0} = \alpha_{1,n} a_3^{n-1} b_2^2 + \alpha_{2,n} a_3^{n-1} b_2^2 + \cdots + \alpha_{n-1,n} a_3 b_2^{2(n-1)}$.

In the three items above, we have proved that, for each $n = 2, \ldots, 10$, there exists $j$ such that $\alpha_{j,n} \neq 0$. Consequently, we find that the inequalities $a_1^{2n,1} \neq 0$, $a_2^{2n-1,0} \neq 0$, $a_1^{2n-1,0} \neq 0$ are true generically for $n = 2, \ldots, 10$, and each arbitrary constant can be used to annihilate one normal form coefficient.

*(4)* $a_1^{2n,0} = \alpha_{1,n} a_3^{n+1} + \alpha_{2,n} a_3^n b_2^2 + \cdots + \alpha_{n,n} a_3^2 b_2^{2n-2}$.
*(5)* $a_1^{2n+1,1} = \alpha_{1,n} a_3^{n+1} + \alpha_{2,n} a_3^{n-1} b_2^2 + \cdots + \alpha_{n+1,n} a_3 b_2^{2n}$.

*(6)* $a_2^{2n,1} = \alpha_{1,n}a_3^n + \alpha_{2,n}a_3^{n-1}b_2^2 + \cdots + \alpha_{n+1,n}b_2^{2n}$.
*(7)* $a_2^{2n,0} = b_2(\alpha_{1,n}a_3^n + \alpha_{2,n}a_3^{n-1}b_2^2 + \cdots + \alpha_{n,n}b_2^{2n-2})$.
*(8)* $a_2^{2n+1,1} = b_2(\alpha_{1,n}a_3^n + \alpha_{2,n}a_3^{n-1}b_2^2 + \cdots + \alpha_{n+1,n}b_2^{2n})$.

In the items (4)-(8), we have found that for each $n = 1,\ldots,10$, there exists $j$ such that $\alpha_{j,n} \neq 0$. Consequently, the inequalities $a_1^{2n,0} \neq 0$, $a_1^{2n+1,1} \neq 0$, $a_2^{2n,1} \neq 0$, $a_2^{2n,0}$, $a_2^{2n+1,1} \neq 0$, hold generically for $n = 1,\ldots,10$.

To apply Theorem 38, we will assume that $a_2^{2,1} \neq 0$; $a_1^{2n,1} \neq 0$ and $a_1^{2n-1,0} \neq 0$ for $n = 2,\ldots,10$; $a_1^{2n,0} \neq 0$ and $a_1^{2n+1,1} \neq 0$ for $n = 1,\ldots,10$. Then, we obtain the following hypernormal form up to order 43:

$$v^* = y\partial_x + \left(a_3^*x^3 + a_4^*x^4 + a_5^*x^5 + \sum_{n=2,n\neq 4}^{43} b_n^*x^{n-1}y\right)\partial_y.$$

Moreover,

$$\begin{aligned} b_2^* &= b_2, \quad a_3^* = a_3, \quad b_3^* = b_3, \quad a_4^* = a_4, \quad a_5^* = a_5, \\ b_5^* &= b_5 - (20a_3b_2b_3b_4 + 5a_6b_2^3 + 45a_3a_6b_2 - \\ &\quad -5a_4b_2^2b_4 + 63a_3a_4b_4)/\left(9a_3^2 + a_3b_2^2\right). \end{aligned}$$

We remark that there is one possibility of Theorem 38 which can not be used, because $a_1^{2,1} = 0$ (i. e., the normal form coefficient $a_5^*$ can not be annihilated). Instead, we have used the arbitrary constant $A_{2,1}$ to annihilate $b_4^*$.

Anyway, we can get different hypernormal forms by applying different items of Theorem 38. For instance, assuming the generic conditions $a_2^{2n,0} \neq 0$, $a_2^{2n,1} \neq 0$ for $n = 1,\ldots,10$, and also $a_2^{2n+1,0} \neq 0$, $a_2^{2n+1,1} \neq 0$ for $n = 1,\ldots,9$, we obtain the following hypernormal form up to order 41:

$$v^* = y\partial_x + \left(b_2^*xy + b_3^*x^2y + + \sum_{n=3}^{41} a_n^*x^n\right)\partial_y.$$

This hypernormal form, up to order 12, is considered in Yuan & Yu [13].

**Example 4:** Case III.

Finally, we will consider $a_2 = a_3 = a_4 = 0$, $a_5 \neq 0$, $b_2 \neq 0$. We have computed the constants that appear in the hypothesis of Theorem 41. They satisfy: $a_1^{3,1} = 0$, $a_1^{h,1} \neq 0$ for $h = 2,\ldots,20$, $h \neq 3$ and $a_1^{h,0}, a_2^{h,1}, a_2^{h,0} \neq 0$ for $h = 2,\ldots,20$. Hence, a hypernormal form up to order 43 is:

$$v^* = y\partial_x + \left(b_2^*xy + b_3^*x^2y + b_5^*x^4y + \sum_{n=5}^{43} a_n^*x^n\right)\partial_y,$$

and the expressions for the first coefficients are:

$$\begin{aligned} b_2^* &= b_2, \quad b_3^* = b_3, \quad a_5^* = a_5, \\ b_5^* &= b_5 - 5b_3b_4/\left(2b_2\right), \quad a_6^* = a_6, \\ a_7^* &= a_7 - 2a_5b_4/b_2, \quad a_8^* = a_8 - \left(6a_6b_2b_4 + a_5b_3b_4\right)/\left(2b_2^2\right). \end{aligned}$$

## 7. FURTHER SIMPLIFICATIONS BY REPARAMETRIZING TIME

In this last section, we will show the improvements that can be achieved if we use $\mathcal{C}^\infty$-equivalence.

Our goal is to show how the field (21) can be transformed into (22) by using not only coordinate transformations, but also reparametrizing the time. The effect of such reparametrization is to multiply the vector by a local nonzero function of the state variables.

We will present without proofs two particular cases (corresponding to examples 2,3 in the previous section) to show the refinements that one can achieve.

Notice that we present several possibilities in each case for the simplified normal form under $\mathcal{C}^\infty$-equivalence. One of them agrees with the one obtained by Loray [9], Strózyna & Zoladek [10].

### Case I: $a_2 \neq 0, \ b_2 \neq 0$.

THEOREM 46. *Assume $a_2 \neq 0$, $b_2 \neq 0$. Then, (21) is $\mathcal{C}^\infty$–equivalent to (22), where*

*1.* $a_3^* = a_5^* = b_5^* = a_6^* = b_6^* = 0$.

*2. We can choose one of the following statements:*

*(2-a)* $b_3^* = a_4^* = 0$,

*(2-b)* $b_3^* = b_4^* = 0$,

*(2-c)* $b_4^* = a_4^* = 0$.

*3. For all $n \geq 3$, we can obtain*

*(3-a) if $A_n \neq 0$, or $B_n \neq 0$, then $a_{3n}^* = b_{3n}^* = 0$,*

*(3-b) if $A_n = B_n = 0$, then $a_{3n}^* = 0$ or $b_{3n}^* = 0$.*

*4. For all $n \geq 2$, we can obtain*

*(4-a) if* $C_n \neq 0$, *then* $a^*_{3n+1} = b^*_{3n+1} = 0$,

*(4-b) if* $C_n = 0$, *then* $a^*_{3n+1} = 0$ *or* $b^*_{3n+1} = 0$.

*5.For all* $n \geq 2$, *we can obtain*

*(5-a) if* $D_n \neq 0$, *then* $a^*_{3n+2} = b^*_{3n+2} = 0$,

*(5-b) if* $D_n = 0$, *then* $a^*_{3n+2} = 0$ *or* $b^*_{3n+2} = 0$,

*where*

$$\begin{aligned} A_n &= a_2 a_2^{2n,1}, \\ B_n &= (3n+1)a_2 a_2^{2n-1,0} - 3n b_2 a_1^{2n-1,0}, \\ C_n &= (3n+2)a_2 a_2^{2n,0} - (3n+1) b_2 a_1^{2n,0}, \\ D_n &= (3n+3)a_2 a_2^{2n+1,1} - (3n+2) b_2 a_1^{2n+1,1}, \end{aligned}$$

*and the constants* $a_j^{h,i}$ *agree with those computed in Section 5 (recall that they only depend on* $a_2$, $b_2$*).*

*Remark:* Using the symbolic program *MapleV*, we have checked that $A_n \neq 0$, for $3 \leq n \leq 30$; $C_n = 0$, for $2 \leq n \leq 30$; and $D_n \neq 0$, for $2 \leq n \leq 30$. Then, a 93th-order hypernormal form using $C^\infty$–equivalence,assuming $a_2, b_2 \neq 0$, is given by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= a_2^* x^2 + b_2^* xy + b_3^* x^2 y + \sum_{j=2}^{30} b^*_{3j+1} x^{3j} y. \end{aligned}$$

There are different possibilities, for instance

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= a_2^* x^2 + b_2^* xy + \sum_{j=1}^{30} b^*_{3j+1} x^{3j} y. \end{aligned}$$

Moreover, our approach allows us to provide the expressions for the hypernormal form coefficients. For instance, in this last system, they are given by:

$$\begin{aligned} a_2^* &= a_2, \\ b_2^* &= b_2, \\ b_4^* &= \left(-40a_2a_3b_3 - 32a_2a_4b_2 + 35{a_3}^2b_2 - 8a_2{b_2}^2b_3 + 6a_3{b_2}^3 + 40{a_2}^2b_4\right)/40{a_2}^2. \end{aligned}$$

## Case II: $a_2 = 0,\ a_3 \neq 0,\ b_2 \neq 0$.

THEOREM 47. *Assume $a_2 = 0,\ a_3 \neq 0,\ b_2 \neq 0$. Then, (21) is $\mathcal{C}^\infty$–equivalent to (22), where*

*1.$a_2^* = 0$, $b_2^* = b_2$, $a_3^* = a_3$,*
*2.$b_3^* = 0$ or $a_4^* = 0$,*
*3.for $n \geq 1$, we can get:*

*(3-a) if $A_n \neq 0$, then $b_{4n}^* = a_{4n+1}^* = 0$,*
*(3-b) if $A_n = 0$, then $b_{4n}^* = 0$ or $a_{4n+1}^* = 0$,*

*4.for $n \geq 1$, we can get:*

*(4-a) if $B_n \neq 0$, then $b_{4n+1}^* = a_{4n+2}^* = 0$,*
*(4-b) if $B_n = 0$, then $b_{4n+1}^* = 0$ or $a_{4n+2}^* = 0$,*

*5.for $n \geq 1$, we can get:*

*(5-a) if $C_n \neq 0$, then $b_{4n+2}^* = a_{4n+3}^* = 0$,*
*(5-b) if $C_n = 0$, then $b_{4n+2}^* = 0$ or $a_{4n+3}^* = 0$,*

*6.for $n \geq 1$, we can get:*

*(6-a) if $D_n \neq 0$, then $b_{4n+3}^* = a_{4n+4}^* = 0$,*
*(6-b) if $D_n = 0$, then $b_{4n+3}^* = 0$ or $a_{4n+4}^* = 0$.*

*Here,*

$$\begin{aligned} A_n &= (4n+2)a_3a_2^{2n,1} - 4nb_2a_1^{2n,1}, \\ B_n &= (4n+3)a_3a_2^{2n,0} - (4n+1)b_2a_1^{2n,0}, \\ C_n &= (4n+4)a_3a_2^{2n+1,1} - (4n+2)b_2a_1^{2n+1,1}, \\ D_n &= (4n+5)a_3a_2^{2n+1,0} - (4n+3)b_2a_1^{2n+1,0}, \end{aligned}$$

*where the constants $a_j^{h,i}, h \geq 2, i = 0,1, j = 1,2$ are those defined in Theorem 38.*

*Remark:* In generic conditions for $a_3, b_2$, we have checked that $A_n \neq 0$, $C_n \neq 0$, $B_n = D_n = 0$, for $1 \leq n \leq 30$. Then, a 123th-order hypernormal form using $C^\infty$–equivalence is

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= b_2^*xy + a_3^*x^3 + \sum_{j=1}^{61} b_{2j+1}^* x^{2j} y. \end{aligned}$$

## APPENDIX

### A.1. PROOFS OF CASE I: $R < 2S - 1$.

We will focus here mainly on the case I.1: $s < r < 2s-1$. In fact, the case I.2 is almost identical.

Recall that, as the first nonzero normal form coefficients are $a_r$, $b_s$, we have $N_1 = \cdots = N_{r-2} = 0$, $N_{r-1} \neq 0$ and $L_1 = \cdots = L_{s-2} = 0$, $L_{s-1} \neq 0$.

(A) Role of $A_{2n,1} \in \mathbb{R}$, $n \geq 1$.

We start analyzing how the arbitrary constant $A_{2n,1}$, $n \geq 1$, appears in $v^*$.

We will transform (21) using a generator $U = \sum_{k\geq 1} U_k$, which depends only on $A_{2n,1} \in \mathbb{R}$. As we will see below, this generator is given, degree by degree, as:

- $U_1 = \cdots = U_{2n-1} = 0$.
- $U_{2n} = A_{2n,1} f_{2n,1}$.
- $U_k$ is given in (16), taking $A_{h,i} = 0$ for $h \neq 2n$, $i \neq 1$, for $k = 2n+1, \ldots, 2n+n(r+1)-2$.
- $U_k$ depends nonlinearly on $A_{2n,1}$ for $k \geq 2n + n(r+1) - 1$.

As $N_1 = \cdots = N_{r-2} = 0$, $L_1 = \cdots = L_{s-2} = 0$, in the sum defining $U_k$, the nonzero summands are those corresponding to $n_j \geq s-1$. Also, $n_j' = n_j + 1$ whenever $s-1 \leq n_j \leq r-2$. Then, applying Theorem 13, we obtain $\widetilde{v}_k = \mathrm{Proj}_{\mathcal{C}_k}[v,U] = 0$ for $k = 2, \ldots, n(r+1)-1$. So, in this case, taking $\lambda = n(r+1)-1$, we have $\mathcal{J}^\lambda[v,U] = 0$.

From the structure of $U$, it is easily obtained that $\mathrm{Proj}_{\mathcal{R}_k}[v,U] = 0$ for $k = \lambda+1, \ldots, \lambda+2n-1$. Then, taking $\delta = \lambda + 2n = 2n + n(r+1) - 1$, we obtain that $\widetilde{v}_k = \mathrm{Proj}_{\mathcal{C}_k}[v,U] = [v,U]_k$ is given in (20) for $k = \lambda+1, \ldots, \delta-1$. In particular,

$$\widetilde{v}_{\lambda+1} = A_{2n,1}\mathrm{Proj}_{\mathcal{C}_{\lambda+1}} N_{r-1} M N_{r-1} \overset{n-1}{\cdots} M N_{r-1} f_{2n,1} \in \mathcal{H}_{\lambda+1;0}. \qquad \text{(A.1)}$$

Moreover, we obtain from (20) that $\Pi_1 \widetilde{v}_{\lambda+1} = \cdots = \Pi_1 \widetilde{v}_{\lambda+s-1} = 0$. Also,

$$\widetilde{v}_{\lambda+s} = A_{2n,1}\mathcal{P}(L_{s-1} M N_{r-1} \overset{n}{\cdots} M N_{r-1} f_{2n,1}) \in \mathcal{H}^*_{\lambda+s;-1} \bigcap \mathcal{C}_{\lambda+s}. \qquad \text{(A.2)}$$

Notice that $A_{2n,1}$ appears linearly in the generator and in the transformed vector field, up to order $\delta - 1$. So, $U_k^{NL} = 0$ for $k = 1, \ldots, \delta-1$.

In the $\delta$-order terms of the generator, $A_{2n,1}$ can appear nonlinearly: $U_\delta = U_\delta^L + U_\delta^{NL}$, where

$$[v_1, U_\delta^L] \; = \; -\mathrm{Proj}_{\mathcal{R}_\delta}\left([v_2, U_{\delta-1}^L] + \cdots + [v_{\delta-1}, U_2^L]\right) =$$

$$= -\mathrm{Proj}_{\mathcal{R}_\delta} \sum_{m=s}^{\delta} (N_{m-1} U^L_{\delta-m+1} + L_{m-1} U^L_{\delta-m+1}),$$

$$\begin{aligned}[v_1, U^{NL}_\delta] &= -\mathrm{Proj}_{\mathcal{R}_\delta} \left( \sum_{j\geq 2} \frac{1}{j!} T^j_U(v) \right) = -\mathrm{Proj}_{\mathcal{R}_\delta} \tfrac{1}{2} [[v,U],U] = \\ &= -\tfrac{1}{2} \mathrm{Proj}_{\mathcal{R}_\delta} [\widetilde{v}_{\lambda+1}, U_{2n}].\end{aligned}$$

Consequently

$$\begin{aligned} U^L_\delta &= M \sum_{m=s}^{\delta} (N_{m-1} U^L_{\delta-m+1} + L_{m-1} U^L_{\delta-m+1}) \in \mathcal{H}^*_{\delta;-1}, \\ U^{NL}_\delta &= \frac{1}{2} A_{2n,1} M (\widetilde{L}_\lambda f_{2n,1} + \widetilde{N}_\lambda f_{2n,1}), \end{aligned}$$

where $\widetilde{L}_\lambda$, $\widetilde{N}_\lambda$ correspond to the vector field $\widetilde{v}_{\lambda+1}$, defined in (A.1). As they depend linearly on $A_{2n,1}$, we find that $U^{NL}_\delta$ depends quadratically on $A_{2n,1}$. Moreover, $\widetilde{N}_\lambda = 0$ because $\Pi_1 \widetilde{v}_{\lambda+1} = 0$.

In summary,

$$U^{NL}_\delta = \frac{1}{2} A_{2n,1} M \widetilde{L}_\lambda f_{2n,1} \in \mathcal{H}_{\delta;2n-2}. \tag{A.3}$$

For $k > \delta$, we can write $U_k = U^L_k + U^{NL}_k$, where $U^L_k$, $U^{NL}_k$ are obtained from (8), (9), respectively. In this way, $U^L_k$ are given in Theorem 12 (taking $A_{h,i} = 0$ for $h \neq 2n$, $i \neq 1$). Hence $U^L_k \in \mathcal{H}^*_{k;-1}$ for $k > \delta$, because $\mathcal{H}^*_{k;j} = \mathcal{H}^*_{k;-1}$ for $j < 0$.

Next lemma summarizes all the concerning with the linear appearance of the arbitrary constant in both, the generator and the transformed vector field.

Lemma 48. *Let us consider the generator $U = \sum_{k\geq 1} U_k$ given before. Denote $\lambda = n(r+1) - 1$, $\delta = 2n + n(r+1) - 1$, $\mu = (2n-1)(r+1) + s - 1$. Then:*

*(a) $U^L_k = 0$, for $k = 1, \ldots, 2n-1$.*

*Consider $q \geq 0$, and let $\alpha, \beta \in \mathbb{N}_0$ such that $q = \alpha(r-1) + \beta(s-1) + \gamma$, $0 \leq \beta(s-1) + \gamma < r-1$, $0 \leq \gamma < s-1$. Then $U^L_{2n+q} \in \mathcal{H}^*_{2n+q;2n-1-2\alpha-\beta}$.*

*(b) $U^{NL}_k = 0$, for $k = 1, \ldots, \delta - 1$.*

*Consider $q \geq 0$, and let $\alpha, \beta \in \mathbb{N}_0$ as in item (a).*

*Then $U^{NL}_{\delta+q} \in \mathcal{H}^*_{\delta+q;2n-2-2\alpha-\beta}$.*

*(c) $\mathrm{Proj}_{\mathcal{C}_k} \left\{ [v_2, U^{NL}_{k-1}] + \cdots + [v_{k-1}, U^{NL}_2] + \sum_{j\geq 2} \frac{1}{j!} T^j_U(v) \right\} = 0$, for $k = 2, \ldots, \mu$.*

*Proof.* For each $q \geq 0$, $\alpha$ is the quotient of the division $\dfrac{q}{r-1}$. Then, we can apply at the most $\alpha$ times the operator $N_{r-1}$ to $f_{2n,1}$ in order to get terms of order $2n+q$ (in other words, applying $N_{r-1}$ more than $\alpha$ times, we will get terms of order greater than $2n+q$). If we denote $\tilde{\alpha}$ to the rest of the above division, we find $q = \alpha(r-1) + \tilde{\alpha}$. Note that $0 \leq \tilde{\alpha} < r-1$.

Analogously, $\beta$, $\gamma$ are, respectively, the quotient and rest of the division $\dfrac{\tilde{\alpha}}{s-1}$. So, $\beta$ denotes how many times we must apply $L_{s-1}$. Moreover, $0 \leq \gamma < s-1$ and $q = \alpha(r-1) + \tilde{\alpha} = \alpha(r-1) + \beta(s-1) + \gamma$. As we are assuming $r-1 < 2(s-1)$ and then $\beta = 0$ or $\beta = 1$.

**(a)** Applying Theorem 12, we get $U^L_{2n} = A_{2n,1} f_{2n,1}$. Also, we find $U^L_{2n+q} \in \mathcal{H}_{2n+q;2n+q-1-|\mathbf{n}'|}$ for $q \geq 1$, because $f_{2n,\mathbf{n};1,\mathbf{n}'} \in \mathcal{H}_{2n+|\mathbf{n}|;2n+|\mathbf{n}|-1-|\mathbf{n}'|}$, $\widetilde{f}_{2n,\mathbf{n};1,\mathbf{n}'} \in \mathcal{H}_{2n+|\mathbf{n}|;2n+|\mathbf{n}|-|\mathbf{n}'|}$, and

$$\begin{aligned} U^L_{2n+q} &= A_{2n,1}\Bigg( \sum_{\substack{l>0,\mathbf{n}\in\mathcal{I}_l,\mathbf{n}'\sim\mathbf{n}\\ |\mathbf{n}|=q,|\mathbf{n}'|<2n+q}} f_{2n,\mathbf{n};1,\mathbf{n}'} + \\ &\quad + \sum_{\substack{l>0,\mathbf{n}\in\mathcal{I}_l,\mathbf{n}'\sim\mathbf{n}\\ |\mathbf{n}|=q,|\mathbf{n}'|=2n+q}} M(2n+q)\circ \mathrm{Proj}_{\mathcal{R}_{2n+q}} \widetilde{f}_{2n,\mathbf{n};1,\mathbf{n}'} \Bigg). \end{aligned}$$

As we must take $|\mathbf{n}| = q = \alpha(r-1) + \beta(s-1) + \gamma$, we get $|\mathbf{n}| - |\mathbf{n}'| \geq -2\alpha - \beta$, because $n_j \geq s-1$ and $n'_j = n_j + 1$ whenever $s-1 \leq n_j \leq r-2$. Then, $U^L_{2n+q} \in \mathcal{H}^*_{2n+q;2n-1-2\alpha-\beta}$.

**(b)** We will use induction.

- The result for $q = 0$ follows from (A.3).
- Assume that the result is also true for each $\tilde{q} < q$. As $\mathrm{dif}(U_k) = \mathrm{dif}(U^L_k)$, we get $\mathrm{dif}(U^L_k) \leq \mathrm{dif}(U^{NL}_k)$ for $k \geq 1$. Hence $\mathrm{dif}(T_U(v)) = \mathrm{dif}(T_{U^L}(v))$. Moreover,

$$[v_1, U^{NL}_{\delta+q}] = -\mathrm{Proj}_{\mathcal{R}_{\delta+q}} \left\{ [v_2, U^{NL}_{\delta+q-1}] + \cdots + [v_{\delta+q-1}, U^{NL}_2] + \sum_{j\geq 2} \frac{1}{j!} T^j_U(v) \right\}.$$

Take $\kappa = \delta + q$. Then:

$$\mathrm{dif}(U^{NL}_\kappa) \geq \mathrm{dif}\left( [v_2, U^{NL}_{\kappa-1}] + \cdots + [v_{\kappa-1}, U^{NL}_2] + \mathcal{J}_\kappa \left\{ \sum_{j\geq 2} \frac{1}{j!} T^j_U(v) \right\} \right) - 1 \geq$$

$$\geq \min\left\{\operatorname{dif}\left([v_2, U_{\kappa-1}^{NL}] + \cdots + [v_{\kappa-1}, U_2^{NL}]\right); \operatorname{dif}\left(\mathcal{J}_\kappa\left(\sum_{j\geq 2}\frac{1}{j!}T_U^j(v)\right)\right)\right\} - 1. \tag{A.4}$$

On the one hand,

$$\operatorname{dif}\left([v_2, U_{\kappa-1}^{NL}] + \cdots + [v_{\kappa-1}, U_2^{NL}]\right) = \min_{j=s,\ldots,q+1}\operatorname{dif}([v_j, U_{\kappa-j+1}^{NL}]) = \min\left\{\operatorname{dif}([v_s, U_{\kappa-s+1}^{NL}]); \operatorname{dif}([v_r, U_{\kappa-r+1}^{NL}])\right\}.$$

In the last equality, we have used the following properties (see Lemmas 17, 19): $\operatorname{dif}([v,U]) \geq \operatorname{dif}(v) + \operatorname{dif}(U)$; $\operatorname{dif}(v_s) = 0$; $\operatorname{dif}(v_k) \geq 0$ for $k = s+1, \ldots, r-1$; $\operatorname{dif}(v_r) = -1$; $\operatorname{dif}(v_k) \geq -1$ for $k > r$; and $\operatorname{dif}(U_k^L) \leq \operatorname{dif}(U_j^L)$ for $k \geq j$.

From the hypothesis of induction, we find:

$$\begin{aligned}\operatorname{dif}(U_{\kappa-s+1}^{NL}) &= \operatorname{dif}(U_{\delta+q-s+1}^{NL}) \geq 2n - 2\alpha - \beta - 1,\\ \operatorname{dif}(U_{\kappa-r+1}^{NL}) &= \operatorname{dif}(U_{\delta+q-r+1}^{NL}) \geq 2n - 2\alpha - \beta.\end{aligned}$$

So, $\operatorname{dif}([v_s, U_{\kappa-s+1}^{NL}]) \geq 2n - 2\alpha - \beta - 1$, and then

$$\operatorname{dif}\left([v_2, U_{\kappa-1}^{NL}] + \cdots + [v_{\kappa-1}, U_2^{NL}]]\right) \geq 2n - 2\alpha - \beta - 1.$$

On the other hand, as $\operatorname{dif}(U_k) \leq \operatorname{dif}(U_j)$ for $k \geq j$, and $\operatorname{dif}(\mathcal{J}_\kappa T_U^n(v)) \geq \operatorname{dif}(\mathcal{J}_\kappa T_U^2(v))$ for $n \geq 2$, we get

$$\operatorname{dif}\left(\mathcal{J}_\kappa \sum_{j\geq 2}\frac{1}{j!}T_U^j(v)\right) = \operatorname{dif}\left(\mathcal{J}_\kappa \sum_{j\geq 2}\frac{1}{j!}T_{U^L}^j(v)\right) = \operatorname{dif}\left([[v, U^L], U^L]_\kappa\right).$$

Also, we have $[v, U^L] = \widetilde{v}_{\lambda+1} + \cdots + \widetilde{v}_{\lambda+s} + \cdots$, where $\operatorname{dif}(\widetilde{v}_k) \geq 0$ for $k = \lambda+1, \ldots, \lambda+s-1$, and $\operatorname{dif}(\widetilde{v}_k) \geq -1$ for $k \geq \lambda + s$. Then,

$$\begin{aligned}\operatorname{dif}(\mathcal{J}_\kappa T_{U^L}^2(v)) &= \min\left\{\operatorname{dif}([\widetilde{v}_{\lambda+1}, U^L]_\kappa); \operatorname{dif}([\widetilde{v}_{\lambda+s}, U^L]_\kappa)\right\} =\\ &= \min\left\{\operatorname{dif}([\widetilde{v}_{\lambda+1}, U_{\kappa-\lambda}^L]); \operatorname{dif}([\widetilde{v}_{\lambda+s}, U_{\kappa-\lambda-s+1}^L])\right\}.\end{aligned}$$

From item (a), we obtain $\operatorname{dif}(U_{\kappa-\lambda}^L) = \operatorname{dif}(U_{2n+q}^L) \geq 2n - 1 - 2\alpha - \beta$. Then, $\operatorname{dif}([\widetilde{v}_{\lambda+1}, U_{\kappa-\lambda}^L]) \geq 2n-1-2\alpha-\beta$. Moreover, as $\operatorname{dif}(U_{\kappa-\lambda-s+1}^L) > 2n-1-2\alpha-\beta$, we get $\operatorname{dif}([\widetilde{v}_{\lambda+s}, U_{\kappa-\lambda-s+1}^L]) > 2n-2-2\alpha-\beta$ and consequently, $\operatorname{dif}(\mathcal{J}_\kappa T_{U^L}^2(v)) \geq 2n - 1 - 2\alpha - \beta$.

From (A.4), we get $\operatorname{dif}(U_{\delta+q}^{NL}) \geq 2n-2-2\alpha-\beta$, and then $U_{\delta+q} \in \mathcal{H}_{\delta+q;2n-2-2\alpha-\beta}^*$.

**(c)** It is enough to observe that

$$1 \leq \mathrm{dif}\left( [v_2, U_{\mu-1}^{NL}] + \cdots + [v_{\mu-1}, U_2^{NL}] + \mathcal{J}_\mu \left\{ \sum_{j\geq 2} \frac{1}{j!} T_U^j(v) \right\} \right) \leq$$
$$\leq \mathrm{dif}\left( [v_2, U_{k-1}^{NL}] + \cdots + [v_{k-1}, U_2^{NL}] + \mathcal{J}_k \left\{ \sum_{j\geq 2} \frac{1}{j!} T_U^j(v) \right\} \right),$$

for $k = 2, \ldots, \mu$ (in the second inequality, we have used item (b)). ∎

Item (c) of the above lemma shows that the constant $A_{2n,1} \in \mathbb{R}$ appears linearly in the transformed vector field $v^*$ up to order $\mu$. This transformed vector field can be expressed as

$$v^* = v_1 + v_s + \cdots + v_\lambda + (v_{\lambda+1} + \widetilde{v}_{\lambda+1}) + \cdots + (v_\mu + \widetilde{v}_\mu) + \sum_{k\geq \mu+1} v_k^*, \quad \text{(A.5)}$$

where $v_k, \widetilde{v}_k \in \mathcal{C}_k$. From this expression, we get

Lemma 49. *Let $\lambda = n(r+1) - 1$, $\mu = (2n-1)(r+1) + s - 1$. Then, the vector field (21) is $\mathrm{C}^\infty$–conjugate to (22), where*

$$\begin{aligned} a_k^* &= a_k, \text{ for } k = 2, \ldots, \lambda + s - 1, \\ b_k^* &= b_k, \text{ for } k = 2, \ldots, \lambda, \\ a_k^* &= a_k + \phi_k^{2n,1}(A_{2n,1}), \text{ for } k = \lambda + s, \ldots, \mu \\ b_k^* &= b_k + \psi_k^{2n,1}(A_{2n,1}), \text{ for } k = \lambda + 1, \ldots, \mu, \end{aligned}$$

*where $\phi_k^{2n,1}(A_{2n,1}) = \Pi_1 \widetilde{v}_k$, $\psi_k^{2n,1}(A_{2n,1}) = \Pi_2 \widetilde{v}_k$.*

There are several possibilities of which normal form coefficient can be annihilated by selecting adequately $A_{2n,1}$ (note that functions $\phi_k^{2n,1}$, $\psi_k^{2n,1}$ are linear).

Among these, we will use it to annihilate the normal form coefficients $a_{\lambda+s}^*$, $b_{\lambda+1}^*$, where $A_{2n,1}$ appears for the first time. From (A.1), (A.2), we get the Lemma 22

(B) Role of $A_{2n,0} \in \mathbb{R}$, $n \geq 1$.

In this case, we will perform a transformation on the normal form (21), using a generator $U = \sum_{k\geq 1} U_k$, which depends only on $A_{2n,0} \in \mathbb{R}$. This generator is given, degree by degree, as:

- $U_1 = \cdots = U_{2n-1} = 0$.

- $U_{2n} = A_{2n,0} f_{2n,0}$.
- For $k = 2n+1, \dots, 2n+n(r+1)+s-3$, we take $U_k$ as in (16), but with $A_{h,i} = 0$, for each $h \neq 2n$, $i \neq 0$.
- $U_k$ depends nonlinearly on $A_{2n,0}$ for $k \geq 2n+n(r+1)+s-2$.

In the sum defining $U_k$, we must take $n_j \geq s-1$, and also $n'_j = n_j + 1$ whenever $s-1 \leq n_j \leq r-2$. Also $\widetilde{v}_k = \mathrm{Proj}_{\mathcal{C}_k}[v, U] = 0$, for $k = 2, \dots, n(r+1)+s-2$. So, we deduce that the vector field remains unalterated up to order $n(r+1)+s-2$. Using the nomenclature of the end of Section 2, we have $\lambda = n(r+1)+s-2$. On the other hand, $\mathrm{Proj}_{\mathcal{R}_k}[v, U] = 0$ for $k = \lambda+1, \dots, \lambda+2n-1$. Then, $\widetilde{v}_k = \mathrm{Proj}_{\mathcal{C}_k}[v, U]$ is given in (20) for $k = \lambda+1, \dots, \lambda+2n-1$. From (20) we also get

$$\widetilde{v}_{\lambda+1} = A_{2n,0} \mathrm{Proj}_{\mathcal{C}_{\lambda+1}} \mathcal{P}(L_{s-1} M N_{r-1} \overset{n}{\cdots} M N_{r-1} f_{2n,0}). \tag{A.6}$$

Moreover, $\Pi_1 \widetilde{v}_{\lambda+1} = \cdots = \Pi_1 \widetilde{v}_{\lambda+r-s} = 0$, and

$$\widetilde{v}_{\lambda+r-s+1} = A_{2n,0} N_{r-1} M N_{r-1} \overset{n}{\cdots} M N_{r-1} f_{2n,0}. \tag{A.7}$$

Notice that $U_k^{NL} = 0$ for $k = 1, \dots, \lambda+2n-1$. So, up to this order, $A_{2n,0}$ appears linearly. This arbitrary constant can appear nonlinearly in the terms of order $\delta = 2n+\lambda = 2n+n(r+1)+s-2$ in the generator. The $\delta$-order terms are $U_\delta = U_\delta^L + U_\delta^{NL}$, where

$$\begin{aligned} [v_1, U_\delta^L] &= -\mathrm{Proj}_{\mathcal{R}_\delta} \left( [v_2, U_{\delta-1}^L] + \cdots + [v_{\delta-1}, U_2^L] \right), \\ [v_1, U_\delta^{NL}] &= -\frac{1}{2} \mathrm{Proj}_{\mathcal{R}_\delta} [\widetilde{v}_{\lambda+1}, U_{2n}], \end{aligned}$$

that is, $U_\delta^L$ is given in (16), and $U_\delta^{NL} = \frac{1}{2} A_{2n,0} M \widetilde{L}_\lambda f_{2n,0} \in \mathcal{H}_{\delta;2n-1}$ (which depends quadratically on $A_{2n,0}$).

The proof of next lemma is analogous to the one of Lemma 48.

Lemma 50. *Consider the generator $U = \sum_{k\geq 1} U_k$ given before. Denote $\lambda = n(r+1)+s-2$, $\delta = 2n+n(r+1)+s-2$, $\mu = 2n(r+1)+s-3$. Then:*

*(a) $U_k^L = 0$, for $k = 1, \dots, 2n-1$.*

*Consider $q \geq 0$ and let $\alpha, \beta \in \mathbb{N}_0$ determined by the relations $q = \alpha(r-1)+\beta(s-1)+\gamma$, $0 \leq \beta(s-1)+\gamma < r-1$, and $0 \leq \gamma < s-1$. Then $U_{2n+q}^L \in \mathcal{H}^*_{2n+q;2n-2\alpha-\beta}$.*

*(b) $U_k^{NL} = 0$, for $k = 1, \dots, \delta-1$.*

*$U_{\delta+j}^{NL} \in \mathcal{H}^*_{\delta+j;2n-1}$, for $j = 0, \dots, r-s-1$.*

*Consider $q \geq 0$ and let $\alpha, \beta \in \mathbb{N}_0$ as indicated in item (a). Then $U_{\delta+r-s+q}^{NL} \in \mathcal{H}^*_{\delta+r-s+q;2n-2-2\alpha-\beta}$.*

*(c)* $\mathrm{Proj}_{\mathcal{C}_k}\left\{[v_2, U_{k-1}^{NL}] + \cdots + [v_{k-1}, U_2^{NL}] + \sum_{j\geq 2} \frac{1}{j!} T_U^j(v)\right\} = 0$, *for* $k = 2, \ldots, \mu$.

The above lemma assures that $A_{2n,0}$ appears linearly in the transformed vector field $v^*$ up to order $\mu$. In fact, this transformed vector field can be expressed as in (A.5), but now the values of $\lambda$, $\mu$ are different. In this case, we conclude:

LEMMA 51. *Let* $\lambda = n(r+1) + s - 2$, $\mu = 2n(r+1) + s - 3$. *Then, the vector field (21) is* $C^\infty$*–conjugate to (22), with*

$$\begin{aligned} a_k^* &= a_k, \text{ for } k = 2, \ldots, \lambda + r - s, \\ b_k^* &= b_k, \text{ for } k = 2, \ldots, \lambda, \\ a_k^* &= a_k + \phi_k^{2n,0}(A_{2n,0}), \text{ for } k = \lambda + r - s + 1, \ldots, \mu, \\ b_k^* &= b_k + \psi_k^{2n,0}(A_{2n,0}), \text{ for } k = \lambda + 1, \ldots, \mu, \end{aligned}$$

*where* $\phi_k^{2n,0}(A_{2n,0}) = \Pi_1 \widetilde{v}_k$, $\psi_k^{2n,0}(A_{2n,0}) = \Pi_2 \widetilde{v}_k$.

Using (A.6), (A.7), we are able to determine how $A_{2n,0}$ appears for the first time in the expressions of the normal form coefficients. This is just the case presented in Lemma 23.

(C) Role of $A_{2n+1,1} \in \mathbb{R}$, $n \geq 1$.

Following the same procedure of items (A), (B), we consider a generator $U = \sum_{k\geq 1} U_k$ such that

- $U_1 = \cdots = U_{2n} = 0$.
- $U_{2n+1} = A_{2n+1,1} f_{2n+1,1}$.
- For $k = 2n+2, \ldots, 2n + n(r+1) + s - 1$, we take $U_k$ as in (16), but with $A_{h,i} = 0$, for each $h \neq 2n+1$, $i \neq 1$.
- $U_k$ depends nonlinearly on $A_{2n+1,1}$ for $k \geq 2n + n(r+1) + s$.

Reasoning as above, we take $\lambda = n(r+1)+s-1$, $\delta = \lambda+2n+1 = 2n+n(r+1)+s$. One can prove that the vector field does not change up to order $\lambda$. Moreover, $\mathrm{Proj}_{\mathcal{R}_k}[v, U] = 0$, and then $\widetilde{v}_k = \mathrm{Proj}_{\mathcal{C}_k}[v, U]$, for $k = \lambda+1, \ldots, \lambda+2n$. From their expressions (which are given in (20), taking $A_{h,i} = 0$ for $h \neq 2n+1$, $i \neq 1$), we obtain

$$\widetilde{v}_{\lambda+1} = A_{2n+1,1} \mathrm{Proj}_{\mathcal{C}_{\lambda+1}} \mathcal{P}(L_{s-1} M N_{r-1} \overset{n}{\cdots} M N_{r-1} f_{2n+1,1}), \quad \text{(A.8)}$$

$$\widetilde{v}_{\lambda+r-s+1} = A_{2n+1,1} N_{r-1} M N_{r-1} \overset{n}{\cdots} M N_{r-1} f_{2n+1,1}. \quad \text{(A.9)}$$

We remark that $\Pi_1 \widetilde{v}_{\lambda+1} = \cdots = \Pi_1 \widetilde{v}_{\lambda+r-s} = 0$.

As in previous items, we get $U_k^{NL} = 0$ for $k = 1, \dots, \delta - 1$. Moreover, the following lemma holds:

LEMMA 52. *Let $U = \sum_{k \geq 1} U_k$ the generator defined before. Denote $\lambda = n(r+1) + s - 1$, $\delta = 2n + n(r+1) + s$, $\mu = 2n(r+1) + s - 1$. Then:*

*(a) $U_k^L = 0$, for $k = 1, \dots, 2n$.*

*Consider $q \geq 0$, and let $\alpha, \beta \in \mathbb{N}_0$ such that $q = \alpha(r-1) + \beta(s-1) + \gamma$, $0 \leq \beta(s-1) + \gamma < r - 1$, $0 \leq \gamma < s - 1$. Then $U_{2n+1+q}^L \in \mathcal{H}^*_{2n+1+q;2n-2\alpha-\beta}$.*

*(b) $U_k^{NL} = 0$, for $k = 1, \dots, \delta - 1$.*

*$U_{\delta+j}^{NL} \in \mathcal{H}^*_{\delta+j;2n-1}$, for all $j = 0, \dots, r - s - 1$.*

*Consider $q \geq 0$, and let $\alpha, \beta \in \mathbb{N}_0$ as indicated in item (a). Then $U_{\delta+r-s+q}^{NL} \in \mathcal{H}^*_{\delta+r-s+q;2n-2-2\alpha-\beta}$.*

*(c)*

$$\mathrm{Proj}_{\mathcal{C}_k}\left\{[v_2, U_{k-1}^{NL}] + \dots + [v_{k-1}, U_2^{NL}] + \sum_{j \geq 2} \frac{1}{j!} T_U^j(v)\right\} = 0, \text{ for } k = 2, \dots, \mu.$$

We have obtained that, using a generator which depends only on $A_{2n+1,1}$, the transformed vector field $v^*$ is expressed as in (A.5), with the values of $\lambda$, $\mu$ that appear in the statement of the above lemma. Moreover:

LEMMA 53. *Let $\lambda = n(r+1) + s - 1$, $\mu = 2n(r+1) + s - 1$. Then, the vector field (21) is $C^\infty$–conjugate to (22), where*

$$\begin{aligned}
a_k^* &= a_k, \text{ for } k = 2, \dots, \lambda + r - s, \\
b_k^* &= b_k, \text{ for } k = 2, \dots, \lambda, \\
a_k^* &= a_k + \phi_k^{2n+1,1}(A_{2n+1,1}), \text{ for } k = \lambda + 1 + r - s, \dots, \mu, \\
b_k^* &= b_k + \psi_k^{2n+1,1}(A_{2n+1,1}), \text{ for } k = \lambda + 1, \dots, \mu,
\end{aligned}$$

*where $\phi_k^{2n+1,1}(A_{2n+1,1}) = \Pi_1 \widetilde{v}_k$, $\psi_k^{2n+1,1}(A_{2n+1,1}) = \Pi_2 \widetilde{v}_k$.*

In particular, using (A.8), (A.9), we determine how the arbitrary constant can appear for the first time in the normal form coefficients, that corresponds to the situation of Lemma 24.

(D) Role of $A_{2n-1,0} \in \mathbb{R}$, $n \geq 2$.

Here, we consider the generator $U = \sum_{k \geq 1} U_k$, with

- $U_1 = \dots = U_{2n-2} = 0$.
- $U_{2n-1} = A_{2n-1,0} f_{2n-1,0}$.

• For $k = 2n, \ldots, 2n + (n-1)(r+1) + 2s - 4$, we take $U_k$ as in (16), but with $A_{h,i} = 0$, for each $h \neq 2n-1$, $i \neq 0$.
• $U_k$ depends nonlinearly on $A_{2n-1,0}$ for $k \geq 2n + (n-1)(r+1) + 2s - 3$.

In this last case, we take $\lambda = (n-1)(r+1) + 2s - 2$, $\delta = \lambda + 2n - 1 = 2n + (n-1)(r+1) + 2s - 3$.

From Lemma 6, we have $\mathrm{Proj}_{\mathcal{C}_{2l-1+|\mathbf{n}|}} N_{n_l} M N_{n_{l-1}} \cdots M N_{n_1} f_{2l-1,0} = 0$, for $\mathbf{n} \in \mathcal{I}_l$, and we can deduce easily that $\mathcal{J}^{\lambda}[v,U] = 0$. Also, $\mathrm{Proj}_{\mathcal{R}_k}[v,U] = 0$, and consequently $\widetilde{v}_k = \mathrm{Proj}_{\mathcal{C}_k}[v,U]$ for $k = \lambda+1, \ldots, \delta-1$. The last ones are given in (20), taking $A_{h,i} = 0$ for $h \neq 2n-1$, $i \neq 0$. From these expressions, we get:

$$\widetilde{v}_{\lambda+1} = A_{2n-1,0}\mathrm{Proj}_{\mathcal{C}_{\lambda+1}} \mathcal{P}(L_{s-1} M L_{s-1} M N_{r-1} \overset{n-1}{\cdots} M N_{r-1} f_{2n-1,0}). \quad \text{(A.10)}$$

Moreover, $\Pi_1 \widetilde{v}_{\lambda+1} = \cdots = \Pi_1 \widetilde{v}_{\lambda+r-s} = 0$, and

$$\widetilde{v}_{\lambda+r-s+1} = A_{2n-1,0} \mathcal{P}(L_{s-1} M N_{r-1} \overset{n}{\cdots} M N_{r-1} f_{2n-1,0}). \quad \text{(A.11)}$$

Also, $U_k^{NL} = 0$ for $k = 1, \ldots, \delta-1$. In this case, we have:

Lemma 54. *Let $U = \sum_{k\geq 1} U_k$ the generator defined before. Denote $\lambda = (n-1)(r+1) + 2s - 2$, $\delta = 2n + (n-1)(r+1) + 2s - 3$, $\mu = 2(n-1)(r+1) + r + s - 2$. Then:*

*(a) $U_k^L = 0$, for $k = 1, \ldots, 2n-2$.*

*Consider $q \geq 0$, and let $\alpha, \beta \in \mathbb{N}_0$ such that $q = \alpha(r-1) + \beta(s-1) + \gamma$, $0 \leq \beta(s-1) + \gamma < r-1$, $0 \leq \gamma < s-1$. Then $U_{2n-1+q}^L \in \mathcal{H}^*_{2n-1+q;2n-1-2\alpha-\beta}$.*

*(b) $U_k^{NL} = 0$, for $k = 1, \ldots, \delta-1$.*

*$U_{\delta+j} \in \mathcal{H}^*_{\delta+j;2n-2}$, for $j = 0, \ldots, r-s-1$.*

*Consider $q \geq 0$, and let $\alpha, \beta \in \mathbb{N}_0$ as indicated in item (a). Then $U_{\delta+r-s+q} \in \mathcal{H}^*_{\delta+r-s+q;2n-3-2\alpha-\beta}$.*

*(c) $\mathrm{Proj}_{\mathcal{C}_k} \left\{ [v_2, U_{k-1}^{NL}] + \cdots + [v_{k-1}, U_2^{NL}] + \sum_{j\geq 2} \frac{1}{j!} T_U^j(v) \right\} = 0$, for $k = 2, \ldots, \mu$.*

The transformed vector field is given in (A.5) with the values of $\lambda$, $\mu$ that appear in the statement of the lemma. In the present case, we find:

Lemma 55. *Let $\lambda = (n-1)(r+1) + 2s - 2$, $\mu = 2(n-1)(r+1) + 2s + r - s - 2$. Then, the vector field (21) is $\mathrm{C}^\infty$–conjugate to (22), with*

$$\begin{aligned}
a_k^* &= a_k, \ \text{for } k = 2, \ldots, \lambda + r - s, \\
b_k^* &= b_k, \ \text{for } k = 2, \ldots, \lambda, \\
a_k^* &= a_k + \phi_k^{2n-1,0}(A_{2n-1,0}), \ \text{for } k = \lambda + r - s + 1, \ldots, \mu, \\
b_k^* &= b_k + \psi_k^{2n-1,0}(A_{2n-1,0}), \ \text{for } k = \lambda + 1, \ldots, \mu,
\end{aligned}$$

*where* $\phi_k^{2n-1,0}(A_{2n-1,0}) = \Pi_1 \widetilde{v}_k,\ \psi_k^{2n-1,0}(A_{2n-1,0}) = \Pi_2 \widetilde{v}_k$.

As a particular case, from (A.10), (A.11), we obtain Lemma 25.

The proofs of Lemmas 27, 28, 29 of the case I.2 are analogous to the ones of Lemmas 22, 23, 24 of the case I.1. Only, the proof of Lemma 30 is slightly different to those of the Lemma 25 in the above subcase, because we must now use Lemma 6. We do not include them for the sake of brevity.

## A.2. PROOFS OF CASE II: $R = 2S - 1$.

In the situation we analyze now, the first nonzero normal form coefficients are $a_{2s-1}$, $b_s$. In this case, we must pay attention to operators $MN_{2s-2}$ and $ML_{s-1}ML_{s-1}$, because both decrease the difference in 2 unities and increase the order in $2s-2$ unities.

Notice that we must use indices $\mathbf{n}$ and $\mathbf{n}' \sim \mathbf{n}$ with $n_j \geq s-1$; and $n'_j = n_j + 1$ whenever $s-1 \leq n_j \leq 2s-3$.

Next, we present the Proof of Lemma 34.

*Proof.* Consider a generator $U = \sum_{k\geq 1} U_k$ such that

- $U_1 = \cdots = U_{2n-1} = 0$,
- $U_{2n} = A_{2n,1} f_{2n,1}$.
- For $k > 2n$, we take $U_k = U_k^L + U_k^{NL}$, where $U_k^L$, $U_k^{NL}$ satisfy (8), (9).

Moreover, we take these functions depending only on $A_{2n,1}$.

It can be proved that $\mathrm{Proj}_{\mathcal{C}_k}\left\{[v_2, U_{k-1}^{NL}] + \cdots + [v_{k-1}, U_2^{NL}] + \sum_{j\geq 2} \frac{1}{j!} T_U^j(v)\right\} = 0$, for $k = 2, \ldots, 4ns - s - 1$. Moreover, $U_k^L$ and $\widetilde{v}_k = \mathrm{Proj}_{\mathcal{C}_k}[v, U]$ (with $k \leq 4ns - s - 1$) are given in (16) and (20), taking $A_{h,i} = 0$ for $h \neq 2n$, $i \neq 1$.

Consequently, $\widetilde{v}_2 = \cdots = \widetilde{v}_{2ns-1} = 0$ and $\Pi_1 \widetilde{v}_k = 0$ for $k = 2ns, \ldots, 2ns + s - 2$. Also,

$$\widetilde{v}_{2ns+s-1} = A_{2n,1} \sum_{\substack{\mathbf{n}\in\mathcal{I}_l, \mathbf{n}'\sim\mathbf{n} \\ n_j = s-1, n'_j = s; n_j = 2s-2; n'_j = 2s \\ 2n + |\mathbf{n}| = 2ns+s-1, 1+|\mathbf{n}'| = 2ns+s+1}} \widetilde{f}_{2n,\mathbf{n};1,\mathbf{n}'},$$

$$\widetilde{v}_{2ns} = A_{2n,1} \sum_{\substack{\mathbf{n}\in\mathcal{I}_l, \mathbf{n}'\sim\mathbf{n} \\ n_j = s-1, n'_j = s; n_j = 2s-2; n'_j = 2s \\ 2n + |\mathbf{n}| = 2ns, 1+|\mathbf{n}'| = 2ns+1}} \mathrm{Proj}_{\mathcal{C}_{2ns}} \widetilde{f}_{2n,\mathbf{n};1,\mathbf{n}'}.$$

To finish the proof, it is sufficient to take $a_1^{2n,1} = \frac{1}{A_{2n,1}} \Pi_1 \widetilde{v}_{2ns+s-1}$, $a_2^{2n,1} = \frac{1}{A_{2n,1}} \Pi_2 \widetilde{v}_{2ns}$. ∎

Lemmas 35, 36, 37 can be proved in a similar way.

## A.3. PROOFS OF CASE III: $R > 2S - 1$.

The last case corresponds to the situation when the first nonzero normal form coefficients are $a_r$, $b_s$, with $r > 2s - 1$. Now, we will focus into the use of $ML_{s-1}ML_{s-1}$ instead $MN_{r-1}$.

Moreover, we must use indices $\mathbf{n}$ and $\mathbf{n}' \sim \mathbf{n}$ with $n_j \geq s - 1$, and $n'_j = n_j + 1$ whenever $s - 1 \leq n_j \leq r - 2$.

Next, we show the proof of Lemma 39.

*Proof.* Take a generator $U = \sum_{k \geq 1} U_k$ such that

- $U_1 = \cdots = U_{h-1} = 0$,
- $U_h = A_{h,0} f_{h,0} \in \mathcal{H}_{h;h}$.
- For $k > h$, we take $U_k = U_k^L + U_k^{NL}$, where $U_k^L$ y $U_k^{NL}$ satisfy (8), (9), and they only depend on $A_{h,0}$.

As in the above cases, we can show that $\widetilde{v}_k = \text{Proj}_{\mathcal{C}_k}[v, U]$ $(k = 2, \ldots, 2hs+r-s-2)$ are given in (20), taking $A_{\tilde{h},i} = 0$ for $\tilde{h} \neq h, i \neq 0$. Moreover, applying Lemmas 8, 10, we get $\widetilde{v}_2 = \cdots = \widetilde{v}_{hs+r-s-1} = 0$, and $\Pi_1 \widetilde{v}_k = 0$, for $k < hs + 2r - 2s$. Also, the expressions for the constants are just those appearing in the statement of the lemma. Moreover, we can prove that $A_{h,0}$ appears nonlinearly in the normal form coefficients of order $2hs + 2r - 3s + 1$ and upper. ▌

The proof of Lemma 40 is analogous to those of Lemma 39. The only difference is that we must use now the Lemma 11.

**Acknowledgments**
This work was partially supported by the *Consejería de Educación y Ciencia de la Junta de Andalucía* (TIC-130) and by the *Ministerio de Ciencia y Tecnología, fondos FEDER* (project BFM2001-2608).

## REFERENCES

1. A. Algaba, E. Freire, and E. Gamero, *Hypernormal forms for equilibria of vector fields. Codimension one linear degeneracies*, Rocky Mountain Journal of Mathematics **29** (1999), 13–45.
2. A. Baider and J.A. Sanders, *Further reduction of the Takens–Bogdanov normal form*, Journal of Differential Equations **99** (1992), 205–244.
3. G. Chen and J. Della Dora, *An algorithm for computing a new normal form for dynamical systems*, Journal of Symbolic Computation **29** (2000), 393–418.
4. S. N. Chow and J. K. Hale, *Methods of Bifurcation Theory*, Springer–Verlag, New York (1982).

5. R. Cushman and J. A. Sanders, *Nilpotent normal forms and representation theory of* $sl(2, r)$, Contemporary Mathematics **56** (1986), 31–51.

6. C. Elphick, E. Tirapegui, M.E. Brachet, P. Coullet, and G. Iooss, *A simple global characterization for normal forms of singular vector fields*, Physica D **29** (1987), 95–127.

7. E. Gamero, E. Freire, and E. Ponce, *On the normal forms for planar systems with nilpotent linear part*, International Series of Numerical Mathematics **97** (1991), 123–127.

8. H. Kokubu, H. Oka, and D. Wang, *Linear grading function and further reduction of normal forms*, Journal of Differential Equations **132** (1996), 293–318.

9. F. Loray, *Réduction formelle des singularités cuspidales de champs de vecteurs analytiques*, Journal of Differential Equations **158** (1999), 152–173.

10. E. Strózyna, and H. Zoladek, *The analytic and formal normal form for the nilpotent singularity*, Journal of Differential Equations, **179** (2002), 479–537.

11. S. Ushiki, *Normal forms for singularities of vector fields*, Japan Journal of Applied Mathematics, **1** (1984), 1–37.

12. D. Wang, J. Li, M. Huang, and Y. Jiang, *Unique normal form of Bogdanov-Takens singularities*, Journal of Differential Equations, **163** (2000), 223–238.

13. Y. Yuan and P. Yu, *Computation of simplest normal forms of differential equations associated with a double-zero eigenvalue*, International Journal of Bifurcation and Chaos, **11** (2001), 1307–1330.