Piecewise smooth differential systems with a center: normal form and limit cycle bifurcation

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Outline of the talk

- Background and the related results.
- The main results.
- Sketch proof of the main results.

Background

Consider a planar piecewise smooth differential system:

$$\dot{x} = f(x,y), \quad \dot{y} = g(x,y), \qquad (x,y) \in \Omega, \tag{1}$$

where

- $\Omega \subset \mathbb{R}^2$ a connected open subset,
- f and g are piecewise smooth.

For more details,

- let $h(x,y)$ be a smooth function defined in Ω, such that
  - $\Sigma := \{(x,y) \in \Omega | h(x,y) = 0\}$ being a regular curve, i.e. $\nabla h(x,y) \neq 0$ on Σ, which is called switch curve of system (1).
- Denote by
  - $\Omega_+ = \Omega \cap \{(x,y) | h(x,y) \geq 0\}$,
  - $\Omega_- = \Omega \cap \{(x,y) | h(x,y) \leq 0\}$.

Then system (1) can be written in

$$\dot{x} = f_+(x,y), \quad \dot{y} = g_+(x,y), \qquad (x,y) \in \Omega_+, \tag{2}$$
$$\dot{x} = f_-(x,y), \quad \dot{y} = g_-(x,y), \qquad (x,y) \in \Omega_-, \tag{3}$$

where

- $f_+, g_+, f_-, g_- \in C^r(\Omega)$, with
  - $r \in \mathbb{N} \cup \{\infty, \omega\}$ and $\mathbb{N}$ the set of positive integers.

Denote by $\mathscr{Z}$ the vector field associated to (2) and (3).

As we know, for piecewise smooth differential systems, there are lots of results from different aspects. For instance,

- Local properties of singularities,
- Periodic orbits
- Oscillations
- Regular perturbation
- Singular perturbation
- and so on

Here, we mainly concern:

- The equivalence between Σ–centers (degenerate or nondegenerate) of piecewise smooth systems
- The smoothness of the transformation between two Σ–centers when restricted on each side of the switching line except at the centers.
- The limit cycle bifurcation from the Σ–center via normal form system.

Notations:

- Denote by $\mathscr{X}$ the vector field associated to systems (2) defined in Ω_+.
- Denote by $\mathscr{Y}$ the vector field associated to systems (3) defined in Ω_-.
- Denote by $\mathscr{Z} = (\mathscr{X}, \mathscr{Y})$ the piecewise smooth vector field defined in Ω.
- Let $\mathscr{V}$ be the set of all piecewise smooth vector fields defined in Ω with the switching curve Σ, and endowed with the C^r product topology.

Definition: Contact point, fold point

- A point $p \in \Sigma$ is a Σ–kth contact point for the vector field $\mathscr{X}$ if
$$\mathscr{X}^k h(p) \neq 0 \text{ and } \mathscr{X}^\ell h(p) = 0 \text{ for } \ell = 1, \ldots, k-1,$$
where by definition
  - $\mathscr{X} h(p) = \langle \mathscr{X}(p), \nabla h(p) \rangle$ and
  - $\mathscr{X}^\ell h(p) = \langle \mathscr{X}(p), \nabla(\mathscr{X}^{\ell-1} h)(p) \rangle$ for $\ell \in \mathbb{N}$, and
  - $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors.
- Especially, if $k = 2$ we call $p \in \Sigma$ a Σ–fold point.

Definition: visible and invisible:

- A Σ–kth contact point p is visible for $\mathscr{X}$ if the trajectory passing through p remains in the Ω_+.
- A Σ–kth contact point p is invisible for $\mathscr{X}$ if the trajectory passing through p remains in the Ω_-.

Note:

- For a visible Σ–kth contact point, k must be even.
- For an invisible Σ–kth contact point, k is also even

Definition: contact singularity, fold singularity

- Point $p \in \Sigma$ is a (k,l)–contact singularity of $\mathscr{Z} = (\mathscr{X}, \mathscr{Y})$ if p is a Σ–kth contact point for $\mathscr{X}$ and is a Σ–lth contact point for $\mathscr{Y}$.
- Point $p \in \Sigma$ is a two–fold singularity of $\mathscr{Z} = (\mathscr{X}, \mathscr{Y})$ if $(k,l) = (2,2)$ (i.e. p is a Σ–fold point for both of $\mathscr{X}$ and $\mathscr{Y}$).

Definition: Σ–center

- p is a Σ–center, if
  - $p \in \Sigma$ is invisible for both of $\mathscr{X}$ and $\mathscr{Y}$,
  - all orbits of $\mathscr{Z}$ are closed in a neighborhood of p.
- p is a (k,l)–Σ–center, if
  - p is a (k,l)–contact point and is a center.
- p is called a nondegenerate Σ–center, if
  - p is a two–fold singularity and is a center.
- The center p is called a degenerate Σ–center in all the other cases.

Definition: Equivalence and conjugation of PS VFs.

Given two piecewise smooth vector fields

- $\mathscr{Z} = (\mathscr{X}, \mathscr{Y}) \in \mathscr{V}$ defined in an open set $U \subset \Omega$ with $p \in U \cap \Sigma$
- $\mathscr{Z}_1 = (\mathscr{X}_1, \mathscr{Y}_1) \in \mathscr{V}$ defined in an open set $U_1 \subset \Omega$ with $p_1 \in U_1 \cap \Sigma$.

Recall that $\mathscr{V}$ is the set of piecewise smooth vector fields defined in Ω.

$\mathscr{Z}$ and $\mathscr{Z}_1$ are

- Σ–equivalent if there exists an orientation preserving homeomorphism

$$h : U \longrightarrow U_1$$

  which transforms

  - the orbits of $\mathscr{X}$ in $\Omega_+ \cap U$ to those of $\mathscr{X}_1$ in $\Omega_+ \cap U_1$, and
  - the orbits of $\mathscr{Y}$ in $\Omega_- \cap U$ to those of $\mathscr{Y}_1$ in $\Omega_- \cap U_1$.

- piecewise C^k smooth Σ–equivalent with $k \in \mathbb{N} \cup \{\infty, \omega\}$ if they are Σ–equivalent and the homeomorphism between $\mathscr{Z}$ and $\mathscr{Z}_1$ is C^k when restricted

  - to $(\Omega_+ \cap U) \setminus \{p\}$, and
  - to $(\Omega_- \cap U_1) \setminus \{p_1\}$, respectively.

$\mathscr{Z}$ and $\mathscr{Z}_1$ are

- Σ–conjugate if there exists an orientation preserving homeomorphism $h : U \longrightarrow U_1$ which conjugates
  - the flow of $\mathscr{X}$ in $\Omega_+ \cap U$ and that of $\mathscr{X}_1$ in $\Omega_+ \cap U_1$, and
  - the flow of $\mathscr{Y}$ in $\Omega_- \cap U$ and that of $\mathscr{Y}_1$ in $\Omega_- \cap U_1$.
- piecewise C^k smooth Σ–conjugate if they are Σ–conjugate and the homeomorphism between $\mathscr{Z}$ and $\mathscr{Z}_1$ is C^k when restricted
  - to $(\Omega_+ \cap U) \setminus \{p\}$, and
  - to $(\Omega_- \cap U_1) \setminus \{p_1\}$, respectively.

Related results on background

On the study of equivalence and conjugate between the flows of two piecewise smooth vector fields, there are very few results.

Along this direction, Buzzi, Carvalho and Teixeira in [JMPA, 2014] provided a normal form of a nondegenerate Σ–center through homeomorphisms, which can be stated as follows:

Buzzi, Carvalho and Teixeira Theorem

Let

$\mathscr{Z} = (\mathscr{X}, \mathscr{Y})$ be a piecewise smooth vector field having a nondegenerate Σ–center, where Σ is the x–axis.

Then

$\mathscr{Z}$ is Σ–equivalent to the piecewise smooth vector field

$$\mathscr{Z}_0 = (\mathscr{X}_0, \mathscr{Y}_0)$$

with

$$\begin{aligned} \mathscr{X}_0(x,y) &= (-1, 2x) \qquad \text{for } y \geq 0, \\ \mathscr{Y}_0(x,y) &= (1, 2x) \qquad \text{for } y \leq 0. \end{aligned} \tag{4}$$

Remark:

Since the Σ–equivalence given by Buzzi et. al. was defined through homeomorphism,

- the transformation cannot provide a direct relation between the two vector fields.
- the dynamics of the normal form vector field $\mathscr{Z}_0$ under perturbation cannot be used to study those of $\mathscr{Z}$, because of the absence of regularities of the transformation.

So it is necessary to consider

the piecewise smooth Σ–equivalence.

Next we also assume: Σ is the x–axis, i.e. $h(x,y) = y$

Main results

First one: periods of the periodic orbits of a (k,l)–Σ–center.

Theorem A [JDE 16]

Let k,l be positive even numbers, and $\max\{k,l\} \leq r \in \mathbb{N} \cup \{\infty, \omega\}$.
For $\mathscr{Z} = (\mathscr{X}, \mathscr{Y}) \in \mathscr{V}$ with

the switching line Σ and $\mathscr{X}, \mathscr{Y} \in C^r(\Omega)$,

if the vector field $\mathscr{Z} \in \mathscr{V}$ has a (k,l)–Σ–center at the origin O, then the periods of the periodic orbits of the Σ–center are monotonic.

This theorem will be used in the proof of Proposition 1, which illustrates an example whose flow is Σ–equivalent to that of $\mathscr{Z}_0$ but not Σ–conjugate to that of $\mathscr{Z}_0$.

Second one: normal form of piecewise smooth systems having a (k,l)–Σ–center via the piecewise smooth Σ–equivalence.

Theorem B [JDE 16]

Let

- k,l,r be those defined in Theorem A.
- $\mathscr{X} \in \mathscr{V}$ has a (k,l)–Σ–center at O

Then there exists a homeomorphism h

from a neighborhood U of O to a neighborhood V of O

such that

(a) $\mathscr{X}$ is locally Σ–equivalent to $\mathscr{X}_0$ via h,

(b) h is a C^r diffeomorphism

from $U \cap (\Omega_+ \setminus \{O\}) \longrightarrow V \cap (\Omega_+ \setminus \{O\})$, and resp.

from $U \cap (\Omega_- \setminus \{O\}) \longrightarrow V \cap (\Omega_- \setminus \{O\})$.

Remark:

- Theorem B shows that the Σ–center of piecewise smooth vector fields has some special property than that of smooth vector fields. Because it is usually not possible that a degenerate center of a smooth vector field is equivalent to a nondegenerate center of another smooth system.
- Theorem B is a generalization of Buzzi et. al. [JMPA 2014] in two aspects:
  - our result permits the center to be degenerate, but not for that of Buzzi et. al.;
  - the homeomorphism is a C^r diffeomorphism in the regions separated by the switching line.

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- Theorem B shows the equivalence between a (k,l)–Σ–center of $\mathscr{Z}$ and a nondegenerate Σ–center of $\mathscr{Z}_0$ the normal form of $\mathscr{Z}$.

  But we don't know their relation about conjugation.

- Now we present an example showing that
  - ∃ a nondegenerate Σ–center of a piecewise smooth differential system which is

    Σ–equivalent but not Σ–conjugate to $\mathscr{Z}_0$ in (4).

The piecewise smooth system $\mathscr{Z}_4 = (\mathscr{X}_4, \mathscr{Y}_4)$ with

$$\begin{aligned} \mathscr{X}_4(x,y) &= (-1, 2x) \qquad \text{for } y \geq 0, \\ \mathscr{Y}_4(x,y) &= (1-y, x) \qquad \text{for } y \leq 0. \end{aligned} \tag{5}$$

has a nondegerate Σ–center at the origin.

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Proposition 1

There exist neighborhoods U_4 and V_4 of the origin and a homeomorphism $h_4 : U_4 \to V_4$ such that

$\mathscr{Z}_4$ is Σ–equivalent to $\mathscr{Z}_0$, but it is not Σ–conjugate to $\mathscr{Z}_0$.

Furthermore, the homeomorphism h_4 is

- C^ω in both of the regions $U_4 \cap \{y > 0\}$ and $U_4 \cap \{y < 0\}$,
- and it is C^0 but not C^1 on $y = 0$.

Remark:

- The homeomorphism can be found using the proof of Theorem A.
- The proof of this proposition will not be presented here

Finally we will apply Theorem B to study

limit cycle bifurcation

under small perturbation of the Σ–center.

Theorem C [JDE 16]

Let

- $\mathscr{Z} \in \mathscr{V}$ be a piecewise C^r smooth vector field having a (k,l)–Σ–center at the origin O with $\max\{k,l\} \le r \in \mathbb{N} \cup \{\infty\}$.
- $\mathscr{W}$ be the set of all piecewise C^r smooth vector fields defined in Ω with the switching line Σ.

Then for

- any $n \in \mathbb{N} \cup \{\infty\}$,
- any neighborhood $\mathscr{U} \subset \mathscr{W}$ of $\mathscr{Z}$,

there exists a perturbed vector field $\mathscr{Z}^\varepsilon \in \mathscr{U}$ such that

$\mathscr{Z}^\varepsilon$ has n hyperbolic limit cycles,

which bifurcate from the periodic orbits of the Σ–center of $\mathscr{Z}$.

Proof of Theorem A

Recall Theorem A in short way

If

a piecewise smooth vector field $\mathscr{X} \in \mathscr{V}$ has a (k,l)–Σ–center at the origin O,

then

the periods of the periodic orbits of the Σ–center are monotonic.

Proof

Step 1: Proving that the time that the orbit of the vector field $\mathscr{X}$ starting at $(p,0)$ on the positive x–axis and arriving at the negative x–axis spends is strictly monotonic in p.

Set

$$\mathscr{X} = (f_+(x,y), g_+(x,y)).$$

Assume that

$$f_+(0,0) < 0.$$

Let

- $(x(t;p,0), y(t;p,0))$ be the solution of $\mathscr{X}$ passing through $(p,0)$ with $p > 0$.

Then there exists a smallest positive time $T_+(p)$ such that

$$y(T_+(p);p,0) = 0.$$

Differentiating this last equation with respect to p, we get

$$T'_+(p) = -\frac{\frac{\partial}{\partial p} y(T_+(p);p,0)}{\frac{\partial}{\partial t} y(T_+(p);p,0)} = -\frac{\int_0^{T_+(p)} \frac{\partial^2}{\partial t \partial p} y(t;p,0)dt}{g_+(x(T_+(p);p,0),0)}. \qquad (6)$$

Then some calculations show that

$$T'_+(0) = \lim_{p \to 0} T'_+(p) = -\frac{T'_+(0)}{f_+(0,0)T'_+(0)+1}.$$

This shows that

$$T'_+(0) = -\frac{2}{f_+(0,0)} > 0.$$

So $T_+(p)$ strictly monotonically increases in a neighborhood of $p = 0$ when p increases.

Step 2: Proving that the time $T_-(p)$ that the orbit of the vector field $\mathscr{Y}$ from $(p,0)$ on the positive x–axis to the negative x–axis spends strictly monotonically decreases in p.

The idea of the proof is similar.

Step 3: The period $T(p)$ of the closed orbit passing through the initial point $(p,0)$ with $p>0$ is the difference between the two times, that is

The periods $T(p) = T_+(p) - T_-(p)$ is monotonic.

Theorem A is proved.

Proof of Theorem B

Recall Theorem B

For any vector field $\mathscr{Z} \in \mathscr{V}$ having a (k,l)–Σ–center at the origin, there exists a homeomorphism h such that

(a) the vector field $\mathscr{Z}$ is locally Σ–equivalent to $\mathscr{Z}_0 = (\mathscr{X}_0, \mathscr{Y}_0)$ in (4) via h,

(b) h is a C^r diffeomorphism in respectively the upper and lower regions.

Proof of Theorem B (a).

Step 1: Let

- γ_p^+ be the orbit of $\mathscr{X}$ passing through $(p,0)$, which is a Σ–kth contact point of $\mathscr{X}$ with $k>1$.

Proving that

- γ_p^+ can be locally expressed as $y=\rho(x)$ in a neighborhood of p, which satisfies

$$\frac{d^k\rho}{dx^k}(p)\neq 0,\qquad \frac{d^\ell\rho}{dx^\ell}(p)=0,\qquad \ell=1,\ldots,k-1,$$

Step 2:

Proving that

- k and l are both even for a piecewise smooth vector field $\mathscr{Z} = (\mathscr{X}, \mathscr{Y}) \in \mathscr{V}$ with a (k,l)–Σ–center at $(p,0)$,
- the orbits of $\mathscr{X}$ in Ω_+ and of $\mathscr{Y}$ in Ω_- are all convex in a neighborhood of $(p,0)$.

Step 3: Constructing a homeomorphism which sends the closed orbits of $\mathscr{Z}$ to the closed orbits of $\mathscr{Z}_0$.

By step 2, we can parameterize these orbit arcs of $\mathscr{X}$ by the polar coordinates $(x,y) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$.

So system (2) in Ω_+ near the origin can be written as

$$\frac{dr}{d\theta} = P_+(r,\theta), \tag{7}$$

where $P_+(r,\theta)$ is C^r because the vector field $\mathscr{X}$ is C^r.

Let

- I be the open interval in $\mathbb{R}$ such that
$$I \times \{0\} = U \cap \{x > 0, y = 0\}.$$
- $\rho(\theta, \sigma)$ be the solution of equation (7) with the initial condition $r(0) = \sigma \in I$, which is C^r in its variables.

Define the map $(x, y) = \Phi_+(\theta, \sigma) = (\rho(\theta, \sigma)\cos\theta, \rho(\theta, \sigma)\sin\theta)$:

$$\begin{array}{ccc} [0, \pi] \times I & \longrightarrow & U_+ \\ (\theta, \sigma) & \longrightarrow & \Phi_+(\theta, \sigma), \end{array}$$

which is bijective, and is C^r, where U_+ is the set formed by the intersection of the convex and closed orbits of $\mathscr{X}$ with the half plane $\{y \geq 0\}$.

For $\mathscr{Z}_0 = (\mathscr{X}_0, \mathscr{Y}_0)$ in equation (4), similar to $\mathscr{Z}$.

Define the map for $\mathscr{Z}_0$ by

$$(x', y') = \Psi_+(\theta, \sigma) : [0, \pi] \times I \longrightarrow V_+,$$

which is a C^ω diffeomorphism, where V_+ is the subsets of V with the intersections of the regions $\{y \geq 0\}$.

Define

$$H_+ := \Psi_+ \circ \Phi_+^{-1} : U_+ \longrightarrow V_+.$$

Then

- it defines a C^r diffeomorphism from U_+ to V_+,
- it maps the orbit arcs of $\mathscr{X}$ in U_+ to the ones of $\mathscr{X}_0$ in V_+.

In the lower half plane it is similar to

define a C^r diffeomorphism from U_- to V_-

$$H_- := \Psi_- \circ \Phi_-^{-1} : U_- \longrightarrow V_-,$$

which sends the orbit arcs of $\mathscr{Y}$ in U_- to the ones of $\mathscr{Y}_0$ in V_-.

Set

$$H(x,y) = \begin{cases} H_+(x,y), & (x,y) \in U_+, \\ (0,0), & (x,y) = (0,0), \\ H_-(x,y), & (x,y) \in U_-. \end{cases}$$

This is a homeomorphism that we need, because

$$H_+(x,y) = H_-(x,y), \qquad (x,y) \in U_+ \cap U_-.$$

Proof of Theorem B (b).

For proving the smoothness of H_+ and H_- on the boundary, we extend

$$H_+ : \quad U_+ \to V_+$$

to be a C^r diffeomorphism

$$H'_+ : \quad C_+ \to D_+,$$

where C_+ is the region formed by the orbit arcs of equation (7) defined on $[-\theta_0, \pi+\theta_0]$ with the initial point on I, where $\theta_0 > 0$ is a suitable small number; and D_+ is the same region.

Similarly, extend

$$H_- : \quad U_- \to V_-$$

to be a C^r diffeomorphism

$$H'_- : \quad C_- \to D_-$$

By these extensions, one can check that

- H is C^r smooth on the region $y \geq 0$, because $H = H_+ = H'_+$ when restricted to the upper half plane $y \geq 0$.
- H is C^r smooth on the region $y \leq 0$, because $H = H_- = H'_-$ when restricted to the upper half plane $y \leq 0$.

This proves Theorem B.

Similarly, extend

$$H_- : \quad U_- \to V_-$$

to be a C^r diffeomorphism

$$H'_- : \quad C_- \to D_-$$

By these extensions, one can check that

- H is C^r smooth on the region $y \geq 0$, because $H = H_+ = H'_+$ when restricted to the upper half plane $y \geq 0$.
- H is C^r smooth on the region $y \leq 0$, because $H = H_- = H'_-$ when restricted to the upper half plane $y \leq 0$.

This proves Theorem B.

Proof of Theorem C

Recall Theorem C

Let

- $\mathscr{Z} \in \mathscr{V}$ have a (k,l)–Σ–center at the origin O.
- $\mathscr{W}$ be the set of all piecewise C^r smooth vector fields.

Then for

- any $n \in \mathbb{N} \cup \{\infty\}$,
- any neighborhood $\mathscr{U} \subset \mathscr{W}$ of $\mathscr{Z}$,

there exists a perturbed vector field $\mathscr{Z}^\varepsilon \in \mathscr{U}$ such that $\mathscr{Z}^\varepsilon$ has n hyperbolic limit cycles.

Proof of Theorem C [The proof is constructive]

Since $\mathscr{Z} = (\mathscr{X}, \mathscr{Y}) \in V$ has a (k,l)–Σ–center,

$\Downarrow$ by Theorem B

there exists a piecewise C^r smooth diffeomorphism

$$(u,v) = h(x,y),$$

which sends the orbits of $\mathscr{Z}$ to the orbits of $\mathscr{Z}_0 = (\mathscr{X}_0, \mathscr{Y}_0)$.

By Buzzi et. al. [JMPA 2014], the perturbed vector field

$$\mathscr{Z}_0^{\varepsilon}(u,v) = \begin{cases} \mathscr{X}_0(u,v), & \text{for } v \geq 0, \\ \mathscr{Y}_0(u,v) + \mathscr{Y}_0^{\varepsilon}(u,v), & \text{for } v \leq 0, \end{cases}$$

of the normal form vector field $\mathscr{Z}_0$ has n hyperbolic limit cycles, where

$$\mathscr{Y}_0^{\varepsilon}(u,v) = \begin{pmatrix} 0 \\ \frac{\partial \xi_0^{\varepsilon}}{\partial u}(u) \end{pmatrix},$$

with

$$\xi_0^{\varepsilon}(u) = \begin{cases} \varepsilon g(u)(\varepsilon - u)(2\varepsilon - u)\ldots(n\varepsilon - u), & \text{for } n \in \mathbb{N}, \\ g(u)\sin(\pi\varepsilon^2/u), & \text{for } n = \infty, \end{cases}$$

and

$$g(u) = \begin{cases} 0, & \text{for } u \leq 0, \\ e^{-1/u}, & \text{for } u > 0. \end{cases}$$

Define

$$\mathscr{X}_1 = Dh^{-1}\mathscr{X}_0 \circ h(x,y),$$
$$\mathscr{X}_1^{\varepsilon} = Dh^{-1}\mathscr{X}_0^{\varepsilon} \circ h(x,y),$$

where Dh^{-1} is the Jacobian matrix.

Since $\mathscr{X}_1$ and $\mathscr{X}$ have the same orbits, there exists a positive function $f(x,y)$ such that

$$\mathscr{X} = \mathscr{X}_1 \cdot f(x,y).$$

Set

$$\mathscr{X}^{\varepsilon} = \mathscr{X}_1^{\varepsilon} \cdot f(x,y)$$

Then

- Since $\lim\limits_{\varepsilon\to 0}\mathscr{X}_0^{\varepsilon} = \mathscr{X}_0$, we have $\lim\limits_{\varepsilon\to 0}\mathscr{X}^{\varepsilon} = \mathscr{X}$.

The next relation among these vector fields:

$$\begin{array}{ccccc} & & \text{perturbed} & & \\ & \mathscr{Z}_0 & \longrightarrow & \mathscr{Z}_0^{\varepsilon} & \text{(has } n \text{ limit cycles)} \\ \text{conjugation} & \updownarrow h & & h \updownarrow & \text{conjugation} \\ & \mathscr{Z}_1 & \longrightarrow & \mathscr{Z}_1^{\varepsilon} & \text{(has } n \text{ limit cycles)} \\ \text{equivalence} & \updownarrow & \text{perturbed} & \updownarrow & \text{equivalence} \\ & \mathscr{Z} & \longrightarrow & \mathscr{Z}^{\varepsilon} & \text{(has } n \text{ limit cycles)} \end{array}$$

completes the proof of Theorem C.

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Thanks for your attention!