Unfoldings of saddle-nodes and their Dulac time

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Bifurcation of critical periodic orbits

A singular point p of a polynomial differential system

$$\begin{cases} \dot{x} = p(x,y), \\ \dot{y} = q(x,y), \end{cases}$$

is a center if it has a punctured neighbourhood that consists entirely of periodic orbits surrounding p. The largest open neighbourhood $\mathscr{P}$ with this property is called the period annulus of the center. The period function of the center assigns to each periodic orbit in $\mathscr{P}$ its period.

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Compactifying $X := p(x,y)\partial_x + q(x,y)\partial_y$ (e.g., to the Poincaré disc), the boundary of $\mathscr{P}$ has two connected components: the center itself and a polycycle Π that we call the outer boundary.

Bifurcation of critical periodic orbits

To parametrize the set of periodic orbits in $\mathscr{P}$ we take an analytic transverse section to X on $\mathscr{P}$, for instance an orbit of $X^{\perp}$. If $\{\gamma_s\}_{s\in(0,1)}$ is such a parametrization, then

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is an analytic map on $(0,1)$. A critical period is an isolated critical point of this function, i.e. $\hat{s} \in (0,1)$ such that

$$T'(s) = \alpha(s-\hat{s})^k + \mathrm{o}\big((s-\hat{s})^k\big)$$

with $\alpha \neq 0$ and $k \geqslant 1$.

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with $\alpha \neq 0$ and $k \geqslant 1$. More geometrically, we shall say that $\gamma_{\hat{s}}$ is a critical periodic orbit.

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We call this number the criticality of the outer boundary and its precise definition is the following, where d_H stands for the Hausdorff distance between compact sets.

Bifurcation of critical periodic orbits

Definition

The criticality of $(\Pi_{\hat{a}}, X_{\hat{a}})$ with respect to the deformation X_a is $\mathrm{Crit}\big((\Pi_{\hat{a}}, X_{\hat{a}}), X_a\big) := \inf_{\delta,\varepsilon} N(\delta,\varepsilon)$, where $N(\delta,\varepsilon)$ is the supremum of the number of critical periodic orbits γ of X_a with $d_H(\gamma, \Pi_{\hat{a}}) \leqslant \varepsilon$ and $\|a-\hat{a}\| \leqslant \delta$.

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Definition

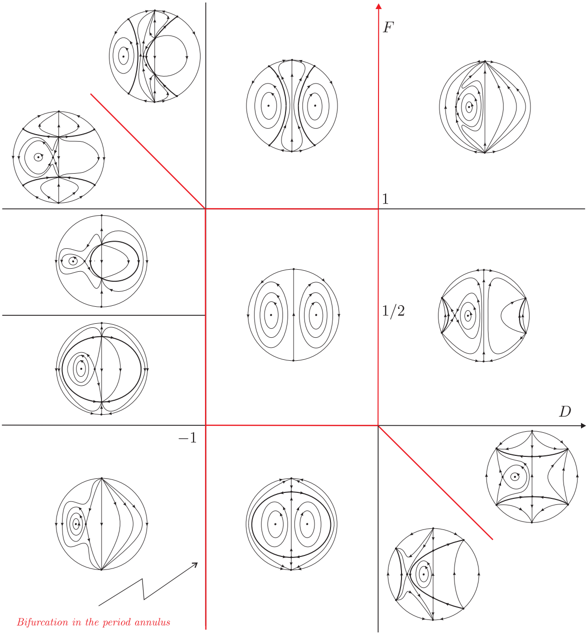
We say that $\hat{a} \in A$ is a local regular value of the period function at the outer boundary of the period annulus if $\mathrm{Crit}\big((\Pi_{\hat{a}}, X_{\hat{a}}), X_a\big) = 0$. Otherwise we say that it is a local bifurcation value of the period function at the outer boundary.

Bifurcation of critical periodic orbits

Since 2001, together with D. Marín and P. Mardešić, we have been developing tools to study the bifurcation of critical periodic orbits from the outer boundary. Our testing ground has been the dehomogenized Loud's family

$$X_a \quad \begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x + Dx^2 + Fy^2, \end{cases}$$

where $a := (D, F) \in \mathbb{R}^2$, which is the most interesting stratum of quadratic centers from the point of view of the period function.



F
1
1/2
D
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Bifurcation in the period annulus

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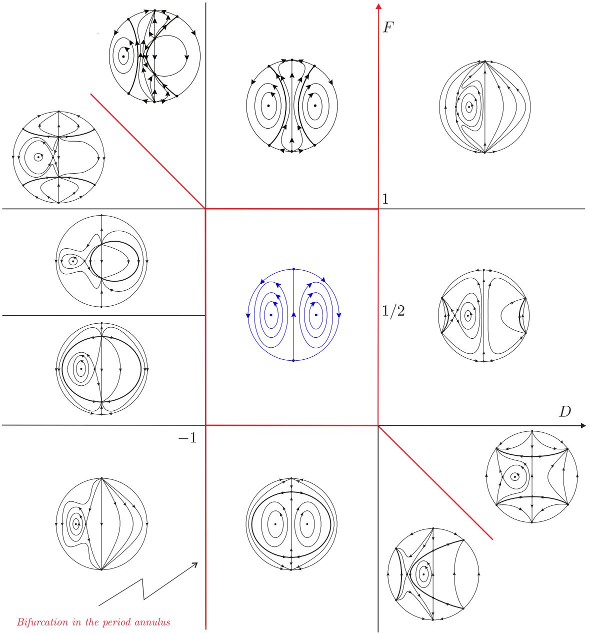

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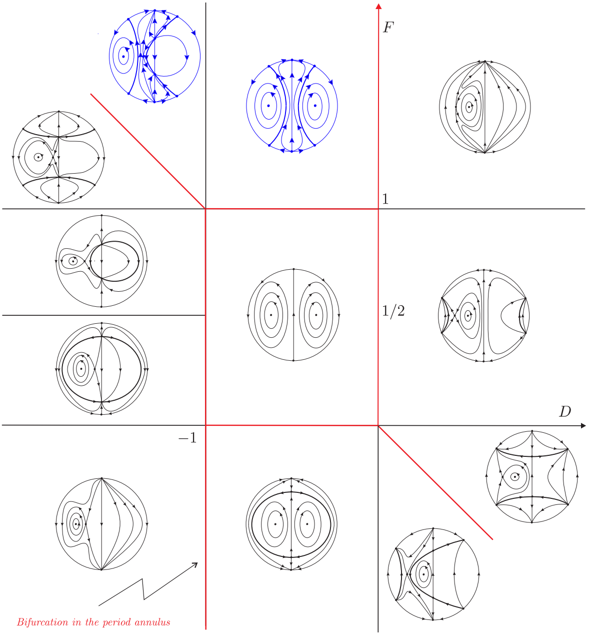

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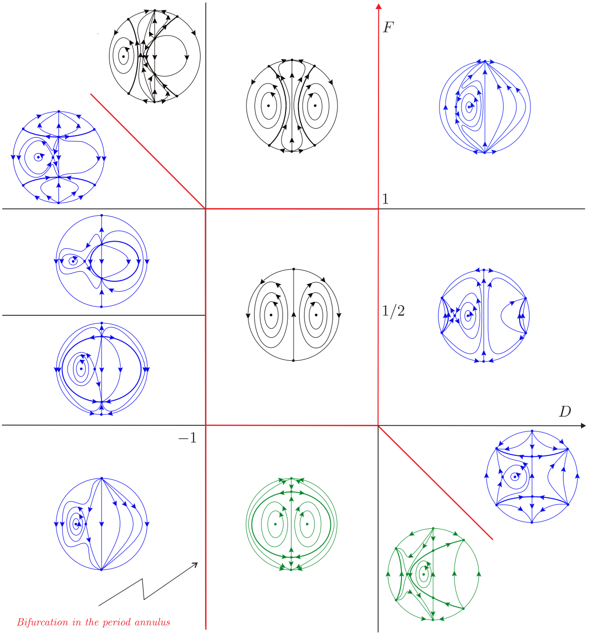

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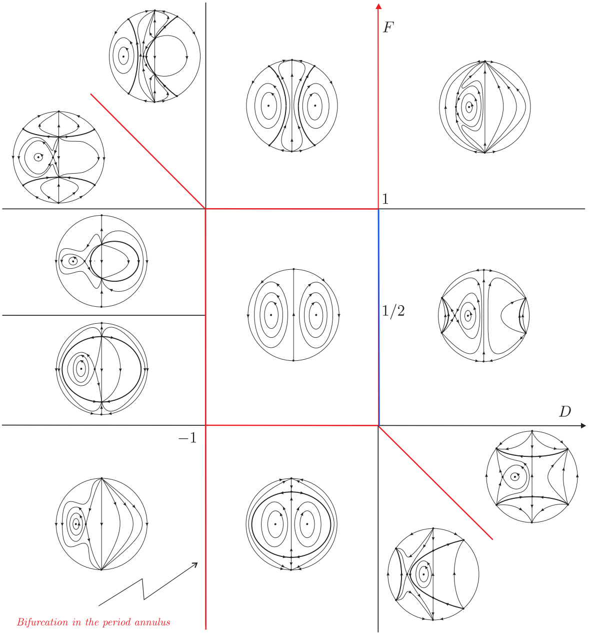

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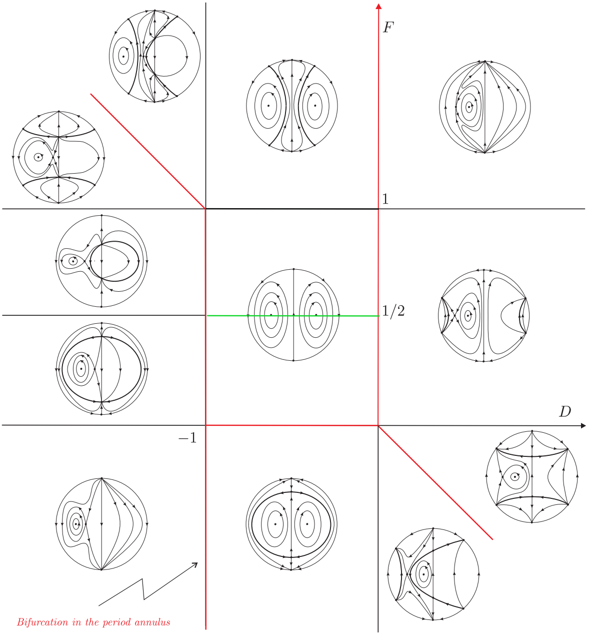

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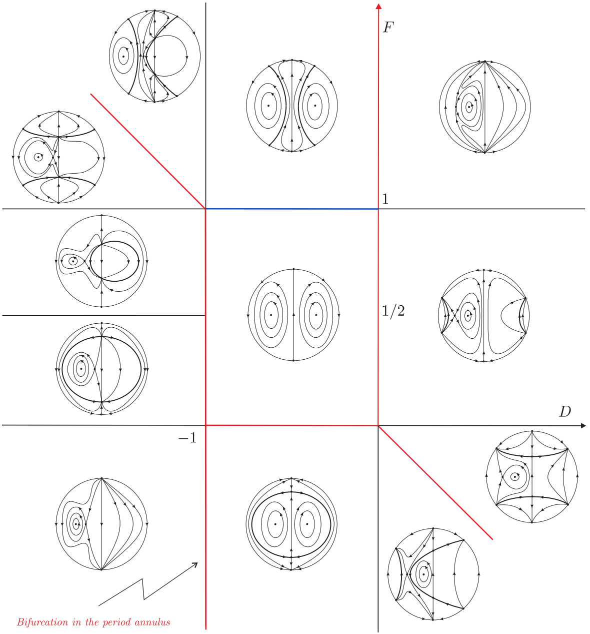
- Mardešić, Marín, Saavedra and Villadelprat, *Unfoldings of saddle-nodes and their Dulac time*, JDE (to appear).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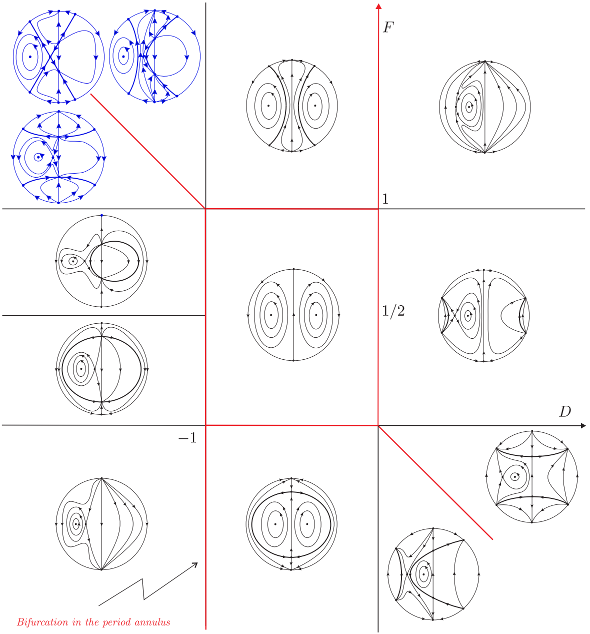

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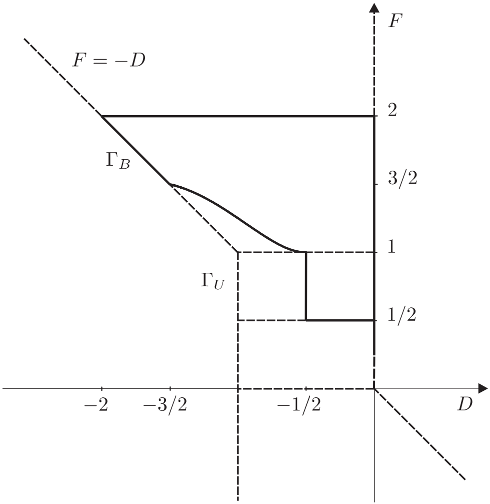
Bifurcation in the period annulus

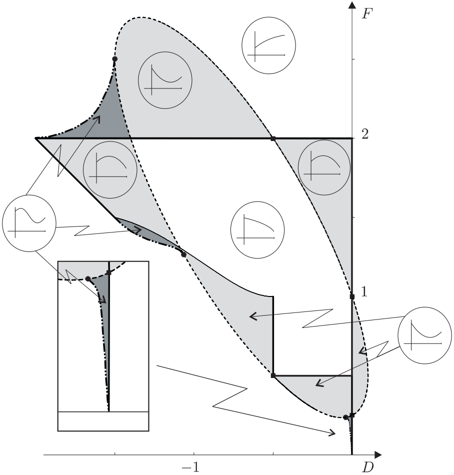
Bifurcation from the polycycle

THEOREM: (2003-06)

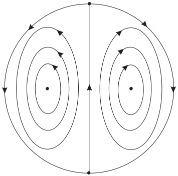
- $\mathbb{R}^2 \setminus \{\Gamma_B \cup \Gamma_U\}$ are local regular values,
- Γ_B are local bifurcation values.



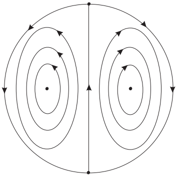
Conjectural bifurcation diagram of the period function



Theorem C: Study at $\{D \in (-1, 0), F = 1\}$



$F < 1$



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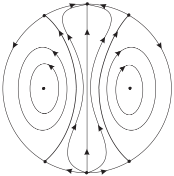

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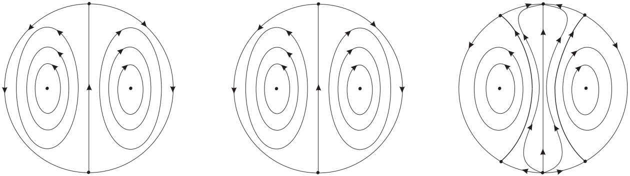

$F < 1$ $\qquad$ $F = 1$ $\qquad$ $F > 1$

Setting $\varepsilon = 2(F - 1)$, one can see that by a local change of coordinates this saddle-node unfolding can be brought to

$$\frac{1}{yU(x,y)}\big(x(x^2 - \varepsilon)\partial_x - (2F - x^2)y\partial_y\big),$$

where $y = 0$ corresponds to the line at infinity.

Theorem B: A more general setting

More generally, in Theorem B we obtain an asymptotic expansion, uniform with respect to the parameters, of the Dulac time of a saddle-node unfolding of the following type:

$$\frac{1}{yU(x,y)}\big(P_\varepsilon(x)\partial_x - V(x)y\partial_y\big), \tag{1}$$

where P_ε, U and V are analytic functions to be described later.

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In fact, we show that any saddle-node unfolding which is locally Darboux integrable is orbitally analytically equivalent to (1).

From Theorem B to Theorem A

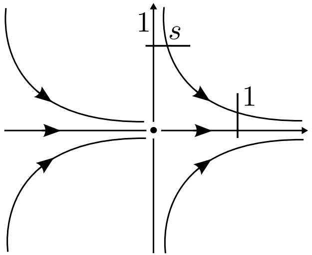
To prove Theorem B we need to develop some results that in principle have nothing to do with the Dulac time.

Since we think that they are interesting on its own we state them separately as Theorem A.

For a better understanding of the statement of Theorem A, we briefly outline without technicalities the underlying ideas that lead to the proof of Theorem B.

From Theorem B to Theorem A

For simplicity, consider $\frac{1}{yU(x,y)}\big(P_\varepsilon(x)\partial_x - V(x)y\partial_y\big)$ with $\varepsilon = 0$ and suppose that the origin is a saddle-node with a hyperbolic sector in the first quadrant.



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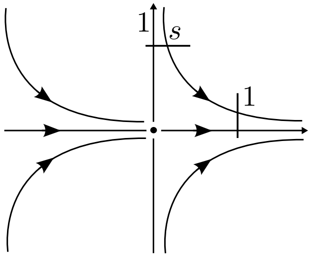

Its Dulac time $T(s)$ can be written as $T = \sum_{n\geqslant 1} T_n$, where each term T_n is in turn the Dulac time of $\frac{1}{y^n U_n(x)}\big(P_0(x)\partial_x - V(x)y\partial_y\big)$, where $U(x,y) = \sum_{n\geqslant 1} U_n(x)y^{n-1}$.

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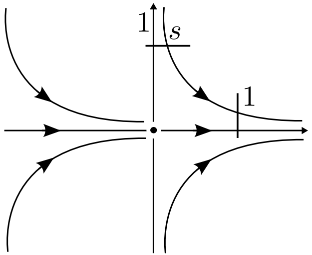

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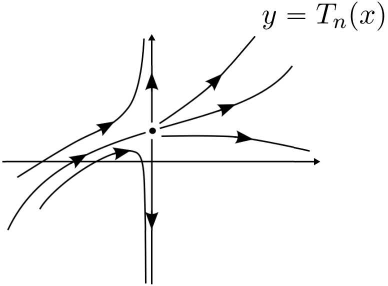
From Theorem B to Theorem A

It turns out that $y = T_n(x)$ is a *trajectory* of the vector field $P_0(x)\partial_x + \big(nV(x)y - U_n(x)\big)\partial_y$ arriving backward in time to the saddle-node at $\left(0, \frac{U_n(0)}{nV(0)}\right)$ through the parabolic sector in $x \geqslant 0$.



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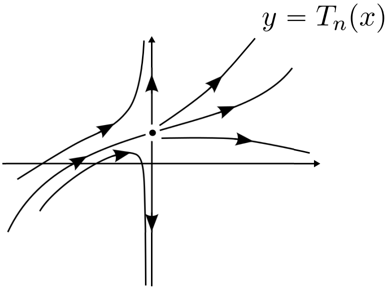

From Theorem B to Theorem A

This is the key point and explains why beginning with the problem of computing the *Dulac time* associated to a *hyperbolic sector*, we end up studying the *trajectories* arriving through a *parabolic sector*.

Assumptions in Theorem A

We consider the unfolding of a saddle-node

$$X = P_\varepsilon(x)\partial_x + \big(\lambda V_a(x)y - U(x)\big)\partial_y, \tag{2}$$

parametrized by $(\varepsilon, a, \lambda, U)$,

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- $P_\varepsilon(x) = P(x;\varepsilon)$ is an analytic function in (x, ε), for $|x| \leqslant r$, such that $P_0(x)$ has a zero of order $\mu + 1 \geqslant 2$ at $x = 0$;

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- $\mathscr{U}$ is the space of series $U(x) = \sum_{j \geqslant 0} u_j x^j \in \mathbb{R}\{x\}$, with convergence radius greater than $r > 0$.

Assumptions in Theorem A

By rescaling, $r = 1$ and $V_a(x) > 0$, for $|x| \leqslant 1$, for all $a \in A$. We endow $\mathscr{U}$ with the norm $\|U\| := \sum_{j \geqslant 0} |u_j|$ and with this norm it becomes a Banach space. We denote $\mathscr{U}_1 := \{U \in \mathscr{U} \,:\, \|U\| \leqslant 1\}$.

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By Weierstrass preparation theorem and rescaling, we can assume that $P_\varepsilon(x)$ is a polynomial of degree $\mu+1$ in x, with $P_0(x) = x^{\mu+1}$.

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Notice that the singularity $(x, y) = (0, U(0)/\lambda)$ is a saddle-node of $X|_{\varepsilon=0}$, whose (real) parabolic sector is contained in the half plane $x \geqslant 0$. We will assume that $P_\varepsilon(x)$ has a real root for $\varepsilon \approx 0$.

Assumptions in Theorem A

In what follows, ϑ_ε will denote the biggest real root of $P_\varepsilon(x)$. Our results refer to this root because we can approach it from the right inside a parabolic sector that does not shrink as ε tends to zero.

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In the sequel, we will assume

(H0) $P_\varepsilon'(\vartheta_\varepsilon) > 0$, for $\varepsilon \approx 0$, so that the singular point

$$(x, y) = \left(\vartheta_\varepsilon, \frac{U(\vartheta_\varepsilon)}{\lambda V_a(\vartheta_\varepsilon)}\right)$$

is a node of X.

Assumptions in Theorem A

The polynomial $P_\varepsilon(x)$ need not be irreducible. We identify the two branches that contain the root $x = \vartheta_\varepsilon$, for $\varepsilon \geqslant 0$ and $\varepsilon \leqslant 0$, and we apply Puiseux theorem to each one, obtaining $\rho_\pm \in \mathbb{N}$ and analytic functions $\sigma_\pm(z) \in \mathbb{R}\{z\}$, such that

$$\vartheta_\varepsilon = \begin{cases} \sigma_-\left((-\varepsilon)^{1/\rho_-}\right), & \text{if } \varepsilon \leqslant 0, \\ \sigma_+\left((+\varepsilon)^{1/\rho_+}\right), & \text{if } \varepsilon \geqslant 0. \end{cases}$$

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We treat the unfolding (2), as $\varepsilon \to 0^+$, or $\varepsilon \to 0^-$. Since the substitution $\varepsilon \longmapsto -\varepsilon$ interchanges both situations, we will restrict to the case $\varepsilon \geqslant 0$

Assumptions in Theorem A

Besides the natural assumption (H0), we need to impose two technical conditions on $P_\varepsilon(x) = P(x;\varepsilon)$. In order to state them precisely, we introduce the function

$$\mathcal{Q}(s,\varepsilon) := \frac{P\big(s+\sigma(\varepsilon);\varepsilon^\rho\big)}{s},$$

which is analytic at $(s,\varepsilon)=(0,0)$ and polynomial in s. Moreover, $\mathcal{Q}(s,0)=s^\mu$ and, on account of (H0), $\mathcal{Q}(0,\varepsilon)=\chi\,\varepsilon^\nu+\ldots$ with $\chi>0$, for some $\nu\in\mathbb{N}$.

Assumptions in Theorem A

(H1) The Newton's diagram of $\mathcal{Q}(s,\varepsilon)$ has only one compact side (connecting the endpoints $(\mu,0)$ and $(0,\nu)$), i.e. $\mathcal{Q}$ admits a Taylor's expansion of the form

$$\mathcal{Q}(s,\varepsilon) = \sum_{\frac{i}{\mu}+\frac{j}{\nu}\geqslant 1} q_{ij} s^i \varepsilon^j.$$

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(H2) The principal (μ,ν)-quasi-homogeneous part of $\mathcal{Q}(s,\varepsilon)$ is positive definite on the first quadrant, i.e.

$$\sum_{\frac{i}{\mu}+\frac{j}{\nu}=1} q_{ij}\sin^i\theta\cos^j\theta>0,\ \text{for all } \theta\in\left[0,\frac{\pi}{2}\right].$$

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Notice that (H2) implies (H0) because $P_\varepsilon'(\vartheta_\varepsilon)=\mathcal{Q}(0,\varepsilon)$.
On the other hand, if $\gcd(\mu,\nu)=1$, then (H1) implies (H2).

Assumptions in Theorem A

Let $y(x) = y(x; x_0, y_0, \varepsilon, a, \lambda, U)$ be the trajectory of

$$X = P_\varepsilon(x)\partial_x + \big(\lambda V_a(x)y - U(x)\big)\partial_y,$$

i.e. the solution of the linear differential equation

$$P_\varepsilon(x)y'(x) = \lambda V_a(x)y(x) - U(x), \tag{3}$$

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with $y(x_0) = y_0$. We have $y(x) = D(x)\frac{y_0}{D(x_0)} + y_L(x)$, where

$$D(x) = D(x; \varepsilon, a, \lambda) := \exp\left(\lambda \int_1^x \frac{V_a(s)}{P_\varepsilon(s)} ds\right)$$

$$y_L(x; x_0, \varepsilon, a, \lambda, U) := D(x; \varepsilon, a, \lambda) \int_x^{x_0} \frac{U(s)}{P_\varepsilon(s)} \frac{ds}{D(s; \varepsilon, a, \lambda)}.$$

Assumptions in Theorem A

Here $D(x)$ is a fundamental solution of the homogeneous equation. Moreover, $y_L(x)$ is the particular solution with initial condition $y_0 = 0$ and it depends linearly on $U \in \mathscr{U}$.

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Our first main result describes how the trajectories of X arrive at the node $(x, y) = \left(\vartheta_\varepsilon, \frac{U(\vartheta_\varepsilon)}{\lambda V_a(\vartheta_\varepsilon})\right)$ given by hypothesis (H0). For convenience, in its statement we use the differential operator

$$\Theta_\lambda = \frac{1}{\lambda} s\partial_s.$$

Statement of the orbital result

$$X = P_\varepsilon(x)\partial_x + \big(\lambda V_a(x)y - U(x)\big)\partial_y,$$

Theorem A (1)

Consider the saddle-node unfolding given in (2) with $\varepsilon \geqslant 0$. Assume that $P_\varepsilon(x)$ satisfies (H1) and (H2).

Statement of the orbital result

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Theorem A (1)

Consider the saddle-node unfolding given in (2) with $\varepsilon \geqslant 0$. Assume that $P_\varepsilon(x)$ satisfies (H1) and (H2). Then, there exist functions $c_j(\varepsilon, \lambda, a, U)$, $j \in \mathbb{Z}^+$, satisfying that, for each $\ell, k \in \mathbb{Z}^+$, $\lambda_0 > 0$ and every compact set $K_a \subset A$,

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(a) …

(b) …

Statement of the orbital result

$$P_\varepsilon(x)y'(x) = \lambda V_a(x)y(x) - U(x), \text{ with } y(x_0) = y_0.$$

Theorem A (2)

(a) for every compact set $K_x \subset (0,1]$, the particular solution y_L of (3) is of the form

$$y_L(s + \vartheta_\varepsilon; x_0, \varepsilon, a, \lambda, U) = \sum_{j=0}^{\ell} c_j(\varepsilon^{1/\rho}, a, \lambda, U)s^j + s^\ell h_\ell(s; x_0, \varepsilon, a, \lambda, U),$$

where $\Theta_\lambda^r h_\ell(s) \to 0$, as $s \to 0^+$, uniformly on $K_x \times \mathcal{A} \times \mathscr{U}_1$, for $r = 0, 1, \ldots, k$;

Statement of the orbital result

$$P_\varepsilon(x)y'(x) = \lambda V_a(x)y(x) - U(x), \text{ with } y(x_0) = y_0.$$

Theorem A (3)

(b) the fundamental homogeneous solution of (3) is of the form $D(s+\vartheta_\varepsilon;\varepsilon,a,\lambda) = s^\ell h_\ell(s;\varepsilon,a,\lambda)$, where $\Theta_\lambda^r h_\ell(s) \to 0$, as $s \to 0^+$, uniformly on $\mathcal{A}$, for $r = 0, 1, \ldots, k$.

Statement of the temporal results

Recall that we consider the saddle-node unfolding

$$\frac{1}{yU_a(x,y)}\left(P_\varepsilon(x)\partial_x - V_a(x)y\partial_y\right).$$

Under assumption (H0), the point $(\vartheta_\varepsilon, 0)$, where ϑ_ε is the biggest root of $P_\varepsilon(x)$, is now a *saddle* of the above differential system. The period annulus is in the quadrant $y \geqslant 0$ and $x \geqslant \vartheta_\varepsilon$.

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In our next result, $\mathcal{T}(s;\varepsilon,a)$ is the Dulac time of the above unfolding between the transverse sections $\{y=1\}$ and $\{x=1\}$, i.e. it is the time that the trajectory starting at $(s+\vartheta_\varepsilon, 1)$ spends to arrive to $\{x=1\}$. We also use $\Theta = \Theta_1$.

Statement of the temporal results

Theorem B

Assume that $P_\varepsilon(x)$ satisfies conditions (H1) and (H2). Then there exist functions $c_j(\varepsilon, a)$, $j \in \mathbb{Z}^+$, satisfying that, for each $\ell, k \in \mathbb{Z}^+$ and every compact set $K_a \subset A$, there exists $\varepsilon_0 > 0$ such that $c_0, \ldots, c_\ell$ are analytic on $[0, \varepsilon_0] \times K_a$; and the Dulac time can be written as

$$\mathcal{T}(s; \varepsilon, a) = \sum_{j=0}^{\ell} c_j(\varepsilon^{1/\rho}, a) s^j + s^\ell h_\ell(s; \varepsilon, a),$$

with $\Theta^r h_\ell(s) \to 0$, as $s \to 0^+$, uniformly on $[0, \varepsilon_0] \times K_a$, for $r = 0, 1, \ldots, k$.

Proof of Theorem B

Consider $\ell, k \in \mathbb{Z}^+$ and a compact set $K_a \subset A$. We decompose the given function $U_a(x,y) = \sum_{n\geqslant 1} U_{n,a}(x) y^{n-1}$, with $U_{n,a} \in \mathscr{U}$, for all $n \in \mathbb{N}$ and $a \in K_a$.

Since $U_a(x,y)$ is absolutely convergent on $|x|, |y| \leqslant 1$, the series $\sum_{n\geqslant 1} \|U_{n,a}\| y^n$ and all its $y\partial_y$ derivatives have convergence radius at least 1.

Consequently,

$$\sum_{n\geqslant 1} n^r \|U_{n,a}\| < \infty, \text{ for all } r \in \mathbb{Z}^+ \text{ and } a \in K_a.$$

Proof of Theorem B

Let $y(x;s)$ be the trajectory of $\frac{1}{yU_a(x,y)}\left(P_\varepsilon(x)\partial_x - V_a(x)y\partial_y\right)$ with initial condition $y(s;s)=1$. Set $\text{¢} := \varepsilon^{1/\rho}$. The Dulac time is

$$\begin{aligned}
\mathcal{T}(s;\text{¢},a) &= \int_{s+\vartheta_\varepsilon}^{1} \frac{U_a\big(x,y(x;s)\big)y(x;s)}{P_\varepsilon(x)}\,dx \\
&= \int_{s+\vartheta_\varepsilon}^{1} \sum_{n\geqslant 1} \frac{U_{n,a}(x)y^n(x;s)}{P_\varepsilon(x)}\,dx.
\end{aligned}$$

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$$
\begin{aligned}
\mathcal{T}(s;\not{c},a) &= \int_{s+\vartheta_\varepsilon}^{1} \frac{U_a\big(x,y(x;s)\big)y(x;s)}{P_\varepsilon(x)}\,dx \\
&= \int_{s+\vartheta_\varepsilon}^{1} \sum_{n\geqslant 1} \frac{U_{n,a}(x)y^n(x;s)}{P_\varepsilon(x)}\,dx.
\end{aligned}
$$

We define

$$
T_n(s) := \int_s^1 \frac{U_{n,a}(x)y^n(x;s)}{P_\varepsilon(x)}dx.
$$

Proof of Theorem B

Then, due to $\partial_s y(x;s) = y(x;s)\frac{V_a(s)}{P_\varepsilon(s)}$,

$$\begin{aligned}\partial_s T_n(s) &= \int_s^1 \frac{U_{n,a}(x)\partial_s y^n(x;s)}{P_\varepsilon(x)}\,dx - \frac{U_{n,a}(s)}{P_\varepsilon(s)} \\ &= \frac{nV_a(s)}{P_\varepsilon(s)}T_n(s) - \frac{U_{n,a}(s)}{P_\varepsilon(s)}\end{aligned}$$

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Hence $T_n(x)$ is the trajectory with initial condition $T_n(1) = 0$ of

$$X = P_\varepsilon(x)\partial_x + \big(\lambda V_a(x)y - U(x)\big)\partial_y,$$

taking $U(x)$ as $U_{n,a}(x)$, $V_a(x)$ as $\frac{V_a(x)}{V_a(0)}$ and λ as $nV_a(0)$.

Proof of Theorem B

Set $\mathcal{T}_n(s;\not\varepsilon,a) := T_n(s+\vartheta_\varepsilon)$. Then, by applying Theorem A,

$$\mathcal{T}_n(s;\not\varepsilon,a) = \sum_{j=0}^{\ell} c_j\big(\not\varepsilon,a,nV_a(0),U_{n,a}\big)s^j + s^\ell h_\ell\big(s;\varepsilon,a,nV_a(0),U_{n,a}\big),$$

with c_j analytic on $(\not\varepsilon,a) \in [0,\varepsilon_0]\times K_a$.

Proof of Theorem B

Set $\mathcal{T}_n(s;\notin,a) := T_n(s+\vartheta_\varepsilon)$. Then, by applying Theorem A,

$$\mathcal{T}_n(s;\notin,a) = \sum_{j=0}^{\ell} c_j\big(\notin,a,nV_a(0),U_{n,a}\big)s^j + s^\ell h_\ell\big(s;\varepsilon,a,nV_a(0),U_{n,a}\big),$$

with c_j analytic on $(\notin,a) \in [0,\varepsilon_0]\times K_a$. Moreover,

$$\gamma_j := \sup\big\{|c_j\big(\varepsilon,a,\lambda,U\big)| : \\ (\varepsilon,a,\lambda,U) \in [0,\varepsilon_0]\times K_a\times[\lambda_0,+\infty)\times\mathscr{U}_1\big\} < +\infty$$

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$$M_\ell^r(s) := \sup\big\{|\Theta_\lambda^r h_\ell\big(s;\varepsilon,a,\lambda,U\big)| : \\ (\varepsilon,a,\lambda,U)\in[0,\varepsilon_0]\times K_a\times[\lambda_0,+\infty)\times\mathscr{U}_1\big\} < +\infty,$$

with $M_\ell^r(s)\to 0$, as $s\to 0^+$, for $r=0,1,\dots,k$.

Proof of Theorem B

In particular, for $(\varepsilon, a, n) \in [0, \varepsilon_0] \times K_a \times \mathbb{N}$ and $r = 0, 1, \ldots, k$,

$$|c_j(\not\varepsilon, a, nV_a(0), U_{n,a})| \leqslant \gamma_j \|U_{n,a}\|$$

$$|\Theta_\lambda^r h_\ell(s; \varepsilon, a, nV_a(0), U_{n,a})| \leqslant M_\ell^r(s) \|U_{n,a}\|.$$

Here, it is crucial that Theorem A holds for λ unbounded and U varying in the Banach space $\mathscr{U}$.

Proof of Theorem B

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$$|\Theta_\lambda^r h_\ell(s; \varepsilon, a, nV_a(0), U_{n,a})| \leqslant M_\ell^r(s) \|U_{n,a}\|.$$

Here, it is crucial that Theorem A holds for λ unbounded and U varying in the Banach space $\mathscr{U}$.
We define at this point the coefficients

$$c_j(\not\varepsilon, a) := \sum_{n \geqslant 1} c_j(\not\varepsilon, nV_a(0), a, U_{n,a}), \text{ for all } j \in \mathbb{Z}^+,$$

which are well-defined because the series are uniformly convergent on $(\not\varepsilon, a) \in [0, \varepsilon_0] \times K_a$

Proof of Theorem B

Similarly, the series

$$h_\ell(s;\varepsilon,a) := \sum_{n\geqslant 1} h_\ell(s;\varepsilon,a,nV_a(0),U_{n,a})$$

is uniformly convergent on $(s,\not\varepsilon,a)\in[0,s_0]\times[0,\varepsilon_0]\times K_a,\ s_0\approx 0$, and it tends to zero, as $s\to 0^+$, uniformly on (ε,a).

Proof of Theorem B

Similarly, the series

$$h_\ell(s;\varepsilon,a) := \sum_{\color{red}{n}\geqslant 1} h_\ell(s;\varepsilon,a,{\color{red}{n}}V_a(0),U_{{\color{red}{n}},a})$$

is uniformly convergent on $(s,\notin,a) \in [0,s_0]\times[0,\varepsilon_0]\times K_a$, $s_0 \approx 0$, and it tends to zero, as $s \to 0^+$, uniformly on (ε,a). Hence,

$$\sum_{n\geqslant 1}\mathcal{T}_n(s;\notin,a) = \sum_{n\geqslant 1}\sum_{j=0}^{\ell} c_j(\notin,a,nV_a(0),U_{n,a})s^j$$

$$+\, s^\ell \sum_{n\geqslant 1} h_\ell(s;\varepsilon,a,nV_a(0),U_{n,a}) = \sum_{j=0}^{\ell} c_j(\notin,a)s^j + s^\ell h_\ell(s;\notin,a)$$

is uniformly convergent on $(s,\notin,a) \in [0,s_0]\times[0,\varepsilon_0]\times K_a$.

Proof of Theorem B

For this reason, we can commute summation and integration in

$$\mathcal{T}(s;\not\varepsilon,a)=\int_{s+\vartheta_\varepsilon}^{1}\sum_{n\geqslant 1}\frac{U_{n,a}(x)y^n(x;s)}{P_\varepsilon(x)}\,dx=\sum_{n\geqslant 1}\mathcal{T}_n(s;\not\varepsilon,a).$$

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Thus $\mathcal{T}(s;\ddagger,a)=\sum_{j=0}^{\ell}c_j(\ddagger,a)s^j+s^\ell h_\ell(s;\ddagger,a)$.

Proof of Theorem B

For this reason, we can commute summation and integration in

$$\mathcal{T}(s;\notin,a) = \int_{s+\vartheta_\varepsilon}^{1} \sum_{n\geqslant 1} \frac{U_{n,a}(x)y^n(x;s)}{P_\varepsilon(x)}\,dx = \sum_{n\geqslant 1} \mathcal{T}_n(s;\notin,a).$$

Thus $\mathcal{T}(s;\notin,a) = \sum_{j=0}^{\ell} c_j(\notin,a)s^j + s^\ell h_\ell(s;\notin,a)$. Finally,

$$\begin{aligned}
&\sum_{n\geqslant 1} \left|\Theta_1^r h_\ell(s;\varepsilon,a,nV_a(0),U_{n,a})\right| \\
&\qquad = V_a^r(0) \sum_{n\geqslant 1} n^r \left|\Theta_\lambda^r h_\ell(s;\varepsilon,a,nV_a(0),U_{n,a})\right| \\
&\qquad \leqslant V_a(0)^r M_\ell^r(s) \sum_{n\geqslant 1} n^r \|U_{n,a}\|
\end{aligned}$$

is uniformly convergent in $(s,\notin,a)$ and tends to zero, as $s \to 0^+$, uniformly on $(\notin,a)$.

Statement of the temporal results

We particularize the unfolding in Theorem B by taking

$$P_\varepsilon(x) = x(x^\mu - \varepsilon) \text{ and } U_a(x,y) = x^m \bar{U}_a(x,y).$$

Corollary B

There exist functions $c_j(\varepsilon, a)$, $j \in \mathbb{Z}^+$, satisfying that for each $\ell, k \in \mathbb{Z}^+$ and every compact set $K_a \subset A$, there exists $\varepsilon_0 > 0$ such that $c_0, \ldots, c_\ell$ are continuous on $[-\varepsilon_0, \varepsilon_0] \times K_a$, and

$$\mathcal{T}(s;\varepsilon,a) = \sum_{j=0}^{\ell} c_j(\varepsilon,a) s^j + s^\ell h_\ell(s;\varepsilon,a)$$

with $\Theta^r h_\ell(s) \to 0$, as $s \to 0^+$, uniformly on $[-\varepsilon_0, \varepsilon_0] \times K_a$, for $r = 0, 1, \ldots, k$. Moreover, $c_j(\varepsilon, a) = 0$, for $\varepsilon \leqslant 0$ and $j = 0, 1, \ldots, m-1$.