On the classification of quasi-homogeneous differential systems in dimensions two and three

Jesús Suárez Pérez del Río

Departamento de Matemáticas- Universidad de Oviedo - Spain

(In collaboration with B. García, A. Lombardero and J. Llibre)

IV Symposium on Planar Vector Fields- Lleida, 2016

Basic references

My talk is based mainly on these papers:

B. García, J. Llibre and J.S. Pérez del Río.
Planar quasi-homogeneus polynomial differential systems and their integrability.
J. Differ. Equ., 255 (2013).

B. García, A. Lombardero and J.S. Pérez del Río
Classification and counting of planar quasi-homogeneous differential systems through their weight vectors
preprint,(2015).

Outline

- The planar quasi–homogeneous polynomial differential systems have been studied from many different points of view, mainly for their integrability (H. Yoshida (1983,88,89),..., A.Algaba, C. García, M.Reyes (2010,11)), for their centers (see, for example, J.Llibre, C. Pessoa (2009)), for their normal forms (A.Algaba, C. García, M.A. Teixeira (2010)), for their limit cycles (W. Li, J. Llibre, J. Yang and Z. Zhang (2009)).
- But there was not an algorithm for constructing all the quasi–homogeneous polynomial differential systems of a given degree. We have obtained such algorithm and its explanation is one of the fundamental goals of this exposition.

Basic definitions

- Polynomial differential systems (S):

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

  with $P, Q \in \mathbb{C}[x, y]$.
- Vector field associated to system (1): $X = (P, Q)$
- In the following we denote $l = deg(P)$ and $m = deg(Q)$. We say that the degree of the system or of the vector field is $n = \max\{\deg(P), \deg(Q)\}$.

Quasi-homogeneous vector fields

- A polynomial differential system or the associated vector field $X = (P, Q)$ is *quasi–homogeneous* (in short, QH-systems or QH-vector fields) if there exist s_1, $s_2, d \in \mathbb{N}$ such that for arbitrary $\alpha \in \mathbb{R}^+ = \{a \in \mathbb{R},\ a > 0\}$, one has that

$$P(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_1-1+d}P(x, y), Q(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_2-1+d}Q(x, y).$$

- We call s_1 and s_2 the *weight exponents* of a QH-system, and d the *weight degree* with respect to the weight exponents s_1 and s_2. In the particular case that $s_1 = s_2 = 1$, then the system is the classical *homogeneous polynomial differential system of degree d*. We will denote the QH-systems that are not homogeneous by QHNH-systems.

Weight-vectors

- Let us consider a QH-system with weight exponents s_1 and s_2 and with weight degree d. In what follows we denote by $\mathbf{w} = (s_1, s_2, d)$ the *weight vector* formed with the weight exponents and the weight degree of the system. We say that the weight vector $\mathbf{w}_m = (s_1^*, s_2^*, d^*)$ is the *minimum weight vector* of the system if any other weight vector $\mathbf{w} = (s_1, s_2, d)$ of the system verifies $s_1^* \leq s_1$, $s_2^* \leq s_2$ and $d^* \leq d$.

Creating new weight vectors

Proposition

Given a weight vector (s_1, s_2, d) of a quasi-homogeneous system S, and $r = \frac{p}{q} \in \mathbb{Q}^+$ with p and q coprime, a vector (rs_1, rs_2, d^*) is also a S weight vector if and only if q divides $gcd\,(s_1, s_2, d-1)$ and $d^* = r\,(d-1)+1$.

Example

Given the QH-system

$$\dot{x} = x^2 y + y^2, \qquad \dot{y} = xy^2$$

for which $(3, 6, 10)$ is a weight vector.

- Multiples: $(4, 8, 13)$, $(5, 10, 16)$, $(6, 12, 19)$...
- Factors: $(2, 4, 7)$, $(1, 2, 4)$.

Weight vector families

Given a quasi-homogeneous system S and $\lambda \in \mathbb{Q}^+$, the **weight vector family of the system with ratio λ** is defined as the set of weight vectors of S where the proportion between the exponents s_1 and s_2 is λ,

$$F_{\mathrm{S}}(\lambda) := \left\{ (s_1, s_2, d) \text{ weight vector of S } : \frac{s_1}{s_2} = \lambda \right\}$$

Example

Given the QH-system S : $\dot{x} = x^5 y^2, \qquad \dot{y} = x^4 y^3$

- $(2, 1, 11)$ and $(10, 5, 51)$ belong to the family $F_{\mathrm{S}}(2)$.
- $(7, 2, 33)$ and $(35, 10, 161)$ belong to the family $F_{\mathrm{S}}\left(\frac{7}{2}\right)$.

Weigh vectors in homogeneous systems

- A quasi homogenous system is homogeneous if and only if has a weight vector (s_1, s_2, d) with $s_1 = s_2$, that is, it has a family of weight vectors $F_S(1)$
- An homogeneous system can have other families of weight vectors, even all families $F_S(\lambda)$ as weight vectors.

Example

Given the QH-system S : $\quad \dot{x} = x^5y^2, \qquad \dot{y} = x^4y^3$

- For all $p, q \in \mathbb{Z}^+$ $(p, q, 4p + 2q + 1)$ is a weight vector of S.

The homogeneous case (1)

Proposition

An homogeneous system of degree n with more than one nonzero monomial in any of its components P or Q verifies that $F_S(1)$ is the only weight vector family, and $\mathbf{w}_m = (1, 1, n)$ is the minimum weight vector of the system.

Examples

- $\dot{x} = x^2y^5, \quad \dot{y} = x^6y + xy^6$ verifies that $\mathbf{w}_m = (1, 1, 7)$.
- $\dot{x} = x^3, \quad \dot{y} = x^2y + y^3$ verifies that $\mathbf{w}_m = (1, 1, 3)$.
- $\dot{x} = x^{10} + x^5y^5, \quad \dot{y} = 0$ verifies that $\mathbf{w}_m = (1, 1, 10)$.

The homogeneous case (2)

Proposition

A homogeneous system of degree n that has only one nonzero monomial in both components P and Q, being

$$P(x,y) = a_{i,n-i}x^i y^{n-i} \qquad Q(x,y) = b_{j,n-j}x^j y^{n-j}$$

for certain $i,\ j \in \{0, 1, ..., n\}$, verifies:

(i) If $j \neq i-1$, then $F_S(1)$ is the only weight vector family, and $\mathbf{w}_m = (1, 1, n)$ is the minimum weight vector of the system.

(ii) If $j = i-1$, then the system has an infinite number of weight vector families, and given any positive rational λ, $F_S(\lambda)$ family exists. As a consequence, $\mathbf{w}_m = (1, 1, n)$ is also the minimum weight vector of the system.

The homogeneous case (3)

Examples

- The system
$$\dot{x} = x^3y^7, \qquad \dot{y} = x^5y^5$$
has a unique family, and $\mathbf{w}_m = (1, 1, 10)$.

- The system
$$\dot{x} = x^3y^7, \qquad \dot{y} = x^2y^8$$
has an infinite number of families, and $\mathbf{w}_m = (1, 1, 10)$. The vector $(5, 8, 67)$, for example, is also a weight vector.

Nonhomogeneous case (QHNH-systems)

- We write the polynomials P and Q of a QH-system through their homogeneous parts:

$$P(x,y) = \sum_{j=0}^{l} P_j(x,y), \quad \text{where} \quad P_j(x,y) = \sum_{i=0}^{j} a_{i,j-i} x^i y^{j-i},$$

$$Q(x,y) = \sum_{j=0}^{m} Q_j(x,y), \quad \text{where} \quad Q_j(x,y) = \sum_{i=0}^{j} b_{i,j-i} x^i y^{j-i}.$$

- From the definition of QH-system we obtain that if $a_{i,j-i}$ is nonzero then

$$(i-1)s_1 + (j-i)s_2 - (d-1) = 0$$

and if $b_{i,j-i}$ is nonzero then

$$is_1 + (j-i-1)s_2 - (d-1) = 0.$$

First consequences

- If we have a QH-system then $P_0 = Q_0 = 0$.
- If $d > 1$ then $b_{01} = a_{10} = 0$, that is, $P_1 = a_{01}y$ and $Q_1 = b_{10}x$.
- All the QH-systems of degree 1 are homogeneous. That is, in the following we can consider that $n \geq 2$.

Uniqueness of nonzero terms

Consider a QHNH-system with weight vector $\mathbf{w} = (s_1, s_2, d)$, where $s_1 \neq s_2$. We have proved that:

- There is a unique $q \in \{0, 1, ..., m\}$ such that $b_{q,m-q} \neq 0$ and a unique $p \in \{0, 1, ..., l\}$ such that $a_{p,l-p} \neq 0$, and

$$(p - q - 1)(s_1 - s_2) = (m - l)s_2.$$

We remark that we could always choose $s_1 > s_2$ by interchanging (if necessary) the variables x and y.

Structure of a homogeneous part of a QHNH system

- Each homogeneous part of P and Q has at most one monomial different from 0.
- If X_{n-t} is the homogeneous part of X of degree $n-t$ and it is nonzero, then X_{n-t} satisfies one of the following statements:

(a) $X_{n-t} = (a_{i^*,n-t-i^*} x^{i^*} y^{n-t-i^*}, 0)$,

(b) $X_{n-t} = (0, b_{j^*,n-t-j^*} x^{j^*} y^{n-t-j^*})$,

(c) $X_{n-t} = (a_{\mathbf{i^*,n-t-i^*}} x^{i^*} y^{n-t-i^*}, b_{\mathbf{i^*-1,n-t-i^*+1}} x^{i^*-1} y^{n-t-i^*+1})$.

Particular case: $d = 1$

Consider the QHNH-systems with weight vector $\mathbf{w} = (s_1, s_2, 1)$, $s_1 > s_2$. Then the following statements hold.

(a) $l > m = 1$, $s_1 = ls_2$ and $\mathbf{w}_m = (l, 1, 1)$.

(b) The only coefficients of the system that can be different from zero are $a_{1,0}$, $a_{0,l}$ and $b_{0,1}$.

- Hence the general expression of the system in this case is

$$\dot{x} = a_{1,0}x + a_{0,l}y^l, \quad \dot{y} = b_{0,1}y$$

Searching for the QHNH-systems of degree n

- The QHNH-systems with $d = 1$ can be obtained using the above result.
- Then, in order to obtain all the QHNH-systems of a fixed degree, we need to obtain all the QHNH-vector fields with $d > 1$. To do this we will construct an algorithm.

Properties of QHNH-systems with $d > 1$

If X is a QHNH-vector field of degree n then for each $\tilde{t}$ $(0 < \tilde{t} < n)$ such that $X_n X_{n-\tilde{t}} \neq 0$ there exists only one $p \in \{0, 1, \ldots, n+1\}$ and only one $q \in \{0, 1, \ldots, n-\tilde{t}+1\}$ such that

- $k = q - p \geq 1$ and $k \leq n - \tilde{t} - p + 1$;
- One has that s_1,s_2 and d verify the linear system $(p-1)s_1 + (n-p)s_2 = d-1$, $(q-1)s_1 + (n-\tilde{t}-q)s_2 = d-1$, that is, $s_1 = (\tilde{t}+k)(d-1)/D$ and $s_2 = k(d-1)/D$, where $D = (p-1)\tilde{t} + (n-1)k$.
- The minimun weight vector of X is $\mathbf{w}_m = ((\tilde{t}+k)/s, k/s, 1 + D/s)$, where s is the greatest common divisor of $\tilde{t}$ and k.

Determining Linear Systems

- Initial equations (associated to X_n and other non zero homogeneous part X_{n-t}):

$$e_p^0[0] \equiv (p-1)s_1 + (n-p)s_2 + 1 - d = 0,$$
$$e_p^t[k] \equiv (p+k-1)s_1 + (n-t-p-k)s_2 + 1 - d = 0.$$

- All possible equations
  For every $p \in \{0, 1, \ldots, n-1\}$ and $t \in \{1, \ldots, n-p\}$ we define the set of equations

$$A_p(t) = \{e_p^t[k] : k = 1, \ldots, n-t-p+1\}.$$

  If we fix $p \in \{0, 1, \ldots, n-1\}$ we denote

$$E_p = \{e_p^0[0]\} \cup A_p(1) \cup \ldots \cup A_p(n-p).$$

Goal of the algorithm

Our goal is to obtain sets of linear equations that verify:

- Each set contains the equation $e_p^0[0]$ and at most one equation of each $A_p(t)$.
- The set of all these equations defines a compatible linear system the unknowns being s_1 and s_2.
- If we add some other equation the increased linear system will be incompatible.

We denote such linear systems as the *maximal linear systems* associated to p. Every one of these maximal linear systems will provide a QHNH- system of degree n.

About the linear systems

- The linear system E_p when $p = 0$ has two equations that can be omitted because they are never satisfied: From $A_0(n)$ the equation $e_0^n[1]$ $(-s_2 = d - 1)$, and from $A_0(n-1)$ the equation $e_0^{n-1}[1]$ $(0 = d - 1)$.
- Therefore, in what follows we must consider

$$\mathcal{E}_p = E_p \text{ if } p > 0, \text{ and } \mathcal{E}_0 = E_0 \setminus \{e_0^n[1], e_0^{n-1}[1]\}.$$

- Compatibility: The linear system defined by the three equations $e_p^0[0]$, $e_p^{t_1}[k_1]$ and $e_p^{t_2}[k_2]$ is compatible if and only if

$$k_1 t_2 = k_2 t_1 .$$

The algorithm (I)

Fixed $p \in \{0, 1, \dots, n-1\}$ the following algorithm allows the determination of all the compatible *maximal linear systems* associated to the set of equations $\mathcal{E}_p$.

- **Step 1.** We choose the equation $e_p^0[0]$ as the first equation of a maximal linear system.
- **Step 2.** We fix $t \in \{1, \dots, n-p\}$ and an equation of $A_p(t) \cap \mathcal{E}_p$, i.e. an equation of the form $e_p^t[k]$ with a $k \in \{1, \dots, n-t-p+1\}$. From the previous results we know that the resolution of the linear system defined by $e_p^0[0]$ and $e_p^t[k]$, allows us to obtain the values of s_1 and s_2, and furthermore the minimum weight vector $\mathbf{w}_m$.

The algorithm (II)

- **Step 3.** For each $t^* \in \{1, \dots, n-p\}$ with $t \neq t^*$ we determine the value $k_{t^*} \in \{1, \dots, n-t^*-p+1\}$, (if it exists), such that the equation $e_p^{t^*}[k_{t^*}]$ satisfies the compatibility condition with the equations of Step 1 and Step 2.
- **Step 4.** We consider all the equations

$$\mathcal{E}_{p,t,k} = \bigcup_{t^* \in \{1,\dots,n-p\} \setminus \{t\}} \{e_p^{t^*}[k_{t^*}] : k_{t^*} t = k t^*\} \cup \{e_p^t[k], e_p^0[0]\},$$

that we have obtained in the Steps 1, 2 and 3 and we construct the associated vector field.
- **Step 5.** We remove from $\mathcal{E}_p$ the equations $\mathcal{E}_{p,t,k} \setminus \{e_p^0[0]\}$.
- **Step 6.** We go back to Step 1 if it is possible.

Obtaining QHNH-systems

- The QHNH-system of degree $n \geq 2$ corresponding to the set of equations $\mathcal{E}_{p,t,k}$ is
$$X_{p,t,k} = X_n + X_{n-t}^{t,k} + \sum_{t^* \in \{1,\ldots,n-p\} \setminus \{t\} \text{ and } k_{t^*} t = kt^*} X_{n-t^*}^{t^*,k_{t^*}},$$
$X_{n-l}^{l,s} = (a_{p+s,n-l-p-s} x^{p+s} y^{n-l-p-s}, b_{p+s-1,n-l-s-p+1} x^{p+l-1} y^{n-l-s-p+1})$,
$X_n = X_{n-0}^{0,0}$.
- In this expression we must consider conditions in the coefficients such that X_n and at least other homogeneous part of X is nonzero.
- The minimum weight vector of $X_{p,t,k}$ is $\mathbf{w} = (s_1, s_2, d)$, where $s_1 = (t+k)/s$, $s_2 = k/s$, $d = 1 + D/s$, where $D = (p-1)t + (n-1)k$ and $s = gcd(t,k)$.

How many algebraic forms are there?

Planar quasi-homogeneous polynomial differential systems

Non-homogeneous

Several different degrees

Example

$$\dot{x} = y^4 + xy^2 + x^2$$
$$\dot{y} = y^3 + xy$$

Homogeneous

Always same degree

Example

$$\dot{x} = x^4 + x^2y^2$$
$$\dot{y} = x^3y + xy^3 + y^4$$

A single general form, linked to $\mathbf{w}_m = (1, 1, n)$

How many algebraic forms are there?

Planar quasi-homogeneous polynomial differential systems

Non-homogeneous

Several different degrees

Example

$$\dot{x} = y^4 + xy^2 + x^2$$
$$\dot{y} = y^3 + xy$$

How many algebraic forms are there?

Homogeneous

Always the same degree

Example

$$\dot{x} = x^4 + x^2y^2$$
$$\dot{y} = x^3y + xy^3 + y^4$$

A single general form, linked to $\mathbf{w}_m = (1, 1, n)$

Euler's totient function

Given $n \in \mathbb{Z}^+$, the number of positive integers less than or equal to n, and also coprime with n,

$$\varphi(n) = |\{r \in \mathbb{N} \,:\, 1 \leq r \leq n\,,\; gcd(n, r) = 1\}|$$

is called **Euler's totient function**.

The values of $\varphi(n)$ can be calculated with the formula

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Example

$$\varphi(100) = \varphi\left(2^2 5^2\right) = 100\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) = 100 \cdot \frac{1}{2} \cdot \frac{4}{5} = 40$$

Counting Theorem

Theorem

The number $c(n)$ of non-homogeneous QH-systems of degree n is given by:

$$c(1) = 0, \quad c(2) = 6, \quad c(3) = 16$$

$$c(n) = 2c(n-1) - c(n-2) + 2\varphi(n+1)$$

or, equivalently,

$$c(n) = 10n - 14 + 2\sum_{k=4}^{n}\sum_{j=5}^{k+1}\varphi(j)$$

for $n \geq 4$, where φ is the Euler's totient function.

Counting Theorem

Theorem

The number $c(n)$ of non-homogeneous QH-systems of degree n is given by:

$$c(1) = 0, \quad c(2) = 6, \quad c(3) = 16$$

$$c(n) = 2c(n-1) - c(n-2) + 2\varphi(n+1)$$

or, equivalently,

$$c(n) = 10n - 14 + 2\sum_{k=4}^{n}\sum_{j=5}^{k+1}\varphi(j)$$

for $n \geq 4$, where φ is the Euler's totient function.

Counting Theorem

Theorem

The number $c(n)$ of non-homogeneous QH-systems of degree n is given by:

$$c(1) = 0, \quad c(2) = 6, \quad c(3) = 16$$

$$c(n) = 2c(n-1) - c(n-2) + 2\varphi(n+1)$$

or, equivalently,

$$c(n) = 10n - 14 + 2\sum_{k=4}^{n}\sum_{j=5}^{k+1}\varphi(j)$$

for $n \geq 4$, where φ is the Euler's totient function.

Counting Theorem

Theorem

The number $c(n)$ of non-homogeneous QH-systems of degree n is given by:

$$c(1) = 0, \quad c(2) = 6, \quad c(3) = 16$$

$$c(n) = 2c(n-1) - c(n-2) + 2\varphi(n+1)$$

or, equivalently,

$$c(n) = 10n - 14 + 2\sum_{k=4}^{n}\sum_{j=5}^{k+1}\varphi(j)$$

for $n \geq 4$, where φ is the Euler's totient function.

Counting Theorem proof sketch

$c(n)$: number of non-homogeneous QH-systems of degree n

$$c(n) = m(1) + m(2) + m(3) + \cdots + m(n)$$

where $m(k)$ is the number of equivalence classes of the set

$$E(k) = \left\{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \; : \; 2 \leq a + b \leq k + 1\right\}$$

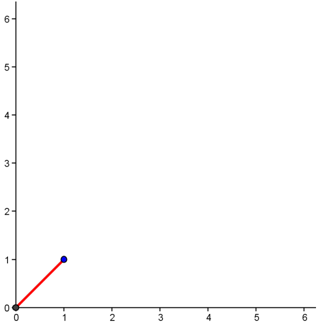
subject to the equivalence relation

$$(a, b) \approx (c, d) \Longleftrightarrow ad = bc.$$

Counting Theorem proof sketch

$c\,(n)$: number of non-homogeneous QH-systems of degree n

$$c\,(n) = m\,(1) + m\,(2) + m\,(3) + \cdots + m\,(n)$$

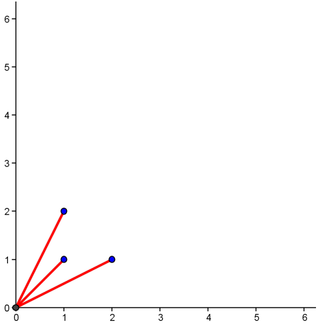


$$m\,(1) = \left| E\,(1) \,/ \approx \right| = 1$$

Counting Theorem proof sketch

$c(n)$: number of non-homogeneous QH-systems of degree n

$$c(n) = m(1) + m(2) + m(3) + \cdots + m(n)$$

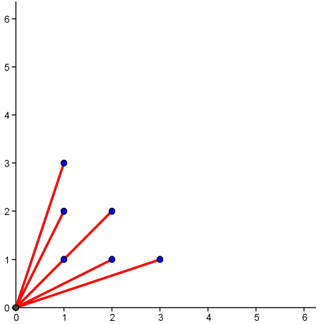


$$m(2) = |E(2) / \approx| = 3$$

Counting Theorem proof sketch

$c(n)$: number of non-homogeneous QH-systems of degree n

$$c(n) = m(1) + m(2) + m(3) + \cdots + m(n)$$

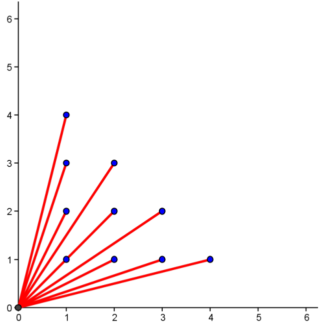


$$m(3) = |E(3)/\approx| = 5$$

Counting Theorem proof sketch

$c(n)$: number of non-homogeneous QH-systems of degree n

$$c(n) = m(1) + m(2) + m(3) + \cdots + m(n)$$

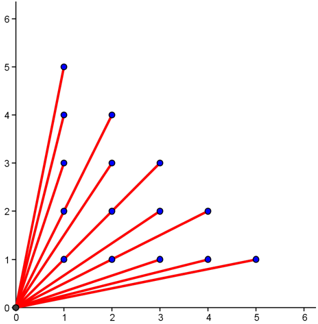


$$m(4) = |E(4) / \approx| = 9$$

Counting Theorem proof sketch

$c(n)$: number of non-homogeneous QH-systems of degree n

$$c(n) = m(1) + m(2) + m(3) + \cdots + m(n)$$

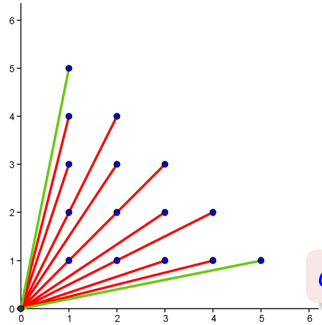


$$m(5) = |E(5)/\approx| = 11$$

Counting Theorem proof sketch

$c(n)$: number of non-homogeneous QH-systems of degree n

$$c(n) = m(1) + m(2) + m(3) + \cdots + m(n)$$



$$m(5) = |E(5)/\approx| = 11$$

New classes: $2 = \varphi(6)$

$$\Downarrow$$

$$m(k) = m(k-1) + \varphi(k+1)$$

$$\Downarrow$$

$$c(n) = 2c(n-1) - c(n-2) + 2\varphi(n+1)$$

Calculations

Number of non-homogeneous QH-systems for a few different degrees (the number of systems grows asymptotically as n^3)

n	systems	n	systems	n	systems
1	0	**11**	420	**25**	3940
2	6	**12**	534	**50**	28414
3	16	**13**	660	**100**	214802
4	34	**14**	802	**500**	25634774
5	56	**15**	960	**1000**	203858766
6	90	**16**	1150	**5000**	25360684978
7	132	**17**	1352	**1e+4**	2.0276e+11
8	186	**18**	1590	**1e+5**	2.0265e+14
9	248	**19**	1844	**1e+6**	2.0264e+17
10	330	**20**	2122	**1e+7**	2.0264e+20

The program *counterQH.m*

- What it does
  - Prompts the user to input a degree n.
  - Returns the number of different non-homogeneous QH-systems of degree n.
- Properties
  - Employs the recursive formula of the Counting Theorem.
  - It is not limited by the n degree introduced.

The program *qh.m*

- What it does?
  - Prompts the user for a system degree n.
  - The output is a .tex document containing a list with all different non-homogeneous QH-systems of degree n.
  - Also provides the $\mathbf{w}_m$ vector of each system.
- Properties
  - Based on the algorithm published by García, Llibre and Pérez del Río (2013).
  - Limited to degrees $n < 385$.

A *qh.m* output

Non-homogeneous quasi-homogeneous polynomial differential systems of degree 6

1) $\dot{x} = a_{0,6}y^6 + a_{1,4}xy^4 + a_{2,2}x^2y^2 + a_{3,0}x^3$

$\dot{y} = b_{0,5}y^5 + b_{1,3}xy^3 + b_{2,1}x^2y$

Minimum weight vector of the system: $(2, 1, 5)$

2) $\dot{x} = a_{5,0}x^5 + a_{3,1}x^3y + a_{1,2}xy^2$

$\dot{y} = b_{6,0}x^6 + b_{4,1}x^4y + b_{2,2}x^2y^2 + b_{0,3}y^3$

Minimum weight vector of the system: $(1, 2, 5)$

3) $\dot{x} = a_{0,6}y^6 + a_{2,3}x^2y^3 + a_{4,0}x^4$

$\dot{y} = b_{1,4}xy^4 + b_{3,1}x^3y$

Minimum weight vector of the system: $(3, 2, 10)$

4) $\dot{x} = a_{4,1}x^4y + a_{1,3}xy^3$

$\dot{y} = b_{6,0}x^6 + b_{3,2}x^3y^2 + b_{0,4}y^4$

Download

Both programs can freely be downloaded at the

http://xixon.epv.uniovi.es/gsd.

web address.

Integrability of QH-systems

- All QH-systems are integrable (see, for instance, I. García (2003), W. Li, J. Llibre, J. Yang and Z. Zhang (2009), Yanxia Hu (2007)).
- One can prove (for example, by using a generalization of the Euler's formula) that if a QH-system has weight exponents s_1 and s_2, then $V = s_1 xQ - s_2 yP$ is an inverse of integrating factor of the system.
- From the results of J. Llibre and Z. Zhang (2002) it is easy to deduce that a quasi–homogenous polynomial differential system has a global analytic first integral if and only if it has a polynomial first integral.
- Using the above results and the classification of QH-systems of degree 2 and 3, we can obtain all the polynomial, rational and global analytic first integral of the QH-systems of degree 2 and 3.

Using the algorithm

The algorithm has been used in:

W. Aziz, J. Llibre and C. Pantazi.
Center of quasi-homogeneous polynomial differential equations of degree three.
Advances in Mathematics, 254 (2014).

H. Liang, J. Huang and Y. Zhao.
Classification of global phase portraits of planar quartic QH polynomial differential systems.
Nonlinear Dyn., 78 (2014).

Using the algorithm

The algorithm has been used in:

Y. Tang, L. Wang and X. Zhang.
Center of planar quintic QH polynomial differential systems.
Discrete Contin. Dyn. Syst., 35 (2015).

Y. Xiong, M. Han and Y. Wang.
Center problems and Limit Cycle Bifurcations in a class of Quasi-Homogeneous Systems.
Int. J. of Bifurcation and Chaos, 25 (2015).

Definition

Let $\mathbb{N}$ denote the set of positive integers. The polynomial differential system

$$\dot{x} = P(x,y,z), \qquad \dot{y} = Q(x,y,z), \qquad \dot{z} = R(x,y,z) \tag{2}$$

or the vector field $X = (P, Q, R)$ is *quasi–homogeneous* if there exist $s_1, s_2, s_3, d \in \mathbb{N}$ such that for arbitrary $\alpha \in \mathbb{R}^+ = \{a \in \mathbb{R},\ a > 0\}$,

$$\begin{aligned} P(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) &= \alpha^{s_1-1+d}P(x,y,z), \\ Q(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) &= \alpha^{s_2-1+d}Q(x,y,z), \\ R(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) &= \alpha^{s_3-1+d}R(x,y,z). \end{aligned} \tag{3}$$

We call s_1, s_2 and s_3 the *weight exponents* of system (1), and d the *weight degree* with respect to the weight exponents s_1, s_2 and s_3.

Weight vectors

- Suppose that system (2) is quasi-homogeneous and with weight exponents s_1, s_2 and s_3 and with weight degree d. In what follows we denote by $\mathbf{w} = (s_1, s_2, s_3, d)$ the *weight vector* formed with the weight exponents and the weight degree of the system. We say that weight vector $\mathbf{w}_m = (s_1^*, s_2^*, s_3^*, d^*)$ is the *minimun weight vector* of the polynomial differential system (2) if any other weight vector $\mathbf{w} = (s_1, s_2, s_3, d)$ of system (2) verifies $s_1^* \leq s_1$, $s_2^* \leq s_2$, $s_3^* \leq s_3$ and $d^* \leq d$.
- A quasi homogenous system is homogeneous if and only if one of its weight vectors is $(1, 1, 1, n)$

The nonhomogeneous case: Homogeneous parts

Now we shall obtain some properties of the coefficients of the quasi–homogeneous polynomial vector fields. We write the polynomials P, Q and R of system (2) in their homogeneous parts

$$P = \sum_{j=0}^{n_1} P_j(x,y,z) = \sum_{j=0}^{n_1} \sum_{p_1=0}^{j} \sum_{p_2=0}^{j-p_1} a_{p_1,p_2,j-p_1-p_2} x^{p_1} y^{p_2} z^{j-p_1-p_2}, \quad (4)$$

$$Q = \sum_{j=0}^{n_2} Q_j(x,y,z) = \sum_{j=0}^{n_2} \sum_{q_1=0}^{j} \sum_{q_2=0}^{j-q_1} b_{q_1,q_2,j-q_1-q_2} x^{q_1} y^{q_2} z^{j-q_1-q_2}, \quad (5)$$

and

$$R = \sum_{j=0}^{n_3} R_j(x,y,z) = \sum_{j=0}^{n_3} \sum_{t_1=0}^{j} \sum_{t_2=0}^{j-t_1} b_{t_1,t_2,j-t_1-t_2} x^{t_1} y^{t_2} z^{j-t_1-t_2}. \quad (6)$$

Equations associated to a nonzero homogeneous part

From (3), (4), (5) and (6) we deduce that the coefficients of a quasi-homogeneous polynomial vector field satisfy for each homogeneous part of degree k the following equations:

$$a_{p_1,p_2,k-p_1-p_2}\alpha^{(p_1-1)s_1+p_2s_2+(k-p_1-p_2)s_3-(d-1)} = a_{p_1,p_2,k-p_1-p_2},$$

$$b_{q_1,q_2,k-q_1-q_2}\alpha^{q_1s_1+(q_2-1)s_2+(k-q_1-q_2)s_3-(d-1)} = b_{q_1,q_2,k-q_1-q_2},$$

$$c_{t_1,t_2,k-t_1-t_2}\alpha^{t_1s_1+t_2s_2+(k-t_1-t_2-1)s_3-(d-1)} = c_{t_1,t_2,k-t_1-t_2}.$$

and we obtain that if the homogeneous part of degree k exists then:

$$\begin{aligned} a_{p_1,p_2,k-p_1-p_2} \neq 0 &\Rightarrow (p_1-1)s_1 + p_2s_2 + (k-p_1-p_2)s_3 = (d-1) \\ b_{q_1,q_2,k-q_1-q_2} \neq 0 &\Rightarrow q_1s_1 + (q_2-1)s_2 + (k-q_1-q_2)s_3 = (d-1) \\ c_{t_1,t_2,k-t_1-t_2} \neq 0 &\Rightarrow t_1s_1 + t_2s_2 + (k-t_1-t_2-1)s_3 = (d-1) \end{aligned} \quad (7)$$

Equations associated to a nonzero homogeneous part

In the following we can assume, without loss of generality, that $s_1 \geq s_2 \geq s_3 > 0$. Therefore, the expression of (7) can be rewritten, using the new variables $\overline{s_1} = s_1 - s_3$, $\overline{s_2} = s_2 - s_3$ and $\overline{d} = d - 1 + s_3$, as:

$$\begin{array}{l} (p_1 - 1)\overline{s_1} + p_2\overline{s_2} + ks_3 - \overline{d} = 0, \\ q_1\overline{s_1} + (q_2 - 1)\overline{s_2} + ks_3 - \overline{d} = 0, \\ t_1\overline{s_1} + t_2\overline{s_2} + ks_3 - \overline{d} = 0. \end{array}$$

Hence, we have that if the homogeneous part of degree k of a quasi-homogeneous vector field is different from zero, then $x_1, x_2 \in \mathbb{N} \cup \{-1\}$ exist such that

$$\begin{array}{l} x_1\overline{s_1} + x_2\overline{s_2} + ks_3 = \overline{d}, \\ -1 \leq x_1 + x_2 \leq k, \\ \bar{s}_1 \geq \bar{s}_2 \geq 0, \bar{s}_1 > 0, s_3 > 0, \overline{d} \geq s_3. \end{array} \qquad (8)$$

System associated to two nonzero homogeneous parts

We remark that if the system is not homogeneous then there are at least two nonzero homogeneous parts that correspond to degrees n and $m < n$. Since for each part equation (8) is verified, we deduce that there are $x_1, x_2, y_1, y_2 \in \mathbb{N} \cup \{-1\}$ such that

$$\begin{aligned} x_1 \overline{s_1} + x_2 \overline{s_2} + n s_3 - \overline{d} = 0, \\ y_1 \overline{s_1} + y_2 \overline{s_2} + m s_3 - \overline{d} = 0, \end{aligned} \tag{9}$$

with $\overline{s}_1 \geq \overline{s}_2 \geq 0, \overline{s}_1 > 0, s_3 > 0, \overline{d} \geq s_3$. The solution of this system is

$$W = (a, b, (Y_1 a + Y_2 b)/(n - m), (X_1 a + X_2 b)/(n - m)), \tag{10}$$

where $Y_i = y_i - x_i$ and $X_i = n y_i - m x_i$, $i = 1, 2$

Proposition

There are some solutions W such that all components $W(j), j = 1, 2, 3, 4$ are positive integers verifying $W(1) \geq W(2) \geq 0$, $W(1) > 0$, $W(3) > 0$ and $W(4) \geq W(3)$ if and only if $Y_1 > 0$ or $Y_1 + Y_2 > 0$.

Searching QHNH systems

- We select the linear systems according to the above proposition.
- We add to each system other compatible equations corresponding to the same homogeneous parts or to a new homogeneous part different from n and m.
- We solve the final systems and determine the associated weight vectors and the quasi-homogeneous systems.

Case n=2

- There are 20 different algebraic forms of tridimensional QHNH systems of degree 2.
- Unlike the planar case, an homogeneous parts of a tridimensional QHNH system can have more than one different term. Example:

$$\begin{aligned} x' &= a_1 xz + a_2 xy, \\ y' &= b_1 z^2 + b_2 yz + b_3 y^2 + b_4 x, \\ z' &= c_1 z^2 + c_2 yz + c_3 y^2 + c_4 x, \\ w_m &= (2, 1, 1, 2) \end{aligned}$$

Conclusions: First part

- In this talk, first of all we have studied the quasi-homogeneous polynomial differential systems and the properties of their weight vectors.
- Using the properties of the quasi-homogeneous systems, we have provided an algorithm for obtaining all QHNH-systems of a given degree.
- Taking into account this algorithm, we have determined, through the Euler's totient function, the exact number of different algebraic forms as a function of the degree of the system. We also present the software implementations required to obtain this number and the corresponding forms.

Conclusions: Second part

- We have commented some applications of the algorithm in our papers and in other recent references: integrability, centers, phase portraits, ...
- We have introduced the tridimensional case and presented the first results about this case. Later we hope to obtain an algorithm and the exact number of algebraic forms in this case.

THANK YOU
FOR
YOUR ATTENTION!!