Characterizing planar polynomial vector fields with an elementary first integral

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The topic

- **Ultimate goal**: Understand polynomial planar vector fields which admit an elementary first integral.
- **Foundation** and starting point: Prelle and Singer.
- **Examples**: Vector fields which admit a Darboux first integral.
- **Lesser goal of this talk**: Elementary vs. Darboux.

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Notation and notions (Part I)

Given: A complex polynomial vector field

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$$

with associated ordinary differential equation

$$\begin{aligned} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{aligned}$$

- A (non-constant) local analytic function $H : U \longrightarrow \mathbb{C}$ is called a *first integral* of X on U if and only if $X(H) = 0$.
- A local analytic function $\mu : U \longrightarrow \mathbb{C}$ is called an *integrating factor* of field X on U if and only if $X(\mu) = -(P_x + Q_y)\mu$; in other words $\mathrm{div}(\mu \cdot X) = 0$.

Notation and notions (Part II)

Elementary extensions of $\mathbb{C}(x, y)$:
Adjoin successively (in any order) algebraic functions, exponentials and logarithms.
Local realization: Analytic functions on a subset of the complex plane.

Darboux functions:

$$F = \exp(R) \cdot \prod S_i^{c_i}$$

with rational functions R and S_i and complex constants c_i.

Note: Existence of Darboux first integral F is equivalent to existence of the first integral

$$\log(F) = R + \sum c_i \log(S_i).$$

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Prelle and Singer (Part I)

Theorem.

(a) If the polynomial vector field X admits an elementary first integral then there exist an integer $m \geq 0$, algebraic functions v, $u_1, \ldots, u_m$ over $\mathbb{K}$ and nonzero constants $c_1, \ldots, c_m \in \mathbb{C}$ such that

$$X(v) + \sum_{i=1}^{m} c_i \frac{X(u_i)}{u_i} = X\left(v + \sum_{i=1}^{m} c_i \log(u_i)\right) = 0,$$

but

$$v + \sum_{i=1}^{m} c_i \log(u_i) \neq \mathrm{const.}$$

The c_i may be chosen linearly independent over the rational numbers $\mathbb{Q}$.

Prelle and Singer (Part II)

Theorem (cont.)

(b) If the vector field X admits an elementary first integral then it admits an integrating factor of the special form

$$\mu = f_1^{-d_1} \cdots f_r^{-d_r},$$

with irreducible and pairwise relatively prime polynomials $f_1, \ldots, f_r$, and exponents $d_1, \ldots, d_r \in \mathbb{Q}$.

Prescribed integrating factors (Part I)

Motivated by Prelle and Singer, statement (b):
Given irreducible, pairwise relatively prime polynomials f_i and nonzero $d_i \in \mathbb{Q}$, try to determine all polynomial vector fields which admit the integrating factor

$$\mu = f_1^{-d_1} \cdots f_r^{-d_r}.$$

(Christopher, Llibre, Pantazi, W. 2006-2011.)

Starting point: If a vector field X admits such an integrating factor then every algebraic curve

$$C_i = \{(x, y);\ f_i(x, y) = 0\}$$

is invariant.
For given $C_1, \ldots, C_r$, determining all polynomial vector fields with these invariant curves can be done via standard (algorithmic) algebra.

Prescribed integrating factors (Part II)

Try to determine all polynomial vector fields admitting integrating factor

$$\mu = f_1^{-d_1} \cdots f_r^{-d_r}.$$

Simple answer in case of *nondegenerate geometry*:

Theorem. Assume that all the curves C_i are nonsingular, all pairwise intersections and all intersections with the line at infinity are transversal, and there are no multiple intersections (including the line at infinity). Then:

(a) All vector fields admiting the integrating factor μ are known explicitly.

(b) They all admit a Darboux first integral.

(c) If one of the exponents d_i is not a positive integer then all vector fields admit a rational first integral.

Prescribed integrating factors (Part III)

Some theory: The vector fields with non-constant Darboux integrating factor

$$\mu = f_1^{-d_1} \cdots f_r^{-d_r}.$$

form a linear space $\mathcal{F}$. The so-called *trivial vector fields* admitting this integrating factor form a subspace $\mathcal{F}^0 \subseteq \mathcal{F}$. To construct a trivial vector field, start with an arbitrary polynomial g, define

$$Z_g = \text{ Hamiltonian vector field of } \quad g/\left(f_1^{d_1-1} \cdots f_r^{d_r-1}\right).$$

Then the vector field

$$f_1^{d_1} \cdots f_r^{d_r} \cdot Z_g = f \cdot X_g - \sum_{i=1}^{r} (d_i - 1) g \frac{f}{f_i} \cdot X_{f_i}$$

(with $X_h = (-h_y,\ h_x)$ the Hamiltonian vector field of a function h) is polynomial and admits the integrating factor μ.

Prescribed integrating factors (Part IV)

Prescribed integrating factor

$$\mu = f_1^{-d_1} \cdots f_r^{-d_r}$$

with rational d_i.

Theorem. The quotient space $\mathcal{F}/\mathcal{F}^0$ is finite dimensional.

Remarks.

- In many instances one has $\mathcal{F} = \mathcal{F}^0$, or $\mathcal{F}/\mathcal{F}^0$ is known explicitly.
- Elements of $\mathcal{F}^0$ admit a rational first integral.

Prescribed Darboux first integrals

Back to today's problem, viz. elementary vs. Darboux first integrals.

Theorem. (Chavarriga, Giacomini, Giné, Llibre 2003.)
Let

$$H(x,y) = f_1^{\lambda_1} \cdots f_r^{\lambda_r} \exp(g/(f_1^{n_1} \cdots f_r^{n_r}))$$

be a Darboux function with $\lambda_1, \cdots, \lambda_r \in \mathbb{C}$, $n_1, \cdots, n_r \in \mathbb{N}_0$ and $g \in \mathbb{C}[x,y]$ coprime with f_i whenever $n_i \neq 0$. Then H is a first integral of the polynomial vector field

$$\widehat{X} = \prod_{k=1}^{r} f_k^{n_k+1} \cdot \left(\sum_{k=1}^{r} \lambda_k X_{f_k}/f_k + Z_g^{(n_1+1,\ldots,n_r+1)} \right)$$

which, in turn, admits the integrating factor $\prod_{k=1}^{r} f_k^{-(n_k+1)}$.

Rational integrating factors

Theorem.

(a) (Chavarriga et al.:) Any polynomial vector field admitting a Darboux first integral admits a rational integrating factor.

(b) (Chavarriga et al.:) If a polynomial vector field admits a Darboux first integral but not a rational first integral then it admits a unique integrating factor.

(c) (Rosenlicht:) If a polynomial vector field admits a rational integrating factor then it admits a Darboux first integral.

An intermediate resumé

- Polynomial vector fields with prescribed Darboux integrating factors are well understood. (Analytic and geometric arguments.)
- They always admit a Liouville first integral (Singer, Christopher)
- Obstructions on the way from special integrating factor $\mu = f_1^{-d_1} \cdots f_r^{-d_r}$ to elementary first integral are relatively well undrstood (analytically and geometrically).
- Elementary vs. Darboux first integrals? It seems that algebraic arguments are the most appropriate.

Some (differential) algebra

Given an algebraic function w over $\mathbb{K} := \mathbb{C}(x, y)$, we denote its *minimal polynomial* by

$$M_w(T) := T^d + \sum_{i=1}^{d} g_i T^{d-i} \in \mathbb{K}[T].$$

Here $-g_1$ is called the *trace* of w and $(-1)^d g_d$ is called the *norm* of w. The other zeros of M_w (in a suitable extension field $\mathbb{F}$ of $\mathbb{K}$) are called the *conjugates* of w.

Lemma. If an algebraic function w is a first integral of a polynomial vector field then all nonconstant coefficients of its minimal polynomial are first integrals of this vector field.

Prelle and Singer, slightly refined

Proposition. If the polynomial vector field X admits an elementary first integral

$$v + \sum_{i=1}^{m} c_i \log(u_i)$$

where $v,\ u_1, \ldots, u_m$ are nonconstant algebraic functions over $\mathbb{K}$ then it admits an algebraic integrating factor μ in $\mathbb{F} := \mathbb{K}[v,\ u_1, \ldots, u_m]$. Moreover, if $\widehat{\mathbb{F}}$ is a normal extension of $\mathbb{F}$ with degree $[\widehat{\mathbb{F}} : \mathbb{K}] = n$ then $\mu^n \in \mathbb{K}$.

Elementary vs. Darboux first integrals

Theorem. Let the polynomial vector field X admit the elementary first integral

$$v + \sum_{i=1}^{m} c_i \log(u_i),$$

where $m > 0$ and v, $u_1, \ldots, u_m$ are nonconstant algebraic functions over $\mathbb{K}$, and furthermore the constants $c_1, \ldots, c_m$ are linearly independent over $\mathbb{Q}$. If X does not admit a rational first integral then the following hold.

(a) Whenever some u_j has non-constant norm, or whenever v has non-constant trace, then X admits a Darboux first integral.

(b) Moreover, if in this setting $\mu = f_1^{-d_1} \cdots f_r^{-d_r}$ is an integrating factor for X then it is uniquely determined (up to a nonzero constant factor) and all d_i are nonnegative integers.

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Sketch of the proof

Let $\mathbb{F}$ be the smallest normal extension of $\mathbb{K}$ which contains v and all u_i, and denote its Galois group by G.
Then

$$X(\sigma(v)) + \sum_{i=1}^{m} c_i \frac{X(\sigma(u_i))}{\sigma(u_i)} = 0$$

for all $\sigma \in G$, and summation yields

$$X\left(\sum_{\sigma\in G}\sigma(v)\right) + \sum_{i=1}^{m} c_i \frac{X\left(\prod_{\sigma\in G}\sigma(u_i)\right)}{\prod_{\sigma\in G}\sigma(u_i)} = 0.$$

Now $R := \sum_{\sigma\in G}\sigma(v)$ is a positive integer multiple of the trace of v, and each $S_i := \prod_{\sigma\in G}\sigma(u_i)$ is a positive integer power of the norm of u_i. Hence R and all S_i are rational.
There remains to verify non-constancy.

Exceptional cases

Remaining problem: Investigate "exceptional" cases admitting an elementary first integral

$$v + \sum_{i=1}^{m} c_i \log(u_i),$$

with all u_i of constant norm, v of constant trace.

Presumably simplest case: $m = 1$ (and $c_1 = 1$ with no further loss of generality). Thus we assume a relation

$$X(v) + \frac{X(u)}{u} = 0$$

with $u \notin \mathbb{K}$ and $v \notin \mathbb{K}$ (otherwise there exists a rational first integral).

The simplest exceptional cases

Proposition. Let X be a polynomial vector field such that

$$X(v) + \frac{X(u)}{u} = 0.$$

with u and v non-constant algebraic functions, u having constant norm and v having constant trace. Moreover assume that X admits no rational first integral. Then:

(a) The intersection $\mathbb{K}[u] \cap \mathbb{K}[v]$ is nontrivial.

(b) If the degree of $\mathbb{K}[v]$ over $\mathbb{K}$ is a prime number, then $\mathbb{K}[v] \subseteq \mathbb{K}[u]$. If both degrees are prime numbers then $\mathbb{K}[v] = \mathbb{K}[u]$.

Quadratic extensions (Part I)

Let X be a polynomial vector field such that

$$X(v) + \frac{X(u)}{u} = 0$$

with u of constant norm and v of constant trace, both contained in a degree two extension of $\mathbb{K}$, but neither contained in $\mathbb{K}$.

Proposition. With no loss of generality one may take u and v to satisfy

$$u^2 + 2g \cdot u + 1 = 0 \text{ and } v = b(g + u)$$

with a non-constant rational function g and a nonzero rational function b.

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Quadratic extensions (Part II)

Let $X = (P,\ Q)$ be a polynomial vector field such that

$$X(v) + \frac{X(u)}{u} = 0$$

with u and v as in the Proposition.

Theorem. There exists a rational function s such that

$$\begin{aligned} P &= \ -s\cdot\left((g^2-1)b_y + (bg-1)g_y\right),\\ Q &= \ s\cdot\left((g^2-1)b_x + (bg-1)g_x\right), \end{aligned}$$

defines a polynomial vector field. If one requires P and Q to have relatively prime entries then s is unique up to a factor in $\mathbb{C}^*$.

Remark. In general such vector fields do not admit a Darboux first integral (which would necessarily be rational).

Cubic extensions (Part I)

- Let $\mathbb{F} = \mathbb{K}[u] = \mathbb{K}[v]$ be a degree three extension, with v of trace zero and u of norm one (with no loss of generality).
- Search for a polynomial vector field X admitting the elementary first integral $v + \log(u)$ but not a rational first integral.

Proposition. Let μ be the integrating factor of X (which is unique up to a nonzero scalar). Then $\mathbb{F} = \mathbb{K}[\mu]$ and there exists $g \in \mathbb{K}$ such that

$$\mu^3 - g = 0.$$

Thus the field extension $\mathbb{F} : \mathbb{K}$ is a cyclic Galois extension of degree three.

Cubic extensions (Part II)

Preparation: The field $\mathbb{F} = \mathbb{K}[\mu]$ is a cyclic Galois extension of $\mathbb{K}$, degree three.

- A general element of $\mathbb{F}$ has the form

$$w := a + b \cdot \mu + c \cdot \mu^2; \quad a,\ b,\ c \in \mathbb{K},$$

- By Hilbert's Theorem 90 we may write

$$u = \frac{\sigma(w)}{w},$$

  for some nonzero w.

- The trace zero element has the form

$$v = r \cdot \mu + s \cdot \mu^2; \quad r, s \in \mathbb{K}.$$

Cubic extensions (Part III)

Lemma. Let w, u and v be as above, and denote by $N(w)$ the norm of w. Then for every derivation Y of $\mathbb{K}$ an identity

$$\frac{Y(u)}{\mu \cdot u} = \frac{1}{N(w)} A_Y + \frac{\mu}{N(w)} B_Y$$

holds, with rational functions A_Y and B_Y. Explicitly one has

$$\begin{aligned} B_Y = \ & gbc^2 \left(\frac{Y(c)}{c} - \frac{Y(b)}{b} + \frac{1}{3}\frac{Y(g)}{g} \right) \\ & + ab^2 \left(\frac{Y(b)}{b} - \frac{Y(a)}{a} + \frac{1}{3}\frac{Y(g)}{g} \right) \\ & + a^2c \left(\frac{Y(a)}{a} - \frac{Y(c)}{c} - \frac{2}{3}\frac{Y(g)}{g} \right) . \end{aligned}$$

Cubic extensions (Part IV)

Proposition. Let w, u and v be as above.

(a) There exists a nontrivial polynomial vector field X which admits the first integral $H := v + \log u$ if and only if the identity

$$Y(s) + \frac{2}{3}s \cdot \frac{Y(g)}{g} + \frac{1}{N(w)}B_Y = 0$$

holds for all derivations of $\mathbb{K}$.

(b) Given any $u \in \mathbb{F}$ with norm one, there exists at most one rational function s such that H (with $v = r \cdot \mu + s \cdot \mu^2$ and arbitrary $r \in \mathbb{K}$) is a first integral for some nontrivial polynomial vector field.

Cubic extensions: An example

Notation and terminology as above.
Let g be arbitrary with $g_x \neq 0$, $a \in \mathbb{C}^*$ constant and set

$$w := a + \mu, \quad N(w) = g + a^3.$$

Fact. For this choice of w there exists no rational function s such that the condition in part (a) of the Proposition can be satisfied for $Y = \partial/\partial x$.

To verify this, choose y_0 suitably and set $\widehat{g}(x) := g(x, y_0)$. The condition reduces to

$$\widehat{s}' + \frac{2}{3}\widehat{s} \cdot \frac{\widehat{g}'}{\widehat{g}} + \frac{a}{3} \cdot \frac{\widehat{g}'}{\widehat{g} \cdot (\widehat{g} + a^3)} = 0$$

for $\widehat{s} := s(\cdot,\ y_0)$.
This linear inhomogeneous differential equation has no rational solution. (Use variation of constants formula and carry out integrations.)

Some open questions

- Are there nontrivial positive examples in the cubic case? (We do not know.)
- What about first integrals $v + \log(u)$ with u and v in a field extension of higher degree?
- What about the general case?

Thank you for your attention!