Invariants and time-reversibility in polynomial systems of ODEs

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FOURTH SYMPOSIUM ON PLANAR VECTOR FIELDS
Lleida, September 5–9, 2016

- K.S. Sibirsky, N. I. Vulpe and other Moldavian mathematicians in 1960-80th.
  - K. S. Sibirsky. *Introduction to the Algebraic Theory of Invariants of Differential Equations.* Nonlinear Science: Theory and Applications. Manchester: Manchester University Press, 1988.
- A generalization to complex systems:
  - Chapter 5 of V. R. and D. S. Shafer, *The Center and Cyclicity Problems: A Computational Algebra Approach*, Birkhüser, Boston, 2009.
- Some contributions in:
  Liu Yi Rong and Li Ji Bin (Theory of values of singular point in complex autonomous differential systems. Sci. China Ser. A, 1989)

Definition

Let k be a field, G be a group of $n \times n$ matrices with elements in k, $A \in G$ and $\mathbf{x} \in k^n$. A polynomial $f \in k[x_1, \dots, x_n]$ is *invariant under* G if $f(\mathbf{x}) = f(A \cdot \mathbf{x})$ for every $A \in G$. An invariant is *irreducible* if it does not factor as a product of polynomials that are themselves invariants.

Example. $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $C_4 = \{E_2, B, B^2, B^3\}$ is a group under multiplication. For $f(\mathbf{x}) = f(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ $f(\mathbf{x}) = f(B \cdot \mathbf{x})$, $f(\mathbf{x}) = f(B^2 \cdot \mathbf{x})$, and $f(\mathbf{x}) = f(B^3 \cdot \mathbf{x})$.
When $k = \mathbb{R}$, B is the group of rotations by multiples of $\frac{\pi}{2}$ radians (mod 2π) about the origin in $\mathbb{R}^2$, and f is an invariant because its level sets are circles centered at the origin, which are unchanged by such rotations.

$$
\begin{aligned}
\frac{d\mathbf{x}}{dt} &= A\mathbf{x}: \\
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2
\end{aligned}
\tag{1}
$$

Let $Q = GL_2(\mathbb{R})$ be the group of all linear invertible transformations of $\mathbb{R}^2$:

$$\mathbf{y} = C\mathbf{x},$$

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det\ C \neq 0.$$

Then,

$$\frac{d\mathbf{y}}{dt} = B\mathbf{y}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = CAC^{-1} = \frac{1}{\det\ C} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},$$

where

$$
\begin{aligned}
d_{11} &= ada_{11} + bda_{21} - aca_{12} - bca_{22} \\
d_{12} &= aba_{11} - b^2a_{21} + a^2a_{12} + aba_{22}, \\
d_{21} &= cda_{11} + d^2a_{21} - c^2a_{12} - cda_{22}, \\
d_{22} &= -bca_{11} - bda_{21} + aca_{12} + ada_{22}.
\end{aligned}
$$

Therefore,

$$\frac{d\mathbf{y}}{dt} = B\mathbf{y}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$b_{11} = \frac{1}{det\ C} d_{11}, \quad b_{12} = \frac{1}{det\ C} d_{12}, \quad b_{21} = \frac{1}{det\ C} d_{21}, \quad b_{22} = \frac{1}{det\ C} d_{22}. \tag{2}$$

$d_{11} = ada_{11} + bda_{21} - aca_{12} - bca_{22}$
$d_{12} = aba_{11} - b^2 a_{21} + a^2 a_{12} + aba_{22},$
$d_{21} = cda_{11} + d^2 a_{21} - c^2 a_{12} - cda_{22},$
$d_{22} = -bca_{11} - bda_{21} + aca_{12} + ada_{22}.$
We look for a homogeneous invariant of degree one:

$$I(\mathbf{a}) = k_1 a_{11} + k_2 a_{12} + k_3 a_{21} + k_4 a_{22}. \tag{3}$$

It should be $I(\mathbf{b}) = I(\mathbf{a})$, that is,

$$k_1 b_{11} + k_2 b_{12} + k_3 b_{21} + k_4 b_{22} = k_1 a_{11} + k_2 a_{12} + k_3 a_{21} + k_4 a_{22}.$$

From (3) and (2)

$$k_3 = \frac{bk_2}{c}, \qquad k_4 = k_1 + \frac{k_2(d-a)}{c}.$$

Thus, $k_2 = k_3 = 0$ $k_4 = k_1$ and $I_1(\mathbf{a}) = a_{11} + a_{22} = trA$.

- Each invariant of degree 2 is of the form:

$$I(\mathbf{a}) = k_1(a_{11}^2 + a_{22}^2 + 2a_{11}a_{22}) + k_2(a_{11}a_{22} - a_{12}a_{21}) =$$

$$k_1 tr^2 A^2 + k_2 \det A.$$

- Any invariant of degree 3 and higher is a polynomial of $tr\,A$ and $\det A$.

Invariants of the rotation group

Consider polynomial systems on $\mathbb{C}^2$ in the form

$$\begin{aligned} \dot{x} &= -\sum_{(p,q)\in\widetilde{S}} a_{pq} x^{p+1} y^q = P(x,y), \\ \dot{y} &= \sum_{(p,q)\in\widetilde{S}} b_{qp} x^q y^{p+1} = Q(x,y), \end{aligned} \tag{4}$$

- $\widetilde{S} \subset \mathbb{N}_{-1} \times \mathbb{N}_0$ is a finite set and each of its elements (p,q) satisfies $p+q \geq 0$.
- ℓ is the cardinality of the set $\widetilde{S}$
- $(a,b) = (a_{p_1,q_1}, a_{p_2,q_2}, \dots, a_{p_\ell,q_\ell}, b_{q_\ell,p_\ell}, \dots, b_{q_2,p_2}, b_{q_1,p_1})$ is the ordered vector of coefficients of system (4),
- $\mathbb{C}[a,b]$ denotes the polynomial ring in the variables a_{pq} and b_{qp}.

Consider the group of rotations

$$x' = e^{-i\varphi} x, \quad y' = e^{i\varphi} y \tag{5}$$

of the phase space $\mathbb{C}^2$ of (4). In (x', y') coordinates

$$\dot{x}' = -\sum_{(p,q)\in\widetilde{S}} a(\varphi)_{pq} x'^{p+1} y'^{q}, \quad \dot{y}' = \sum_{(p,q)\in\widetilde{S}} b(\varphi)_{qp} x'^{q} y'^{p+1},$$

where the coefficients of the transformed system are

$$a(\varphi)_{p_j q_j} = a_{p_j q_j} e^{i(p_j - q_j)\varphi}, \quad b(\varphi)_{q_j p_j} = b_{q_j p_j} e^{i(q_j - p_j)\varphi}, \tag{6}$$

for $j = 1, \dots, \ell$. For any fixed φ the equations in (6) determine an invertible linear mapping U_φ of the space $E(a, b)$ of parameters of (4) onto itself, which we will represent as the block diagonal $2\ell \times 2\ell$ matrix

$$U_\varphi = \begin{pmatrix} U_\varphi^{(a)} & 0 \\ 0 & U_\varphi^{(b)} \end{pmatrix},$$

$U_\varphi^{(a)}$ and $U_\varphi^{(b)}$ are diagonal matrices that act on the coordinates a and b respectively.

Example.

$$\dot{x} = -a_{00}x - a_{-11}y - a_{20}x^3, \quad \dot{y} = b_{1,-1}x + b_{00}y + b_{02}y^3 \qquad (7)$$

$\widetilde{S}$ is the ordered set $\{(0,0),(-1,1),(2,0)\}$, and equation (6) gives $2\ell = 6$ equations

$$\begin{array}{lll} a(\varphi)_{00} = a_{00}e^{i(0-0)\varphi} & a(\varphi)_{-11} = a_{-11}e^{i(-1-1)\varphi} & a(\varphi)_{20} = a_{20}e^{i(2-0)\varphi} \\ b(\varphi)_{00} = b_{00}e^{i(0-0)\varphi} & b(\varphi)_{1,-1} = b_{1,-1}e^{i(1-(-1))\varphi} & b(\varphi)_{02} = b_{02}e^{i(0-2)\varphi} \end{array}$$

so that

$$U_\varphi \cdot (a,b) = \begin{pmatrix} U_\varphi^{(a)} & 0 \\ 0 & U_\varphi^{(b)} \end{pmatrix} \cdot (a,b)^T =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-i2\varphi} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i2\varphi} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i2\varphi} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i2\varphi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{00} \\ a_{-11} \\ a_{20} \\ b_{02} \\ b_{1,-1} \\ b_{00} \end{pmatrix} = \begin{pmatrix} a_{00} \\ a_{-11}e^{-i2\varphi} \\ a_{20}e^{i2\varphi} \\ b_{02}e^{-i2\varphi} \\ b_{1,-1}e^{i2\varphi} \\ b_{00} \end{pmatrix}.$$

The set $U = \{U_\varphi : \varphi \in \mathbb{R}\}$ is a group, a subgroup of the group of invertible $2\ell \times 2\ell$ matrices with entries in $\mathbb{C}$. In the context of U the group operation corresponds to following one rotation with another.

Definition

The group $U = \{U_\varphi : \varphi \in \mathbb{R}\}$ is called the *rotation group* of family (4). A polynomial invariant of the group U is termed an *invariant of the rotation group*, or more simply an *invariant*.

• We wish to identify all polynomial invariants of U.
$f \in \mathbb{C}[a, b]$ is an invariant of $U \iff$ each of its terms is an invariant $\Longrightarrow$ it suffices to find the invariant monomials.

$$\dot{x} = -\sum_{(p,q)\in\widetilde{S}} a_{pq}x^{p+1}y^q = P(x,y), \quad \dot{y} = \sum_{(p,q)\in\widetilde{S}} b_{qp}x^q y^{p+1} = Q(x,y),$$

$$L(\nu) = \binom{p_1}{q_1}\nu_1 + \cdots + \binom{p_\ell}{q_\ell}\nu_\ell + \binom{q_\ell}{p_\ell}\nu_{\ell+1} + \cdots + \binom{q_1}{p_1}\nu_{2\ell}.$$

$$\dot{x} = -a_{00}x - a_{-11}y - a_{20}x^3, \quad \dot{y} = b_{1,-1}x + b_{00}y + b_{02}y^3$$

$$L(\nu) = \binom{0}{0}\nu_1 + \binom{-1}{1}\nu_2 + \binom{2}{0}\nu_3 + \binom{0}{2}\nu_4 + + \binom{1}{-1}\nu_5 + \binom{0}{0}\nu_6.$$

$$\nu = (\nu_1, \nu_2, \ldots, \nu_{2\ell-1}, \nu_{2\ell}), \quad \hat{\nu} = (\nu_{2\ell}, \nu_{2\ell-1}, \ldots, \nu_2, \nu_1),$$

Proposition

$(a,b)^\nu \stackrel{def}{=} a_{p_1q_1}^{\nu_1}\cdots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell p_\ell}^{\nu_{\ell+1}}\cdots b_{q_1p_1}^{\nu_{2\ell}}$ is an invariant $\Leftrightarrow L_1(\nu)=L_2(\nu)$.

$$\mathcal{M}=\{\nu\in\mathbb{N}_0^{2\ell} : L(\nu)=\begin{pmatrix}k\\k\end{pmatrix} \text{ for some } k\in\mathbb{N}_0\}. \tag{8}$$

$(a,b)^\nu$ is invariant under U if and only if $\nu\in\mathcal{M}$. For

$$(a,b)^\nu = a_{p_1q_1}^{\nu_1}\cdots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell p_\ell}^{\nu_{\ell+1}}\cdots b_{q_1p_1}^{\nu_{2\ell}} \in \mathbb{C}[a,b]$$

its conjugate is

$$\widehat{(a,b)^\nu} = a_{p_1q_1}^{\nu_{2\ell}}\cdots a_{p_\ell q_\ell}^{\nu_\ell+1} b_{q_\ell p_\ell}^{\nu_\ell}\cdots b_{q_1p_1}^{\nu_1} \in \mathbb{C}[a,b]$$

Since, for any $\nu\in\mathbb{N}_0^{2\ell}$, $L_1(\nu)-L_2(\nu)=-(L_1(\hat\nu)-L_2(\hat\nu))$, $\Rightarrow (a,b)^\nu$ is invariant under U if and only if its conjugate $\widehat{(a,b)^\nu}$ is.

Definition

The ideal $I_S=\langle (a,b)^\nu-\widehat{(a,b)^\nu} \mid \nu\in\mathcal{M}\rangle$ is called the Sibirsky ideal.

- To find a basis of irreducible invariants it is sufficient to find a basis of the Sibirsky ideal.

An algorithm for computing a generating set of invariants (A. Jarrah, R. Laubenbacher, V.R. J. Symb. Comp. 2003)

$$\dot{x} = -\sum_{(p,q)\in\widetilde{S}} a_{pq}x^{p+1}y^q = P(x,y),$$
$$\dot{y} = \sum_{(p,q)\in\widetilde{S}} b_{qp}x^q y^{p+1} = Q(x,y),$$

$$L(\nu) = \begin{pmatrix} L^1(\nu) \\ L^2(\nu) \end{pmatrix} = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}\nu_1 + \cdots + \begin{pmatrix} p_\ell \\ q_\ell \end{pmatrix}\nu_\ell + \begin{pmatrix} q_\ell \\ p_\ell \end{pmatrix}\nu_{\ell+1} + \cdots + \begin{pmatrix} q_1 \\ p_1 \end{pmatrix}\nu_{2\ell}.$$

- Problem: find a generating set for the Sibirski ideal

$I_S = \langle (a,b)^\nu - \widehat{(a,b)^\nu} | \nu \in \mathcal{M}\rangle$

$\Leftrightarrow$ a basis for

$$\mathcal{M} = \{\nu \in \mathbb{N}_0^{2\ell} : L(\nu) = \begin{pmatrix} k \\ k \end{pmatrix} \text{ for some } k \in \mathbb{N}_0\}.$$

Algorithm

Input: Two sequences of integers $p_1, \dots, p_\ell$ ($p_i \geq -1$) and $q_1, \dots, q_\ell$ ($q_i \geq 0$). (These are the coefficient labels for our system.)
Output: A finite set of generators for I_S (equivalently, the Hilbert basis of $\mathcal{M}$).

1. Compute a reduced Gröbner basis G for the ideal

   $$\mathcal{J} = \langle a_{p_i q_i} - y_i t_1^{p_i} t_2^{q_i}, b_{q_i p_i} - y_i t_1^{q_i} t_2^{p_i} \mid i = 1, \dots, \ell \rangle \\ \subset \mathbb{C}[a, b, y_1, \dots, y_\ell, t_1, t_2]$$

   with respect to any elimination ordering for which

   $$\{t_1, t_2\} > \{y_1, \dots, y_\ell\} > \{a_{p_1 q_1}, \dots, b_{q_1 p_1}\}.$$

2. $I_S = \langle G \cap \mathbb{C}[a, b] \rangle$.
3. The Hilbert basis of $\mathbb{C}[\mathcal{M}]$ is formed by the monomials of I_S and monomials of the form $a_{ik} b_{ki}$

The idea of the proof:

- Show that I_S is the kernel of the ring homomorphism

$$\phi : C[a,b] \mapsto \mathbb{C}[a,b,y_1,\dots,y_\ell,t_1,t_2]$$

  defined by

$$a_{p_i q_i} \mapsto y_i t_1^{p_i} t_2^{q_i},\ \ b_{q_i p_i} \mapsto y_i t_1^{q_i} t_2^{p_i} \mid i = 1,\dots,\ell$$

- Compute the kernel using known algorithm of computational algebra
- Show that the exponents of the generators of I_S and vectors $(1,0,\dots,0,1),\ (0,1,0,\dots,0,1,0),\ (0,\dots,0,1,1,0,\dots,0)$ form a Hilbert basis of $\mathcal{M}$.

$$\dot{x} = x - a_{10}x^2 - a_{01}xy - a_{-12}y^2, \quad \dot{y} = -y + b_{10}xy + b_{01}y^2 + b_{2,-1}x^2.$$

```
i = {a10 - y1 t1, a01 - y2 t2, a12 - y3 t1^(-1) t2^2, b21 - y3 t1^2 t2^(-1),
  b10 - y2 t1, b01 - y1 t2 }
```

$$\left\{\mathrm{a10} - \mathrm{t1\,y1},\ \mathrm{a01} - \mathrm{t2\,y2},\ \mathrm{a12} - \frac{\mathrm{t2}^2\,\mathrm{y3}}{\mathrm{t1}},\ \mathrm{b21} - \frac{\mathrm{t1}^2\,\mathrm{y3}}{\mathrm{t2}},\ \mathrm{b10} - \mathrm{t1\,y2},\ \mathrm{b01} - \mathrm{t2\,y1}\right\}$$

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GroebnerBasis[i, {t1, t2, y1, y2, y3, b10, b01, a10, a01, a12, b21}]
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$$\{-\mathrm{a10}^3\,\mathrm{a12} + \mathrm{b01}^3\,\mathrm{b21},\ \mathrm{a10}^2\,\mathrm{a12\,b10} - \mathrm{a01\,b01}^2\,\mathrm{b21},\ -\mathrm{a01\,a10} + \mathrm{b01\,b10},$$
$$\mathrm{a10\,a12\,b10}^2 - \mathrm{a01}^2\,\mathrm{b01\,b21},\ \mathrm{a12\,b10}^3 - \mathrm{a01}^3\,\mathrm{b21},\ \mathrm{a01\,b21\,y2} - \mathrm{b10}^2\,\mathrm{y3},$$
$$\mathrm{a10\,a12\,y2} - \mathrm{a01\,b01\,y3},\ \mathrm{b01\,b21\,y2} - \mathrm{a10\,b10\,y3},\ \mathrm{a12\,b10\,y2} - \mathrm{a01}^2\,\mathrm{y3},$$
$$\mathrm{a12\,b21\,y2}^2 - \mathrm{a01\,b10\,y3}^2,\ \mathrm{a01\,y1} - \mathrm{b01\,y2},\ \mathrm{a10\,a12\,y1} - \mathrm{b01}^2\,\mathrm{y3},\ \mathrm{b01\,b21\,y1} - \mathrm{a10}^2\,\mathrm{y3},$$
$$\mathrm{b10\,y1} - \mathrm{a10\,y2},\ \mathrm{a12\,b21\,y1\,y2} - \mathrm{a01\,a10\,y3}^2,\ \mathrm{a12\,b21\,y1}^2 - \mathrm{a10\,b01\,y3}^2,$$
$$-\mathrm{a12\,b10} + \mathrm{a01\,t2\,y3},\ -\mathrm{b01}^2\,\mathrm{b21} + \mathrm{a10}^2\,\mathrm{t2\,y3},\ -\mathrm{a10\,a12} + \mathrm{b01\,t2\,y3},$$
$$-\mathrm{a01\,b01\,b21} + \mathrm{a10\,b10\,t2\,y3},\ -\mathrm{a01}^2\,\mathrm{b21} + \mathrm{b10}^2\,\mathrm{t2\,y3},\ -\mathrm{a12\,b21\,y1} + \mathrm{a10\,t2\,y3}^2,$$
$$-\mathrm{a12\,b21\,y2} + \mathrm{b10\,t2\,y3}^2,\ -\mathrm{a01} + \mathrm{t2\,y2},\ -\mathrm{b01} + \mathrm{t2\,y1},\ -\mathrm{a12}^2\,\mathrm{b21} + \mathrm{t2}^3\,\mathrm{y3}^3,$$
$$\frac{\mathrm{a12}^2\,\mathrm{b21}}{\mathrm{t2}} - \mathrm{t2}^2\,\mathrm{y3}^3,\ \frac{\mathrm{a01}}{\mathrm{t2}} - \mathrm{y2},\ \frac{\mathrm{a10\,a12}}{\mathrm{t2}} - \mathrm{b01\,y3},\ \frac{\mathrm{b01}}{\mathrm{t2}} - \mathrm{y1},\ \frac{\mathrm{a12\,b10}}{\mathrm{t2}} - \mathrm{a01\,y3},$$
$$-\mathrm{b21\,y1}^2 + \frac{\mathrm{a10}^2\,\mathrm{y3}}{\mathrm{t2}},\ -\mathrm{b21\,y1\,y2} + \frac{\mathrm{a10\,b10\,y3}}{\mathrm{t2}},\ -\mathrm{b21\,y2}^2 + \frac{\mathrm{b10}^2\,\mathrm{y3}}{\mathrm{t2}},\ \frac{\mathrm{a12\,b21\,y2}}{\mathrm{t2}} - \mathrm{b10\,y3}^2,$$
$$\frac{\mathrm{a12\,b21\,y1}}{\mathrm{t2}} - \mathrm{a10\,y3}^2,\ \mathrm{a12\,t1} - \mathrm{t2}^2\,\mathrm{y3},\ \mathrm{a01\,t1} - \mathrm{b10\,t2},\ \mathrm{b01\,t1} - \mathrm{a10\,t2},$$
$$-\mathrm{b01\,b21} + \mathrm{a10\,t1\,y3},\ -\mathrm{a01\,b21} + \mathrm{b10\,t1\,y3},\ -\frac{\mathrm{a12\,b21}}{\mathrm{t2}} + \mathrm{t1\,y3}^2,\ -\mathrm{b10} + \mathrm{t1\,y2},$$
$$-\mathrm{a10} + \mathrm{t1\,y1},\ -\mathrm{b21\,t2} + \mathrm{t1}^2\,\mathrm{y3},\ \frac{\mathrm{b21}}{\mathrm{t1}} - \frac{\mathrm{t1\,y3}}{\mathrm{t2}},\ \frac{\mathrm{a10}}{\mathrm{t1}} - \mathrm{y1},\ \frac{\mathrm{b10}}{\mathrm{t1}} - \mathrm{y2},\ -\frac{\mathrm{a12}}{\mathrm{t2}^2} + \frac{\mathrm{y3}}{\mathrm{t1}}\}$$

The polynomials that do not depend on t_1, t_2:
$f_1 = a_{01}^3 b_{2,-1} - a_{-12} b_{10}^3$, $f_2 = a_{10} a_{01} - b_{01} b_{10}$, $f_3 = a_{10}^3 a_{-12} - b_{2,-1} b_{01}^3$,
$f_4 = a_{10} a_{-12} b_{10}^2 - a_{01}^2 b_{2,-1} b_{01}$, $f_5 = a_{10}^2 a_{-12} b_{10} - a_{01} b_{2,-1} b_{01}^2$.
Thus,

$$I_S = \mathcal{I} = \langle f_1, \dots, f_5 \rangle.$$

Basis of the invariants:
$a_{01}^3 b_{2,-1}$, $a_{-12} b_{10}^3$, $a_{10} a_{01}$, $b_{01} b_{10}$, $a_{10}^3 a_{-12}$, $b_{2,-1} b_{01}^3$,
$a_{10} a_{-12} b_{10}^2$, $a_{01}^2 b_{2,-1} b_{01}$, $a_{10}^2 a_{-12} b_{10}$, $a_{01} b_{2,-1} b_{01}^2$,
$a_{10} b_{01}$, $a_{01} b_{10}$, $a_{-12} b_{2,-1}$.
Basis of $\mathcal{M}$:
$(0,3,0,1,0,0)$, $(0,0,1,0,3,0)$, $(1,1,0,0,0,0)$, $(0,0,0,0,1,1), \dots,$
$(0,1,0,0,1,0)$, $(0,0,1,1,0,0)$.

Time-reversible systems

$$\frac{d\mathbf{z}}{dt} = F(\mathbf{z}) \quad (\mathbf{z} \in \Omega), \tag{9}$$

$F : \Omega \mapsto T\Omega$ is a vector field and Ω is a manifold.

Definition

A time-reversible symmetry of (9) is an invertible map $R : \Omega \mapsto \Omega$, such that

$$\frac{d(R\mathbf{z})}{dt} = -F(R\mathbf{z}). \tag{10}$$

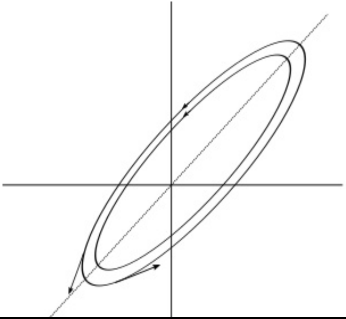
Example

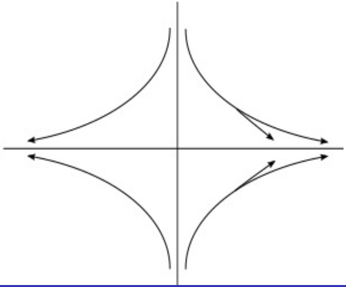
$$\dot{u} = v + vf(u, v^2), \qquad \dot{v} = -u + g(u, v^2), \tag{11}$$

The transformation $u \to u$, $v \to -v$, $t \to -t$ leaves the system unchanged $\Rightarrow$ the u–axis is a line of symmetry for the orbits $\Rightarrow$ no trajectory in a neighbourhood of $(0, 0)$ can be a spiral $\Rightarrow$ the origin is a center.

Here

$$R : u \mapsto u, \;\; v \mapsto -v. \tag{12}$$





Complexification

- $\dot{u} = U(u, v), \quad \dot{v} = V(u, v) \quad x = u + iv$

$$\dot{x} = \dot{u} + i\dot{v} = U + iV = P(x, \bar{x}) \tag{13}$$

- Add to (13) its complex conjugate to obtain the system

$$\dot{x} = P(x, \bar{x}), \ \dot{\bar{x}} = \overline{P(x, \bar{x})}. \tag{14}$$

- The condition of time-reversibility with respect to $Ou = Im\, x$: $P(\bar{x}, x) = -\overline{P(x, \bar{x})}$.

$$\dot{u} = U(u, v), \quad \dot{v} = V(u, v)$$

is time-reversibility with respect to $y = \tan\varphi\, x$ if:

$$e^{2i\varphi}\overline{P(x, \bar{x})} = -P(e^{2i\varphi}\bar{x}, e^{-2i\varphi}x). \tag{15}$$

Consider $\bar{x}$ as a new variable y and allow the parameters of the second equation of (14) to be arbitrary.

- The complex system

  $$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

  is time–reversible with respect to

  $$R : x \mapsto \gamma y, \ \ y \mapsto \gamma^{-1} x$$

  if and only if for some γ

  $$\gamma Q(\gamma y, x/\gamma) = -P(x, y), \quad \gamma Q(x, y) = -P(\gamma y, x/\gamma)\,. \tag{16}$$

In the particular case when $\gamma = e^{2i\varphi}$, $y = \bar{x}$, and $Q = \bar{P}$ the equality (16) is equivalent to the reflection with respect a line and the reversion of time.

$$
\begin{aligned}
\dot{x} &= \quad x - \sum_{(p,q)\in S} a_{pq} x^{p+1} y^q = P(x,y), \\
\dot{y} &= \quad -y + \sum_{(p,q)\in S} b_{qp} x^q y^{p+1} = Q(x,y),
\end{aligned} \tag{17}
$$

where S is the set
$S = \{(p_j, q_j) \,|\, p_j + q_j \geq 0, j = 1, \ldots, \ell\} \subset (\{-1\} \cup \mathbb{N}_0) \times \mathbb{N}_0$, and
$\mathbb{N}_0$ denotes the set of nonnegative integers. The parameters
$a_{p_j q_j}, \; b_{q_j p_j}$ $(j = 1, \ldots, \ell)$ are from $\mathbb{C}$ or $\mathbb{R}$.
$(a, b) = (a_{p_1 q_1}, \ldots, a_{p_\ell q_\ell}, b_{q_\ell p_\ell} \ldots, b_{q_1 p_1})$ is the ordered vector of coefficients of system (17),
$k[a, b]$ the polynomial ring in the variables a_{pq}, b_{qp} over the field k.

The condition of time-reversibility

$$\gamma Q(\gamma y, x/\gamma) = -P(x,y), \quad \gamma Q(x,y) = -P(\gamma y, x/\gamma)\,.$$

yields that system (17) is time–reversible if and only if

$$b_{qp} = \gamma^{p-q} a_{pq}, \qquad a_{pq} = b_{qp}\gamma^{q-p}. \tag{18}$$

$$\Updownarrow$$

$$a_{p_k q_k} = t_k, \quad b_{q_k p_k} = \gamma^{p_k - q_k} t_k \tag{19}$$

for $k = 1, \dots, \ell$. (19) define a surface in the affine space $\mathbb{C}^{3\ell+1} = (a_{p_1 q_1}, \dots, a_{p_\ell q_\ell}, b_{q_\ell p_\ell}, \dots, b_{q_1 p_1}, t_1, \dots, t_\ell, \gamma)$.

- The set of all time-reversible systems is the projection of this surface onto $\mathbb{C}^{2\ell}$).
- To find this set we have to eliminate t_k and γ from (19)

Definition

Let I be an ideal in $k[x_1, \ldots, x_n]$ (with the implicit ordering of the variables $x_1 > \cdots > x_n$) and fix $\ell \in \{0, 1, \ldots, n-1\}$.The *$\ell$–elimination ideal* of I is the ideal $I_\ell = I \cap k[x_{\ell+1}, \ldots, x_n]$. Any point $(a_{\ell+1}, \ldots, a_n) \in \mathbf{V}(I_\ell)$ is called a *partial solution* of the system $\{f = 0 : f \in I\}$.

Elimination Theorem (e.g. Cox, Little, O'Shea, Ideals, Varieties, Algorithms)

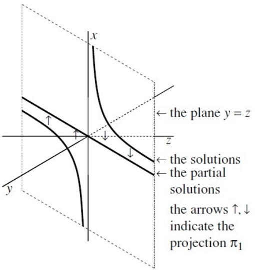
Fix the lexicographic term order on the ring $k[x_1, \ldots, x_n]$ with $x_1 > x_2 > \cdots > x_n$ and let G be a Gröbner basis for an ideal$I$ of $k[x_1, \ldots, x_n]$ with respect to this order. Then for every ℓ, $0 \leq \ell \leq n-1$, the set

$$G_\ell := G \cap k[x_{\ell+1}, \ldots, x_n]$$

is a Gröbner basis for the ℓ–th elimination ideal I_ℓ.

Elimination – projection of the variety on the subspace $x_{\ell+1}, \ldots, x_n$.
The variety of $\mathbf{V}(I_\ell)$ is the Zariski closure of the projection of $\mathbf{V}(I)$.
It is not always possible to extend a partial solution to a solution of the original system.

Example. $\quad xy = 1, \quad xz = 1$. The reduced Gröbner basis of $I = \langle xy - 1, xz - 1\rangle$ with respect to lex with $x > y > z$ is $\{xz - 1, y - z\}$.
$I_1 = \langle y - z\rangle$. $\mathbf{V}(I_1)$ is the line $y = z$ in the (y, z)-plane. Partial solutions corresponding to I_1 are $\{(a, a) : a \in \mathbb{C}\}$. Any partial solution (a, a) for which $a \neq 0$ can be extended to the solution $(1/a, a, a)$, except of $(0, 0)$.



$$H = \langle a_{p_k q_k} - t_k, b_{q_k p_k} - \gamma^{p_k - q_k} t_k \mid k = 1, \ldots, \ell \rangle, \tag{20}$$

Let $\mathcal{R}$ be the set of all time-reversible systems in the family (17).

Theorem (V. R., 2008)

$\overline{\mathcal{R}} = \mathbf{V}(\mathcal{I}_R)$ where $\mathcal{I}_R = \mathbb{C}[a, b] \cap H$, that is, the Zariski closure of the set $\mathcal{R}$ of all time-reversible systems is the variety of the ideal $\mathcal{I}_R$.

Every time–reversible system $(a, b) \in E(a, b)$ belongs to $\mathbf{V}(\mathcal{I}_R)$. The converse is false.

Theorem 1 (V.R. 2008)

Let $\mathcal{R} \subset E(a, b)$ be the set of all time–reversible systems in the family (4), then
(a) $\mathcal{R} \subset \mathbf{V}(I_R)$;
(b) $\mathbf{V}(\mathcal{I}_R) \setminus \mathcal{R} = \{(a, b) \mid \exists (p, q) \in S \text{ such that } a_{pq} b_{qp} = 0 \text{ but } a_{pq} + b_{qp} \neq 0\}$.

(b) means that if in a time-reversible system (4) $a_{pq} \neq 0$ then $b_{qp} \neq 0$ as well. (b) $\Longrightarrow$ the inclusion in (a) is strict, that is $\mathcal{R} \subsetneq \mathbf{V}(\mathcal{I}_R)$.

An algorithm for computing the set of all time-reversible systems

Let

$$H = \langle a_{p_k q_k} - t_k, b_{q_k p_k} - \gamma^{p_k - q_k} t_k \mid k = 1, \dots, \ell \rangle.$$

- Compute a Groebner basis G_H for H with respect to any elimination order with $\{w, \gamma, t_k\} > \{a_{p_k q_k}, b_{q_k p_k} \mid k = 1, \dots, \ell\}$;
- the set $B = G_H \cap k[a, b]$ is a set of binomials; $\mathbf{V}(\langle B \rangle) = \mathbf{V}(\mathcal{I}_R)$ is the Zariski closure of set of all time-reversible systems.

Example

$$\dot{x} = x - a_{10}x^2 - a_{01}xy - a_{-12}y^2, \; \dot{y} = -y + b_{10}xy + b_{01}y^2 + b_{2,-1}x^2.$$

$$H = \langle 1 - w\gamma^4, a_{10} - t_1, b_{01} - \gamma t_1, a_{01} - t_2, \; \gamma b_{10} - t_2, a_{-12} - t_3, \gamma^3 b_{2,-1} - t_3 \rangle$$

```
GroebnerBasis[{a10 - t1, b01 - γ t1, a01 - t2, γ b10 - t2,
  a12 - t3, γ^3 b21 - t3, 1 - w γ^4},
 {w, γ, t1, t2, t3, b10, b01, a10, a01, a12, b21}]
```

$\{-a10^3\, a12 + b01^3\, b21,\ a10^2\, a12\, b10 - a01\, b01^2\, b21,\ -a01\, a10 + b01\, b10,$
$a10\, a12\, b10^2 - a01^2\, b01\, b21,\ a12\, b10^3 - a01^3\, b21,\ -a12 + t3,$
$-a01 + t2,\ -a10 + t1,\ -a12\, b10^2 + a01^2\, b21\, \gamma,\ -b01 + a10\, \gamma,$
$-a10\, a12\, b10 + a01\, b01\, b21\, \gamma,\ -a10^2\, a12 + b01^2\, b21\, \gamma,\ -a01 + b10\, \gamma,$
$-a12\, b10 + a01\, b21\, \gamma^2,\ -a10\, a12 + b01\, b21\, \gamma^2,\ -a12 + b21\, \gamma^3,$
$a12^2\, w - b21^2\, \gamma^2,\ -b10\, b21 + a01\, a12\, w,\ -b10^4 + a01^4\, w,$
$-a10\, b21 + a12\, b01\, w,\ -a10\, b10^3 + a01^3\, b01\, w,\ -a10^2\, b10^2 + a01^2\, b01^2\, w,$
$-a10^3\, b10 + a01\, b01^3\, w,\ -a10^4 + b01^4\, w,\ -b21 + a12\, w\, \gamma,$
$-b10^3 + a01^3\, w\, \gamma,\ -a10\, b10^2 + a01^2\, b01\, w\, \gamma,\ -a10^2\, b10 + a01\, b01^2\, w\, \gamma,$
$-a10^3 + b01^3\, w\, \gamma,\ -b10^2 + a01^2\, w\, \gamma^2,\ -a10\, b10 + a01\, b01\, w\, \gamma^2,$
$-a10^2 + b01^2\, w\, \gamma^2,\ -b10 + a01\, w\, \gamma^3,\ -a10 + b01\, w\, \gamma^3,\ -1 + w\, \gamma^4\}$

The polynomials that do not depend on w, γ, t_1, t_2, t_3:
$f_1 = a_{01}^3 b_{2,-1} - a_{-12} b_{10}^3$, $f_2 = a_{10} a_{01} - b_{01} b_{10}$,
$f_3 = a_{10}^3 a_{-12} - b_{2,-1} b_{01}^3$, $f_4 = a_{10} a_{-12} b_{10}^2 - a_{01}^2 b_{2,-1} b_{01}$,
$f_5 = a_{10}^2 a_{-12} b_{10} - a_{01} b_{2,-1} b_{01}^2$. Thus,

$$\mathcal{I}_R = \langle f_1, \ldots, f_5 \rangle.$$

The polynomials that do not depend on w, γ, t_1, t_2, t_3:
$f_1 = a_{01}^3 b_{2,-1} - a_{-12} b_{10}^3$, $f_2 = a_{10} a_{01} - b_{01} b_{10}$,
$f_3 = a_{10}^3 a_{-12} - b_{2,-1} b_{01}^3$, $f_4 = a_{10} a_{-12} b_{10}^2 - a_{01}^2 b_{2,-1} b_{01}$,
$f_5 = a_{10}^2 a_{-12} b_{10} - a_{01} b_{2,-1} b_{01}^2$. Thus,

$$\mathcal{I}_R = \langle f_1, \ldots, f_5 \rangle.$$

- $\mathbf{V}(\langle f_1, \ldots, f_5 \rangle)$ is the Zariski closure of the set of all time-reversible systems inside of the family

The polynomials that do not depend on w, γ, t_1, t_2, t_3:
$f_1 = a_{01}^3 b_{2,-1} - a_{-12} b_{10}^3$, $f_2 = a_{10} a_{01} - b_{01} b_{10}$,
$f_3 = a_{10}^3 a_{-12} - b_{2,-1} b_{01}^3$, $f_4 = a_{10} a_{-12} b_{10}^2 - a_{01}^2 b_{2,-1} b_{01}$,
$f_5 = a_{10}^2 a_{-12} b_{10} - a_{01} b_{2,-1} b_{01}^2$. Thus,

$$\mathcal{I}_R = \langle f_1, \dots, f_5 \rangle.$$

- $\mathbf{V}(\langle f_1, \dots, f_5 \rangle)$ is the Zariski closure of the set of all time-reversible systems inside of the family
- We have seen that $I_S = \langle f_1, \dots, f_5 \rangle$ and the monomials of f_i together with $a_{10} b_{01}$, $a_{01} b_{10}$, $a_{-12} b_{2,-1}$ generate the subalgebra $\mathbb{C}[\mathcal{M}]$ for invariants of U_φ and the exponents of the monomials form the Hilbert basis of the monoid $\mathcal{M}$.

Theorem 2 (V.R., 2008)

$I_S = \mathcal{I}_R$ and both ideals are prime.

Applications of invariants

1) Classifications of phase portraits (Sibirski, Vulpe and others)
2) Integrability:

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q) = P, \;\; \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}) = Q \quad (21)$$

Look for a series $\Phi(x, y; a_{10}, b_{10}, \dots) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j}$ such that

$$\frac{\partial \Phi}{\partial x} P + \frac{\partial \Phi}{\partial y} Q = g_{11}(xy)^2 + g_{22}(xy)^3 + \cdots , \qquad (22)$$

$g_{11}, g_{22}, \dots$ are polynomials in a_{pq}, b_{qp} *(focus quantities)*.

Theorem

$g_{ss}(a, b) \in \mathbb{C}[\mathcal{M}]$ and have the form $g_{ss} = \sum_{\nu \in \mathcal{M}} g^{(\nu)}((a, b)^{\nu} - \widehat{(a, b)^{\nu}})$.

$\Rightarrow$ every time-reversible system is integrable.

Applications for studying critical periods and small limit cycle bifurcations:

- V. Levandovskyy, V. G. Romanovski, D. S. Shafer (2009) The cyclicity of a cubic system with nonradical Bautin ideal, Journal of Differential Equations, 246 1274-1287.
- M. Han, V. G. Romanovski (2010) Estimating the number of limit cycles in polynomials systems Journal of Mathematical Analysis and Applications, 368, 491-497.
- B. Ferčec, V. Levandovskyy, V. Romanovski, D. Shafer. Bifurcation of critical periods of polynomial systems. Journal of Differential Equations, 2015, vol. 259, 3825-3853

The idea: use the invariants to simplify the focus and period quantities

$$\dot{x} = (x - a_{-12}y^2 - a_{20}x^3 - a_{02}xy^2) \;\; \dot{y} = -(y - b_{2,-1}x^2 - b_{20}x^2y - b_{02}y^3) \tag{23}$$

The algorithm yields
$a_{20}a_{02} - b_{20}b_{02},\; a_{-12}^2 a_{20} b_{20}^2 - a_{02}^2 b_{2,-1}^2 b_{02},\; a_{-12}^2 a_{20}^2 b_{20} - a_{02} b_{2,-1}^2 b_{02}^2,\; -a_{02}^3 b_{2,-1}^2 - a_{-12}^2 b_{20}^3,\; a_{-12}^2 a_{20}^3 - b_{2,-1}^2 b_{02}^3,$
Irreducible invariants:
$c_1 = a_{-12}b_{2,-1},\; c_2 = a_{20}b_{02},\; c_3 = a_{02}b_{20},\; c_4 = b_{20}b_{02},\; c_5 = a_{02}^3 b_{2,-1}^2,\; c_6 = a_{02}^2 b_{2,-1}^2 b_{02},\; c_7 = a_{02} b_{2,-1}^2 b_{02}^2,\; c_8 = b_{2,-1}^2 b_{02}^3,\; c_9 = a_{20}a_{02},\; c_{10} = a_{-12}^2 b_{20}^3,\; c_{11} = a_{-12}^2 a_{20} b_{20}^2,\; c_{12} = a_{-12}^2 a_{20}^2 b_{20},\; c_{13} = a_{-12}^2 a_{20}^3$
The focus quantities of (23) belong to the subalgebra $\mathbb{C}[c_1, \dots, c_{13}]$:

$$g_{kk} = g_{kk}(c_1, \dots, c_{13}) \tag{24}$$

$$F : E(a,b) = \mathbb{A}_{\mathbb{C}}^6 = \mathbb{C}^6 \longrightarrow \mathbb{A}_{\mathbb{C}}^{13} = \mathbb{C}^{13}.$$

$$
\begin{aligned}
g_{22} &= -\, i(a_{20}a_{02} - b_{20}b_{02}),\\
g_{44} &= -\, i(2160a_{20}^3a_{12}^2 + 5760a_{20}^2b_{20}a_{12}^2 + 2160a_{20}b_{20}^2a_{12}^2 - 1440b_{20}^3a_{12}^2\\
&\quad + 1440a_{02}^3b_{21}^2 - 2160a_{02}^2b_{02}b_{21}^2 - 5760a_{02}b_{02}^2b_{21}^2 - 2160b_{02}^3b_{21}^2)\\
g_{55} &= -\, i(-340200a_{20}^2b_{20}a_{12}^3b_{21} - 226800a_{20}b_{20}^2a_{12}^3b_{21} + 113400b_{20}^3a_{12}^3b_{21}\\
&\quad - 113400a_{02}^3a_{12}b_{21}^3 + 226800a_{02}^2b_{02}a_{12}b_{21}^3 + 340200a_{02}b_{02}^2a_{12}b_{21}^3)\\
g_{66} &= -\, i(102060000a_{20}^2b_{20}^2b_{02}a_{12}^2 + 68040000a_{20}b_{20}^3b_{02}a_{12}^2 - 34020000b_{20}^4b_{02}a_{12}^2\\
&\quad + 34020000a_{02}^3b_{20}b_{02}b_{21}^2 - 68040000a_{02}^2b_{20}b_{02}^2b_{21}^2 - 102060000a_{02}b_{20}b_{02}^3b_{2}^{2}
\end{aligned}
$$

$$
\begin{aligned}
&g_{11}^F = 0 \qquad g_{22}^F = -i(-3c_4 + 3c_9) \qquad g_{33}^F = 0\\
&g_{44}^F = -i(1440c_5 - 2160c_6 - 5760c_7 - 2160c_8 - 1440c_{10} + 2160c_{11} + 5760c_{12}\\
&\qquad\qquad + 2160c_{13})\\
&g_{55}^F = -i(-113400c_1c_5 + 226800c_1c_6 + 340200c_1c_7 + 113400c_1c_{10}\\
&\quad - 226800c_1c_{11} - 340200c_1c_{12})\\
&g_{66}^F = -i(34020000c_4c_5 - 68040000c_4c_6 - 102060000c_4c_7 - 34020000c_4c_{10}\\
&\qquad\qquad + 68040000c_4c_{11} + 102060000c_4c_{12}).
\end{aligned}
$$

The Bautin ideal is not radical in the ring

$$\mathbb{C}[a_{-12}, a_{20}, a_{02}, b_{20}, b_{02}, b_{2,-1}]$$

however it is radical in

$$\mathbb{C}[c_1, \dots, c_{13}]$$

$\Rightarrow$ the upper bound for cyclicity is 4.

Time-reversibility in higher dimensional systems

- Problem: Find conditions for complete local integrability (existence of two independent analytic local first integrals in a neighbourhood of the origin) of

$$\begin{aligned} \dot{x}_1 &= \qquad P_1(x_1, x_2, x_3) \\ \dot{x}_2 &= \quad x_2 + P_2(x_1, x_2, x_3) \\ \dot{x}_2 &= -x_3 + P_3(x_1, x_2, x_3) \end{aligned} \tag{25}$$

on $\mathbb{C}^3$, where P_j, $j \in \{1, 2, 3\}$, is an analytic function on a neighborhood of the origin.

The system can have integrals of the form

- $\psi_1(x_1, x_2, x_3) = x_1 + \sum_{i+j+k \geq 2} p_{ijk} x_1^i x_2^j x_3^k$
- $\psi_2(x_1, x_2, x_3) = x_2 x_3 + \sum_{i+j+k \geq 3} k_{ijs} x_1^i x_2^j x_3^k,$

By Llibre, Pantazi and Walcher (Bull. Sci. Math., 136, 342-359, 2012) if a system (25) is time-reversible with respect to a linear invertible transformation which permutes x_2 and x_3 then it is integrable.
We write (25) as

$$\dot{x}_1 = \sum a_{jkl} x_1^j x_2^k x_3^l, \quad \dot{x}_2 = x_2 \sum b_{mnp} x_1^m x_2^n x_3^p, \quad \dot{x}_3 = x_3 \sum c_{qrs} x_1^q x_2^r x_3^s. \tag{26}$$

Let u, v, w be the number of parameters of the first, the second and the third equation, respectively. By (a, b, c) we denote the $(u + v + w)$-tuple of parameters of system (26).
System (26) is time-reversible if there exists an invertible matrix T such that

$$T^{-1} \circ f \circ T = -f. \tag{27}$$

We look for a transformation T in the form

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & 1/\gamma & 0 \end{pmatrix}. \tag{28}$$

(27) is satisfied for T defined by (28) if and only if

$$a_{jkl} = -\gamma^{l-k} a_{jlk}, \quad b_{mnp} = -\gamma^{p-n} c_{mpn}. \tag{29}$$

let $k[a, b, c]$ be the ring of polynomials in parameters of system (26) with the coefficients in a field k and

$$H = \langle 1 - y\gamma,\ a_{jkl} + \gamma^{l-k} a_{jlk},\ b_{mnp} + \gamma^{p-n} c_{mpn} \rangle, \tag{30}$$

where y is a new variable.

Computation of $\mathcal{I} = k[a, b] \cap H$

Theorem (Hu, Han, R., 2013)

The Zariski closure of all time-reversible (with respect to (28)) systems inside the family (26) with coefficients in the field k (k is $\mathbb{R}$ or $\mathbb{C}$) is the variety $\mathbf{V}(I_S)$ of the ideal

$$I_S = k[a, b, c] \cap H. \tag{31}$$

A generating set for I_S is obtained by computing a Groebner basis for H with respect to any elimination order with $\{y, \gamma\} > \{a, b, c\}$ and choosing from the output list the polynomials which do not depend on y and γ.

Corollary

Let I_S be ideal (31) of system (26). Then all systems from $\mathbf{V}(I_S)$ are integrable.

Follows from a result of [V. R., Y. Xia, X. Zhang. J. Differential Equations 257 (2014) 3079–3101].

A generalized time-reversibility

Time-reversibility after some analytic transformations:

- B. Ferčec, J. Giné, V. R., V. Edneral. JMAA, 2016, vol. 434, issue 1, 894-914 (2-dim systems)
- V. R., D.S. Shafer. Applied Mathematics Letters, 2016, vol. 51, 27-33 (3-dim systems)

Suppose $A_j(x, y, z)$ is a homogeneous polynomial function of degree $m > 0$, $j \in \{1, 2, 3\}$.

$$\begin{aligned} \dot{x}_1 &= x_1 A_1(x_1, x_2, x_3) \\ \dot{x}_2 &= x_2(1 + A_2(x_1, x_2, x_3)) \\ \dot{x}_3 &= -x_3(1 + A_3(x_1, x_2, x_3)) \end{aligned} \tag{32}$$

Lemma

Let $F(x,y,z)$ be a homogeneous polynomial function of degree $m > 0$ and for any non-zero constants k_1, k_2, and k_3 set $G(x,y,z) = F(\frac{x}{k_1}, \frac{z}{k_3}, \frac{y}{k_2})$. Define mappings f and g on a neighborhood of the origin by $f = 1 + F$ and $g = 1 - G$. Then there exists a neighborhood N of the origin on which the mapping $\mathscr{F}$ defined by

$$y_1 = \frac{k_1 x_1}{f^{1/m}}, \qquad y_2 = \frac{k_2 x_3}{f^{1/m}}, \qquad y_3 = \frac{k_3 x_2}{f^{1/m}} \tag{33}$$

is an analytic change of coordinates (i.e., is one-to-one onto its image) with analytic inverse $\mathscr{G}$ on $\mathscr{F}(N)$ given by

$$x_1 = \frac{y_1}{k_1 g^{1/m}}, \qquad x_2 = \frac{y_3}{k_3 g^{1/m}}, \qquad x_3 = \frac{y_2}{k_2 g^{1/m}}. \tag{34}$$

Proposition 1

Suppose $A_j(x, y, z)$ is a homogeneous polynomial of degree $m > 0$, $j \in \{1, 2, 3\}$. There exist a local analytic change of coordinates of the form (33) in a neighborhood of the origin, with local inverse (34), and a rescaling of time that transform (32), i.e.

$$\begin{aligned} \dot{x}_1 &= x_1 A_1(x_1, x_2, x_3) \\ \dot{x}_2 &= x_2(1 + A_2(x_1, x_2, x_3)) \\ \dot{x}_3 &= -x_3(1 + A_3(x_1, x_2, x_3)) \end{aligned}$$

into

$$\begin{aligned} \dot{y}_1 &= y_1 B_1(y_1, y_2, y_3) \\ \dot{y}_2 &= -y_2(1 + h(y_1, y_2, y_3)) \\ \dot{y}_3 &= y_3(1 - h(y_1, y_2, y_3)). \end{aligned} \tag{35}$$

The polynomial F of Lemma is $F = \frac{1}{2}(A_2 + A_3)$.

The basis of a computationally efficient way of identifying some completely integrable systems of the form (32) is the following

Theorem (V.R. & D.S. Shafer, 2016)

Suppose $A_j(x, y, z)$ is a homogeneous polynomial of degree $m > 0$, $j \in \{1, 2, 3\}$ and that system (32) is transformed to system (35) by the analytic change of coordinates given by Proposition 1. If

$$B_1(y_1, y_3, y_2) = -B_1(y_1, y_2, y_3) \qquad h(y_1, y_3, y_2) = -h(y_1, y_2, y_3) \tag{36}$$

then system (32) has two functionally independent local analytic first integrals in a neighborhood of the origin.

Example

$$
\begin{aligned}
\dot{x}_1 &= x_1(a_1x_1^2 + a_2x_1x_2 + a_4x_2^2 + a_3x_1x_3 + a_5x_2x_3 + a_6x_3^2) = x_1A_1(\mathbf{x}) \\
\dot{x}_2 &= x_2(1 + b_1x_1^2 + b_2x_1x_2 + b_4x_2^2 + b_3x_1x_3 + b_5x_2x_3 + b_6x_3^2) = \\
&\quad x_2(1 + A_2(\mathbf{x})) \\
\dot{x}_3 &= -x_3(1 + c_1x_1^2 + c_2x_1x_2 + c_4x_2^2 + c_3x_1x_3 + c_5x_2x_3 + c_6x_3^2) = \\
&\quad x_3(1 + A_3(\mathbf{x}))
\end{aligned}
\tag{37}
$$

Theorem (V.R. & D.S. Shafer, 2016)

System (37) admits two analytic local first integrals of the form $\Psi_1(x_1, x_2, x_3) = x_1 + \cdots$ and $\Psi_2(x_1, x_2, x_3) = x_2x_3 + \cdots$ provided the parameter string (a, b, c) lies in the union of varieties $\mathbf{V}(I_1) \cup \mathbf{V}(I_2) \cup \mathbf{V}(I_3) \cup \mathbf{V}(I_4)$ of the ideals:

Ideal	Generators
I_1	$a_1,\ a_5,\ b_1,\ b_2,\ b_3,\ b_5,\ b_6,\ c_1,\ c_2,\ c_3,\ c_4,\ c_5,\ a_3^2 b_4 + a_2^2 c_6,\ \ a_3^2 a_4 + a_2^2 a_6 +$
I_2	$a_1,\ a_5,\ b_1,\ b_5,\ b_6,\ c_1,\ c_4,\ c_5,\ a_3 + b_3,\ a_2 - c_2,\ a_6 b_4 - a_4 c_6 + b_4 c_6,\ a_6 b_3$ $a_4 b_2 + a_4 c_2 - 2b_4 c_2,\ a_4 b_3 + b_3 b_4 + a_4 c_3 - b_4 c_3,\ a_6 b_2 + a_6 c_2 + b_2 c_6 -$ $b_2 b_3 + 3b_3 c_2 - b_2 c_3 + c_2 c_3,\ 2a_6 c_2^2 + 2a_4 c_3^2 + b_2 c_2 c_6 + 7c_2^2 c_6,\ 4b_3^2 b_4 +$ $4b_3 b_4 c_3 + b_2^2 c_6 + 2b_2 c_2 c_6 - 3c_2^2 c_6,\ 4b_4 c_3^2 + b_2^2 c_6 + 6b_2 c_2 c_6 + 9c_2^2 c_6$
I_3	$a_1,\ a_5,\ b_1 - c_1,\ b_5 - c_5,\ a_2 b_3 + a_3 c_2,\ a_3 b_2 + a_2 c_3,\ a_4 b_6 + a_6 c_4,\ a_6 b_4 +$ $b_4 b_6 - c_4 c_6,\ a_2^2 b_6 - a_3^2 c_4,\ a_3^2 a_4 + a_2^2 a_6,\ a_6 b_2^2 + a_4 c_3^2,\ a_3^2 b_4 - a_2^2 c_6,\ b_3^2 b_4$ $b_6 c_2^2 - b_3^2 c_4\ \ a_4 b_3^2 + a_6 c_2^2,\ b_2^2 b_6 - c_3^2 c_4,\ b_4 c_3^2 - b_2^2 c_6,\ a_2 b_2 b_6 + a_3 c_3 c_4,$ $a_2 b_6 c_2 + a_3 b_3 c_4,\ a_2 a_6 b_2 - a_3 a_4 c_3,\ a_3 a_4 b_3 - a_2 a_6 c_2,\ a_3 b_3 b_4 + a_2 c_2 c_6,\ a_3$ $a_6 b_2 c_2 + a_4 b_3 c_3,\ b_2 b_6 c_2 - b_3 c_3 c_4,\ b_3 b_4 c_3 - b_2 c_2 c_6$
I_4	$a_1,\ a_5,\ b_1,\ c_1,\ a_3 + b_3,\ a_2 - c_2,\ a_4 + b_4,\ a_6 - c_6,\ b_5 - c_5,\ b_4 b_6 - c_4 c_6,$ $2b_2 b_6 - 3b_3 c_5 - c_3 c_5 + 2b_2 c_6,\ b_6 c_4 - c_5^2 + b_4 c_6 + 2c_4 c_6,\ 2b_4 c_3 + 2c_3 c_4$ $2b_6 c_2 + b_3 c_5 - c_3 c_5 + 2c_2 c_6,\ b_2 b_3 + 3b_3 c_2 - b_2 c_3 + c_2 c_3,\ 2b_3 b_4 + 2b_3 c$ $b_4 c_5^2 - b_4^2 c_6 - 2b_4 c_4 c_6 - c_4^2 c_6,\ 2c_3 c_4 c_5 - b_2 c_5^2 - 3c_2 c_5^2 + b_2 b_4 c_6 + 3b_4 c$ $2b_3 c_4 c_5 - b_2 c_5^2 + c_2 c_5^2 + b_2 b_4 c_6 - b_4 c_2 c_6 + b_2 c_4 c_6 - c_2 c_4 c_6,\ 4c_3^2 c_4 - 2b$ $6c_2 c_3 c_5 + b_2^2 c_6 + 6b_2 c_2 c_6 + 9c_2^2 c_6,\ 4b_3 c_3 c_4 - 2b_2 c_3 c_5 + 2c_2 c_3 c_5 + b_2^2 c_6$ $4b_3^2 c_4 + 8b_3 c_2 c_5 - 2b_2 c_3 c_5 + 2c_2 c_3 c_5 + b_2^2 c_6 - 2b_2 c_2 c_6 + c_2^2 c_6$

Sketch of the proof. By Proposition 1 and its proof introduce:

$$A_j(\tfrac{y_1}{k_1}, \tfrac{y_3}{k_3}, \tfrac{y_2}{k_2}) = \widehat{A}_j(\mathbf{y}),$$

$f = 1 + F$ and $F = \frac{1}{2}[A_2 + A_3]$ is homogeneous of degree m,

$$u_j(\mathbf{x}) \stackrel{\text{def}}{=} x_j \frac{\partial f}{\partial x_j}(\mathbf{x}), \quad j \in \{1, 2, 3\} \quad u_j(\tfrac{y_1}{k_1}, \tfrac{y_3}{k_3}, \tfrac{y_2}{k_2}) = \widehat{u}_j(\mathbf{y}).$$

$$[\widehat{u}_1 \widehat{A}_1 + \widehat{u}_2(g + \widehat{A}_2) - \widehat{u}_3(g + \widehat{A}_3)] = S.$$

$$B_1 = \widehat{A}_1 - \tfrac{1}{m} S \qquad \text{and} \qquad h = -\tfrac{1}{2}(\widehat{A}_2 - \widehat{A}_3) + \tfrac{1}{m} S.$$

- Write down the system

$$B_1(y_1, y_3, y_2) = -B_1(y_1, y_2, y_3) \qquad h(y_1, y_3, y_2) = -h(y_1, y_2, y_3) \tag{38}$$

- Equal coefficients of similar terms in the above equations, to obtain the system

$$2a_1 = 2a_5 = b_1 - c_1 = b_5 - c_5 = a_1(b_1 + c_1) = c_4 k_2^2 - b_6 k_3^2 = b_5^2 + 2b_6 b_4$$
$$b_2 k_2 - 3c_2 k_2 + 3b_3 k_3 - c_3 k_3 = b_5 b_4 k_2^2 - c_5 c_4 k_2^2 + b_6 b_5 k_3^2 - c_6 c_5 k_3^2 = 4a$$
$$c_3 k_3 = 2a_4 k_2^2 - b_4 k_2^2 - c_4 k_2^2 + 2a_6 k_3^2 + b_6 k_3^2 + c_6 k_3^2 = a_2 b_3 + a_3 b_2 + b_3$$
$$-c_3 c_2 + 2a_5 c_1 - b_5 c_1 - c_5 c_1 = 2a_4 b_2 k_2^3 + 3b_4 b_2 k_2^3 + b_2 c_4 k_2^3 + 2a_4 c_2 k_2^3$$
$$3b_6 b_3 k_3^3 + b_3 c_6 k_3^3 + 2a_6 c_3 k_3^3 - b_6 c_3 k_3^3 - 3c_6 c_3 k_3^3 = 2a_1 b_2 k_2 + 4a_2 b_1 k_2 +$$
$$+4a_2 c_1 k_2 - b_2 c_1 k_2 - c_2 c_1 k_2 + 2a_1 b_3 k_3 + 4a_3 b_1 k_3 + b_3 b_1 k_3 + 2a_1 c_3 k_3 +$$
$$2a_2 b_2 k_2^2 + b_2^2 k_2^2 + 4a_4 b_1 k_2^2 + 2b_4 b_1 k_2^2 + 2b_1 c_4 k_2^2 + 2a_2 c_2 k_2^2 - c_2^2 k_2^2 + 4a$$
$$+2a_3 b_3 k_3^2 + b_3^2 k_3^2 + 4a_6 b_1 k_3^2 + 2b_6 b_1 k_3^2 + 2b_1 c_6 k_3^2 + 2a_3 c_3 k_3^2 - c_3^2 k_3^2 +$$
$$2a_4 b_3 k_2 + 3b_4 b_3 k_2 + 2a_5 b_2 k_2 + 3b_5 b_2 k_2 + b_2 c_5 k_2 + b_3 c_4 k_2 + 2a_4 c_3 k_2 -$$
$$-b_5 c_2 k_2 - 3c_5 c_2 k_2 + 2a_5 b_3 k_3 + 3b_5 b_3 k_3 + 2a_6 b_2 k_3 + 3b_6 b_2 k_3 + b_2 c_6 k_3$$
$$-3c_5 c_3 k_3 + 2a_6 c_2 k_3 - b_6 c_2 k_3 - 3c_6 c_2 k_3 = 0$$

- Let $\mathcal{S}$ be the set of parameters (a, b, c) and k_1, k_2, and k_3 for which system (37) can be transformed to a time-reversible system (35) by a transformation (33)
- Then, such (a, b, c) and k_1, k_2, k_3 should satisfy the above algebraic system, that is, $\mathcal{S}$ is the variety of the ideal I with generators listed above intersected with $K = \{(k_1, k_2, k_3) : k_1 k_2 k_3 \neq 0\}$.

The point (a, b, c, k_1, k_2, k_3) is in the set $\mathcal{S}$ if

$$1 - k_1 u = 0,\ 1 - k_2 v = 0,\ 1 - k_3 w = 0,\ f = 0\ \ \forall f \in I. \qquad (39)$$

Let
$J = \langle I, 1 - k_1 u, 1 - k_2 v, 1 - k_3 w \rangle \subset \mathbb{C}[a, b, c, k_1, k_2, k_3, u, v, w]$.
Then the set of solutions of (39) is the variety of the ideal J. The Zariski closure of the projection of the variety $\mathbf{V}(J)$ onto the space of parameters (a, b, c) is the variety of the six elimination ideal of J, the ideal $J^{(6)}$. By the Elimination Theorem to find $J^{(6)}$ one can compute a Gröbner basis of J with respect to the lex order with $\{k_1, k_2, k_3, u, v, w\} > \{a, b, c\}$ and take from the output polynomials that depend only on a, b, and c, obtaining a basis of $J^{(6)}$. The variety $V = \mathbf{V}(J^{(6)})$ is the Zariski closure of $\pi_6(\mathbf{V}(J))$. Although not all systems corresponding to points of V are time-reversible, all of them admit two analytic first integrals of the form $\Psi_1(x_1, x_2, x_3) = x_1 + \cdots$ and $\Psi_2(x_1, x_2, x_3) = x_2 x_3 + \cdots$, since the set of systems admitting two integrals Ψ_1 and Ψ_2 is an algebraic set [V.R., Y. Xia, X. Zhang, 2014].

- Elimination in Mathematica: Eliminate[I,{k1,k2,k3,u,v,w}]
- Elimination in Singular: eliminate(I,k1*k2*k3*u*v*w)

Singular output:

```
J6[1]=b5-c5
J6[2]=b1-c1
J6[3]=a5
J6[4]=a1
J6[5]=b4*b6-c4*c6
J6[6]=a4*b6+a6*c4
J6[7]=a3*b2-3*a2*b3+b2*b3-3*a3*c2+a2*c3-c2*c3
J6[8]=a6*b4*c5+a4*c5*c6
J6[9]=a6*b4*c4+a4*c4*c6
J6[10]=a2*b6*c2+b2*b6*c2-2*b6*c2^2+a3*b3*c4+2*b3^2*c4-b3*c
J6[11]=b2*b3*c1-c1*c2*c3
.........................................
J6[291]=162*a4^3*b3*b4^2*c3-144*a4^2*b3*b4^3*c3+
320*a4*b3*b4^4*c3+144*a4^3*b4^2*c3^2-136*a4^2*b4^3*c3^2-
64*a4*b4^4*c3^2-320*b3*b4^4*c3*c4+64*b4^4*c3^2*c4+
144*b3*b4^3*c3*c4^2+136*b4^3*c3^2*c4^2-162*b3*b4^2*c3*c4^3
144*b4^2*c3^2*c4^3+891*a2*a4^2*b3*b4^2*c5-...
```

- Use the `minAssGTZ` command of SINGULAR to obtain the decomposition of radical of $J^{(6)}$ as an intersection of prime ideals, which yields the four ideals of the statement of the theorem, so that $\mathbf{V}(J^{(6)}) = \mathbf{V}(I_1) \cup \mathbf{V}(I_2) \cup \mathbf{V}(I_3) \cup \mathbf{V}(I_4)$.

The work was supported by
the Slovenian Research Agency and
by FP7-PEOPLE-2012-IRSES-316338

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by FP7-PEOPLE-2012-IRSES-316338

Thank you for your attention!