Fourth Symposium on Planar Vector Fields

September 5-9th, 2016

The Teixeira singularity degeneracy and its bifurcation in PWL systems

Enrique Ponce

Joint work with Rony Cristiano & Daniel Pagano (UFSC, Florianópolis, Brasil) and Emilio Freire (Univ. de Sevilla),

Introduction

- We consider 3D piecewise linear Filippov differential systems with a separation plane, having a two-fold point with invisible tangencies, that is, the so-called Teixeira singularity (TS-point, for short).
- For some parameter values this singularity undergoes a compound bifurcation: there appears a sliding bifurcation involving a pseudo-equilibrium point and, simultaneously, a bifurcation associated to the birth of a crossing limit cycle.
- After determining a generic canonical form, we show how to characterize such a compound bifurcation.
- Our motivation comes from the natural appearance of TS-points in the control of Boost converters.

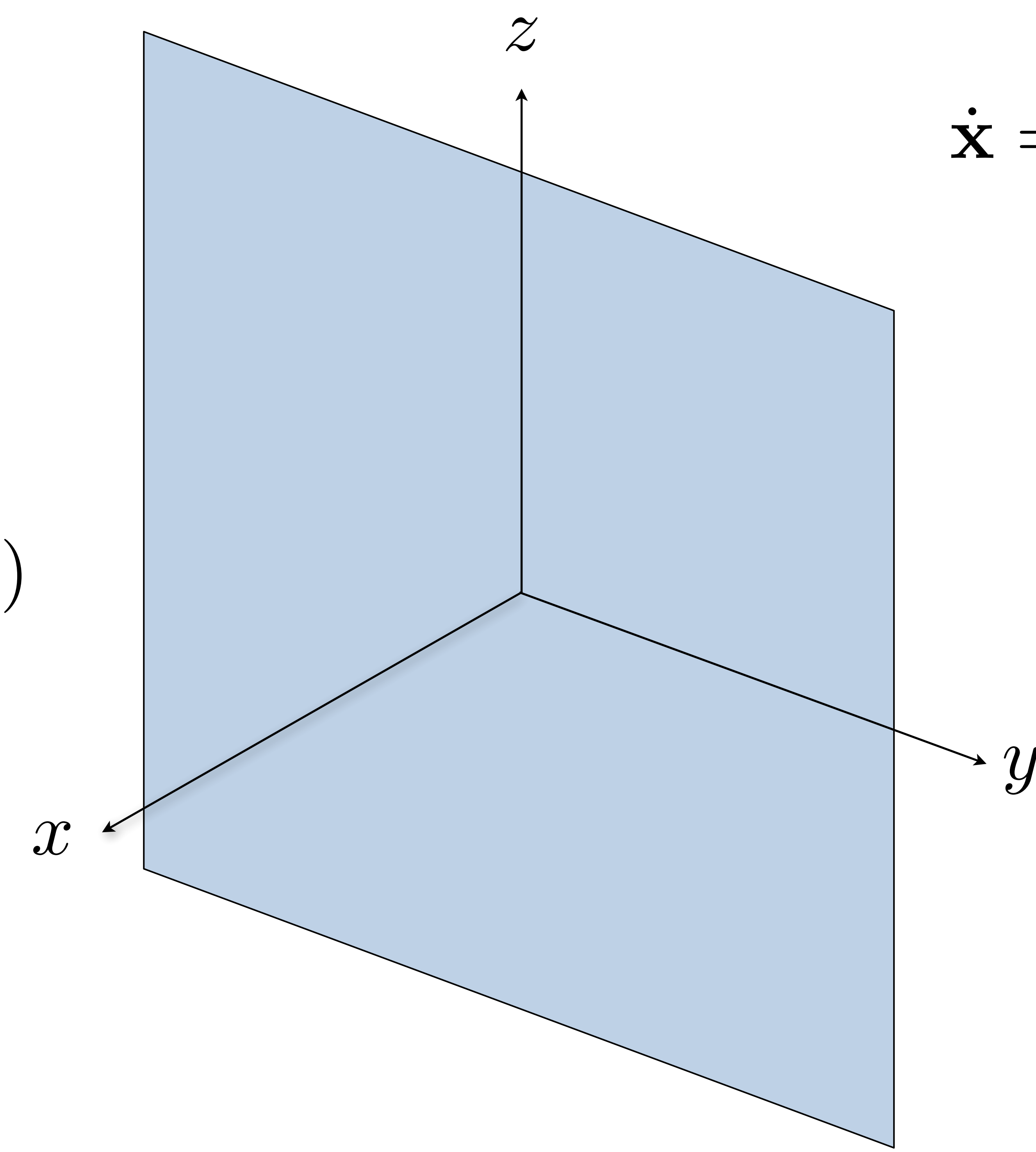
Summary

- A non-trivial yet manageable single-parameter example
- The general DPWL case:
  - Hypotheses for having a TS-point and canonical form
  - The sliding dynamics bifurcation
  - The crossing dynamics bifurcation
- A multi-parametric example
- TS-point in a DC-DC Boost converter
- Conclusions

A non-trivial yet manageable example

$$\dot{\mathbf{x}} = \mathbf{F}^-(\mathbf{x})$$

$$\dot{\mathbf{x}} = \mathbf{F}^+(\mathbf{x})$$



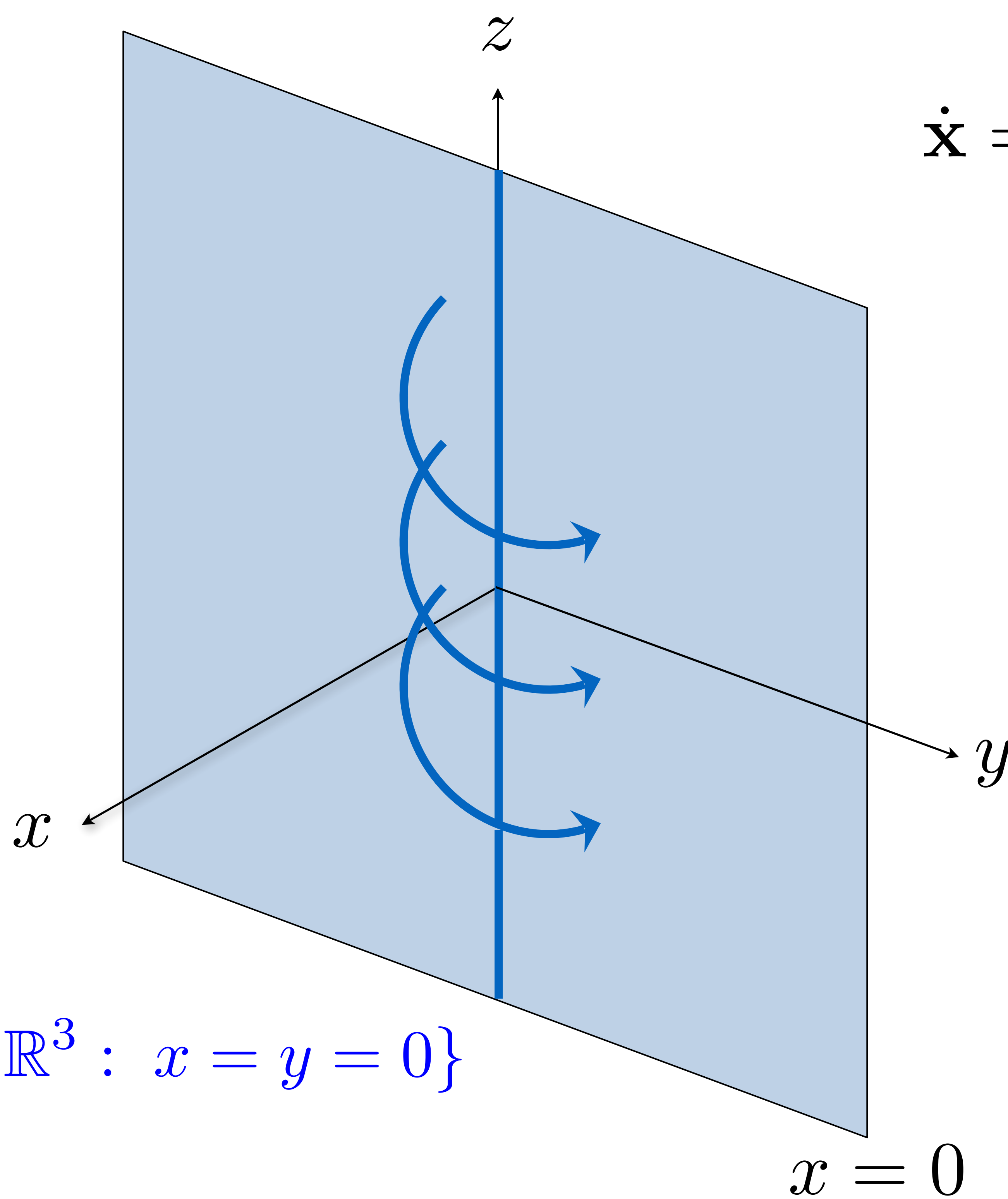
$$\Sigma = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 : x = 0\}$$

A non-trivial yet manageable example

$$\ddot{x} = -\dot{y} = -1$$

$$\dot{\mathbf{x}} = \mathbf{F}^{-}(\mathbf{x})$$

$$\begin{cases} \dot{x} & = & -y \\ \dot{y} & = & 1 \\ \dot{z} & = & -2 \end{cases}$$

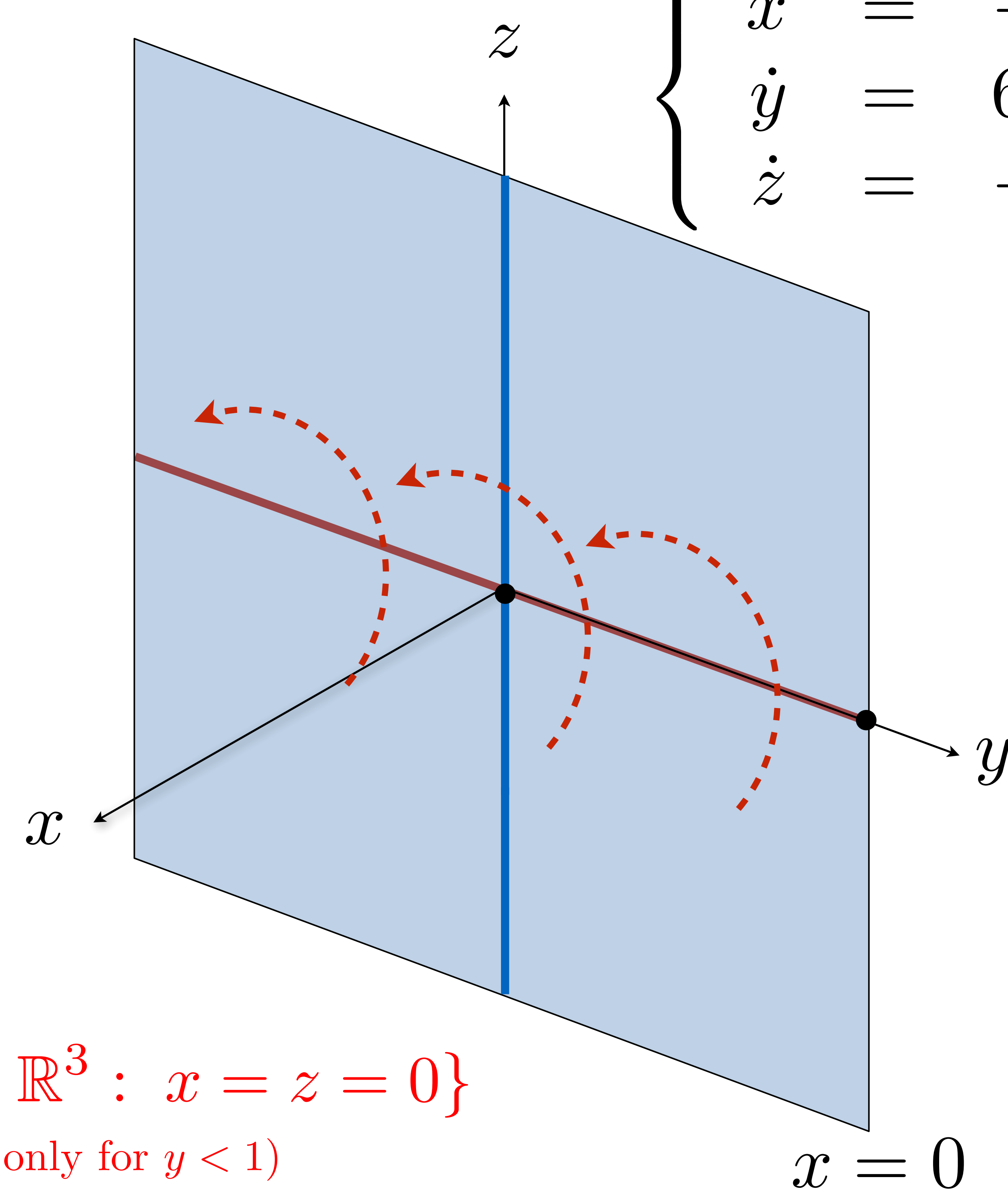


$$T_{+} = \{(x, y, z) \in \mathbb{R}^3 : \ x = y = 0\}$$

A non-trivial yet manageable example

$$\begin{cases} \dot{x} &= -6ax + z \\ \dot{y} &= 6a^3x - a \\ \dot{z} &= -11a^2x - y + 1 \end{cases}$$

$$\begin{cases} \dot{x} &= -y \\ \dot{y} &= 1 \\ \dot{z} &= -2 \end{cases}$$



$$\ddot{x}|_{T_-} = 1 - y$$

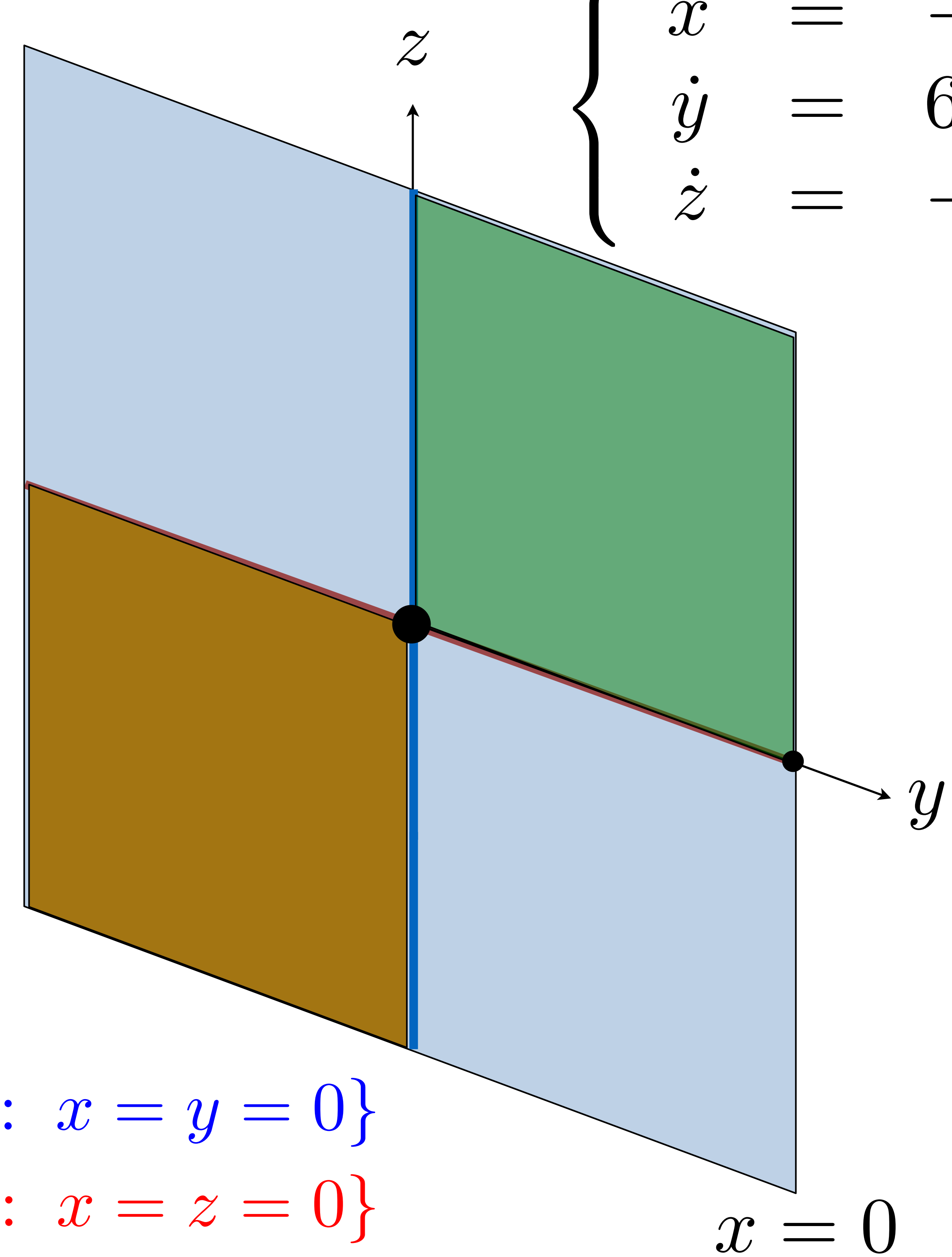
$$T_- = \{(x, y, z) \in \mathbb{R}^3 : \; x = z = 0\}$$

(invisible tangency only for $y < 1$)

A non-trivial yet manageable example

$$\begin{cases} \dot{x} &= -6ax + z \\ \dot{y} &= 6a^3 x - a \\ \dot{z} &= -11a^2 x - y + 1 \end{cases}$$

$$\begin{cases} \dot{x} &= -y \\ \dot{y} &= 1 \\ \dot{z} &= -2 \end{cases}$$



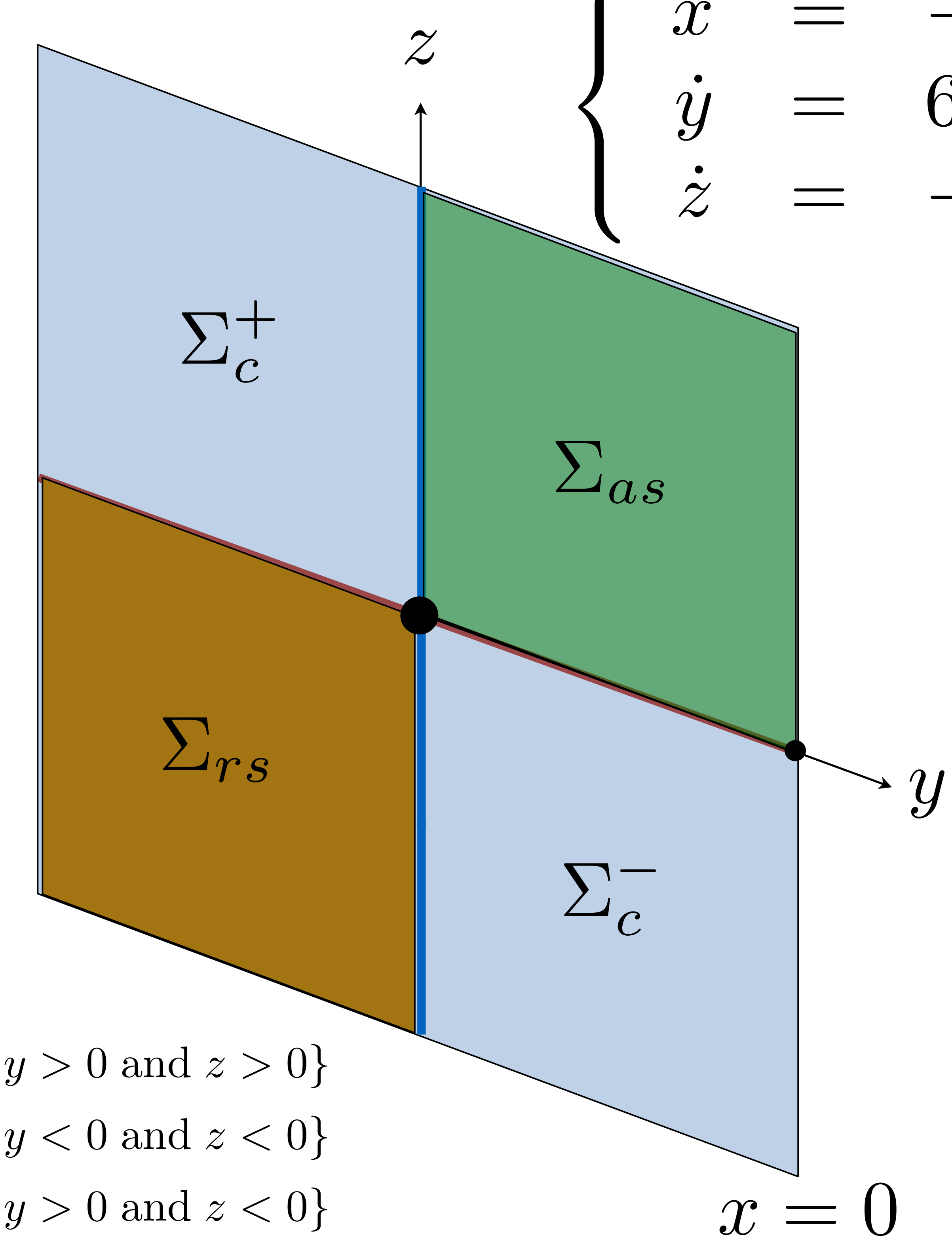
$T_+ = \{(x, y, z) \in \mathbb{R}^3 : \ x = y = 0\}$

$T_- = \{(x, y, z) \in \mathbb{R}^3 : \ x = z = 0\}$

A non-trivial yet manageable example

$$\begin{cases} \dot{x} &= -6ax + z \\ \dot{y} &= 6a^3x - a \\ \dot{z} &= -11a^2x - y + 1 \end{cases}$$

$$\begin{cases} \dot{x} &= -y \\ \dot{y} &= 1 \\ \dot{z} &= -2 \end{cases}$$



$\Sigma_{as} = \{(x,y,z) \in \mathbb{R}^3 : \ x = 0, \ y > 0 \text{ and } z > 0\}$

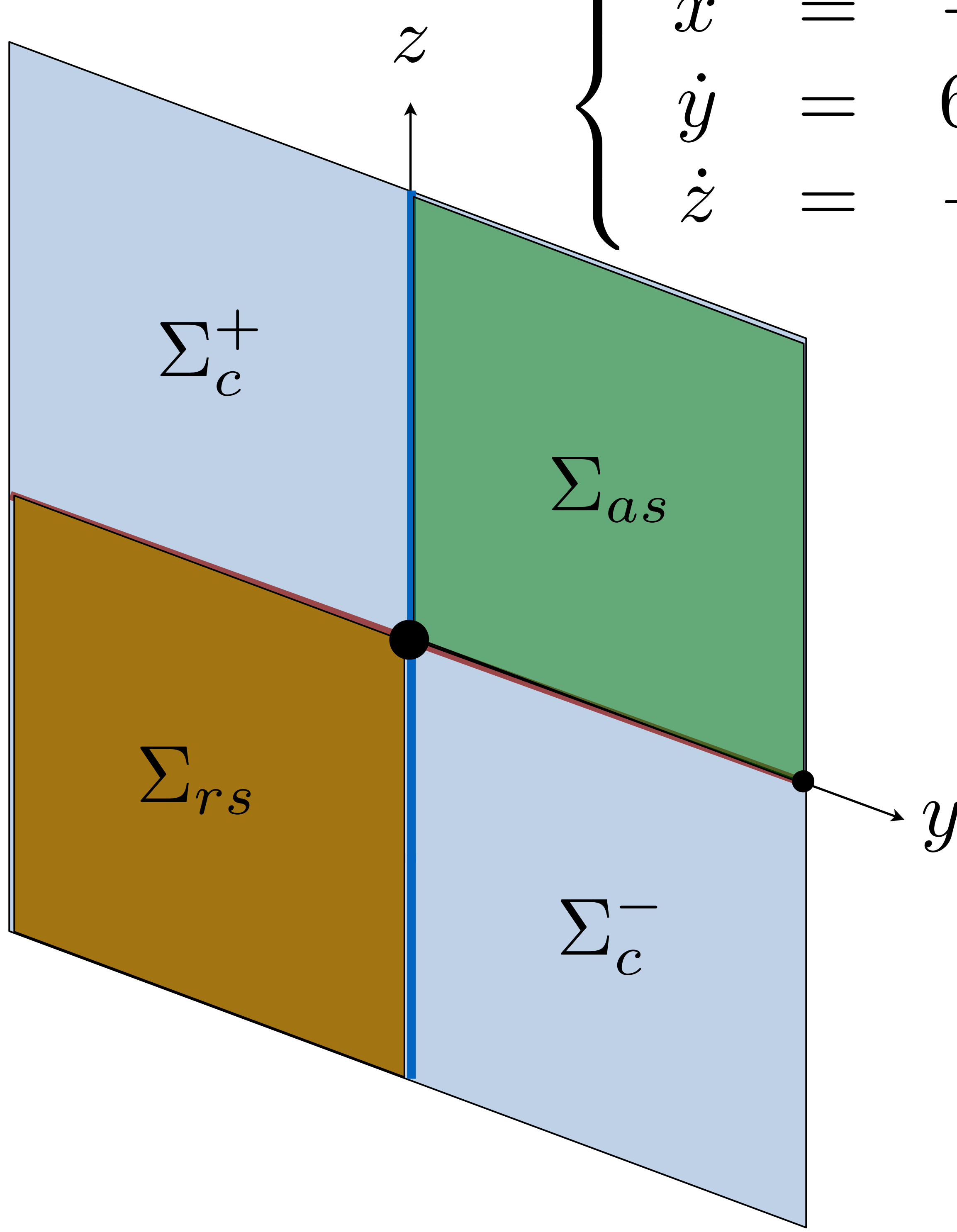
$\Sigma_{rs} = \{(x,y,z) \in \mathbb{R}^3 : \ x = 0, \ y < 0 \text{ and } z < 0\}$

$\Sigma_c^- = \{(x,y,z) \in \mathbb{R}^3 : \ x = 0, \ y > 0 \text{ and } z < 0\}$

$\Sigma_c^+ = \{(x,y,z) \in \mathbb{R}^3 : \ x = 0, \ y < 0 \text{ and } z > 0\}$

The sliding dynamics

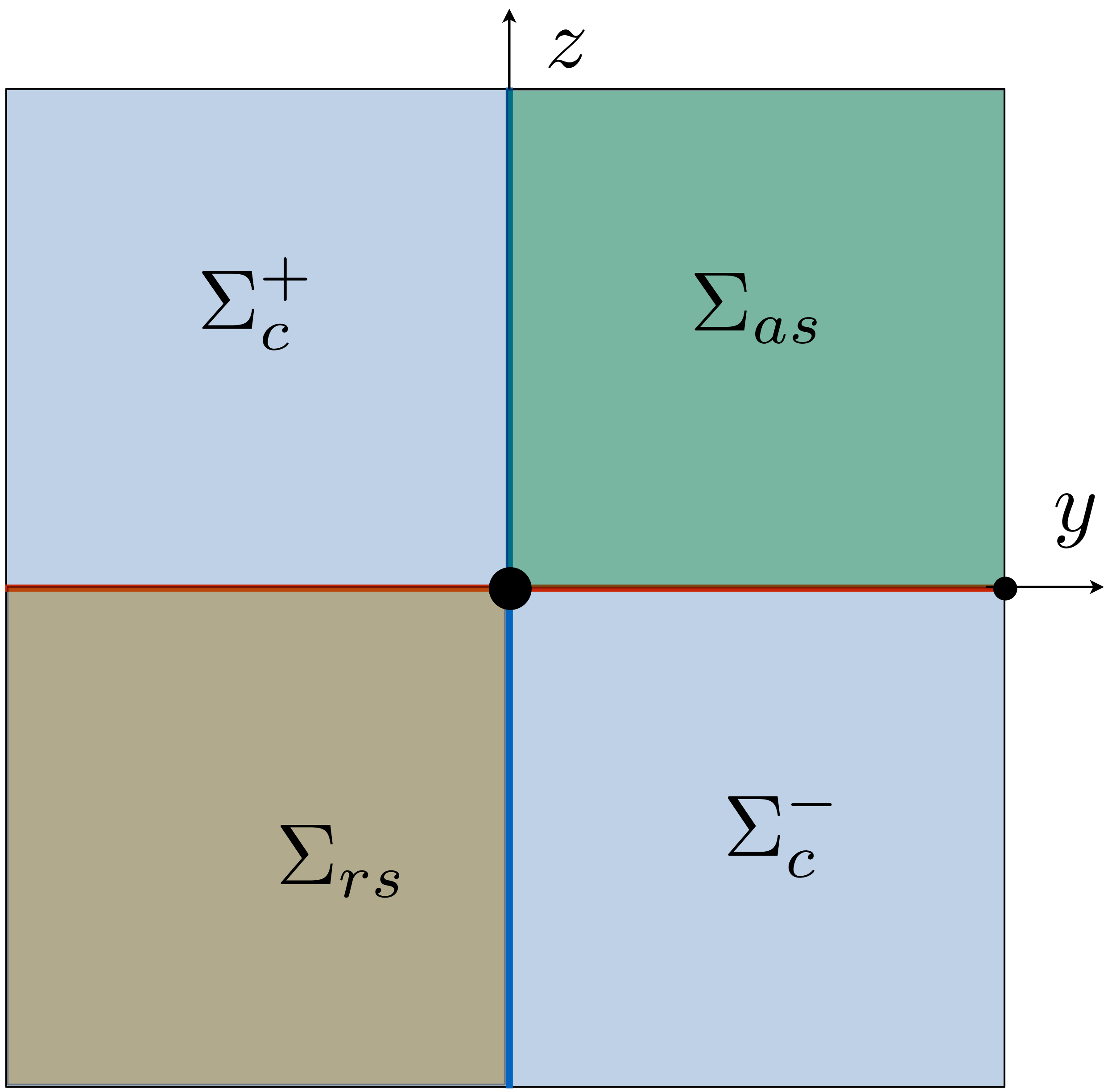
$$\begin{cases} \dot{x} &= -6ax + z \\ \dot{y} &= 6a^3x - a \\ \dot{z} &= -11a^2x - y + 1 \end{cases}$$



$$\begin{cases} \dot{x} &= -y \\ \dot{y} &= 1 \\ \dot{z} &= -2 \end{cases}$$

$$\mathbf{F}^s(\mathbf{x}) = \frac{\langle \mathbf{e}_1, \mathbf{F}^+(\mathbf{x})\rangle \mathbf{F}^-(\mathbf{x}) - \langle \mathbf{e}_1, \mathbf{F}^-(\mathbf{x})\rangle \mathbf{F}^+(\mathbf{x})}{\langle \mathbf{e}_1, \mathbf{F}^+(\mathbf{x}) - \mathbf{F}^-(\mathbf{x})\rangle} = \frac{1}{y+z}\begin{bmatrix} 0 \\ z - ay \\ -2z + y(1-y) \end{bmatrix}$$

The sliding dynamics

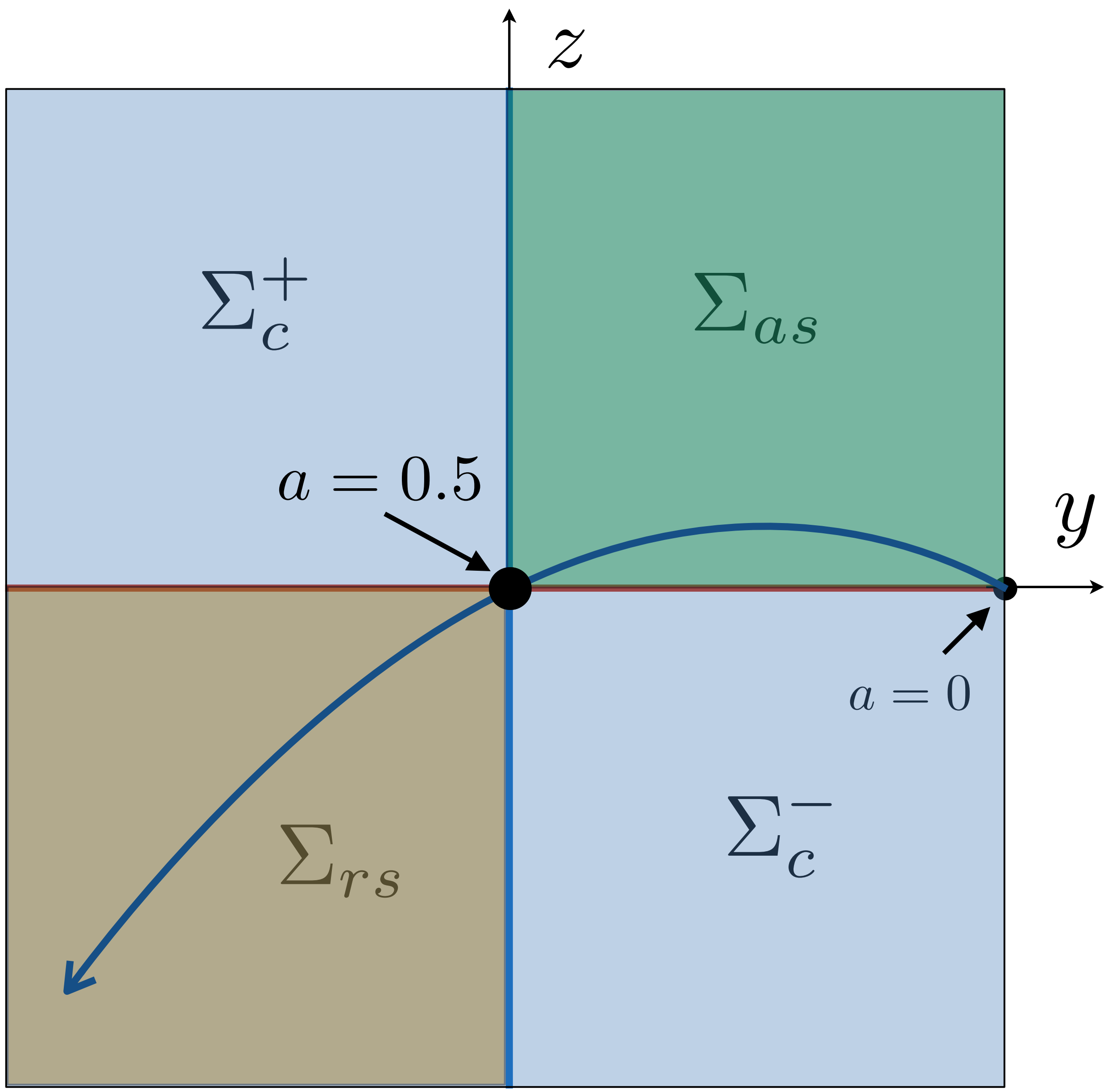


$$\mathbf{F}^s(\mathbf{x}) = \frac{1}{y+z}\left[\begin{array}{c} 0 \\ z - ay \\ -2z + y(1-y) \end{array}\right]$$

Pseudo-equilibrium point at

$$\mathbf{x}(a) = (0, y(a), z(a)) = (0, 1-2a, a(1-2a))$$

The sliding dynamics



$$\mathbf{F}^s(\mathbf{x}) = \frac{1}{y+z}\begin{bmatrix} 0 \\ z - ay \\ -2z + y(1-y) \end{bmatrix}$$

Pseudo-equilibrium point at

$$\mathbf{x}(a) = (0, y(a), z(a)) = (0, 1-2a, a(1-2a))$$

The sliding dynamics

Desingularized sliding v.f.

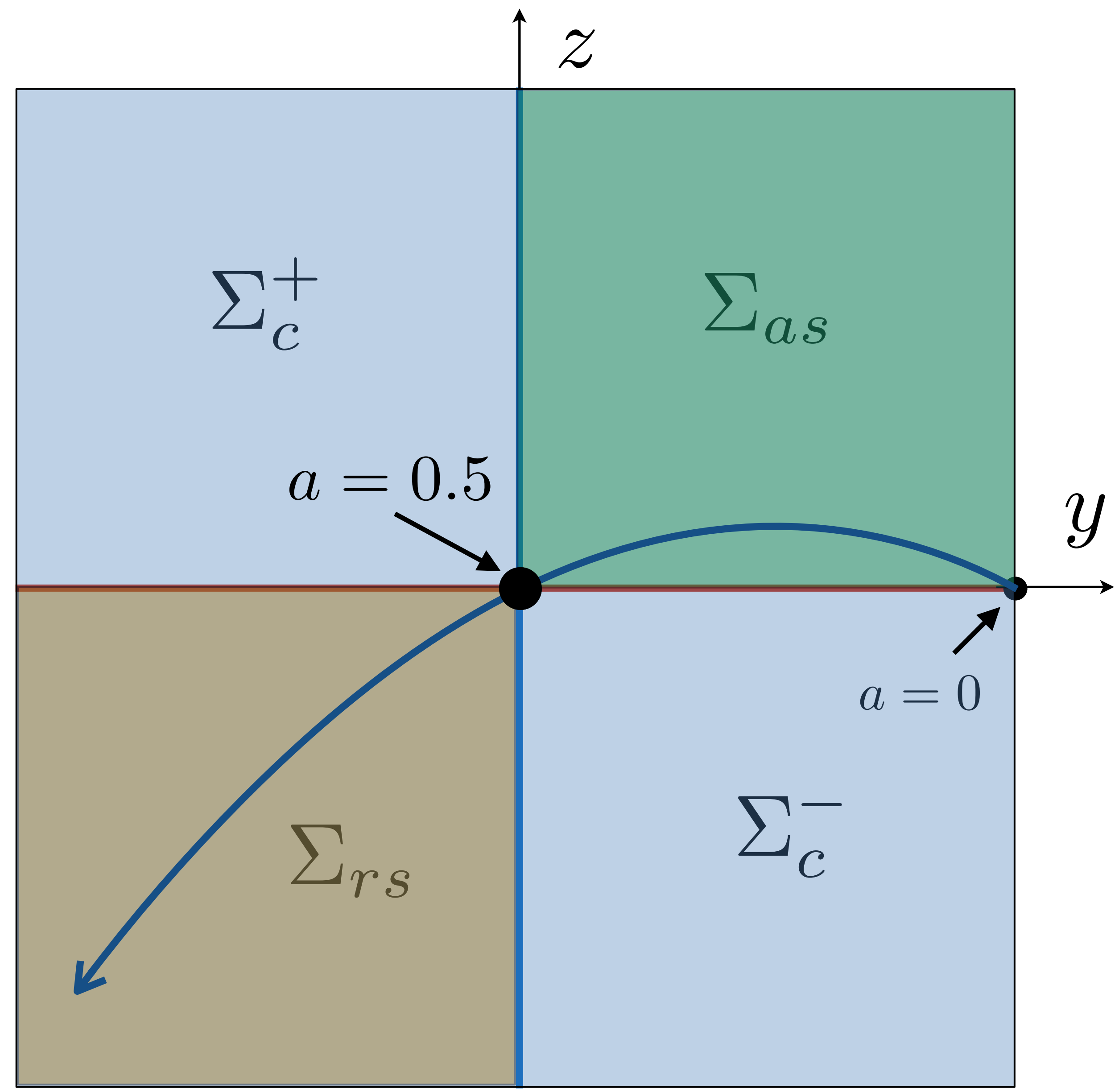
$$\mathbf{F}_d^s(\mathbf{x}) = \pm \begin{bmatrix} 0 \\ z - ay \\ -2z + y(1-y) \end{bmatrix}$$

$\pm(y+z) > 0$

$$\mathbf{F}^s(\mathbf{x}) = \frac{1}{y+z} \begin{bmatrix} 0 \\ z - ay \\ -2z + y(1-y) \end{bmatrix}$$

Pseudo-equilibrium point at

$$\mathbf{x}(a) = (0, y(a), z(a)) = (0, 1-2a, a(1-2a))$$



The sliding dynamics

Desingularized sliding v.f.

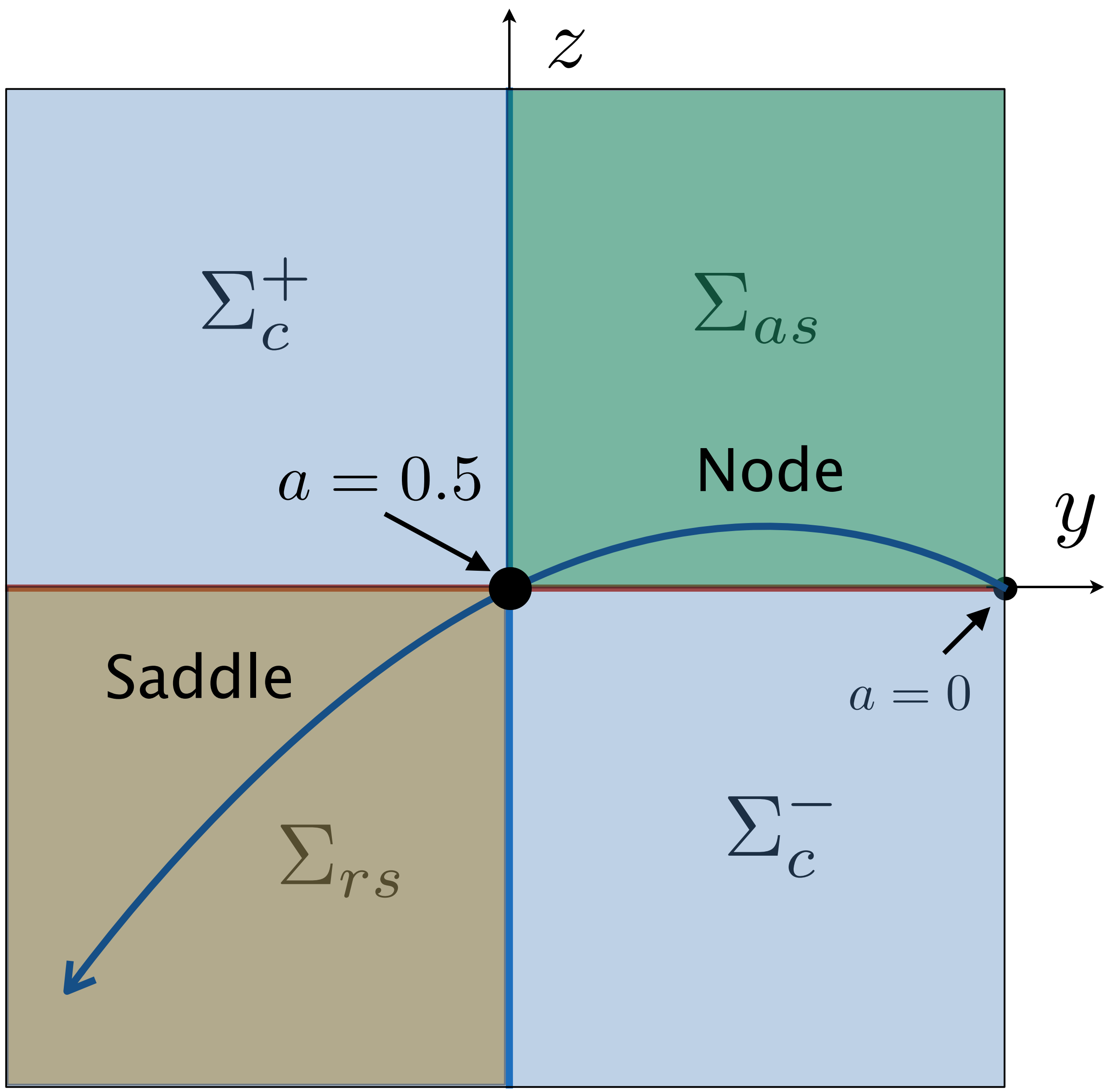
$$\mathbf{F}_d^s(\mathbf{x}) = \pm \begin{bmatrix} 0 \\ z - ay \\ -2z + y(1-y) \end{bmatrix}$$

$\pm(y+z) > 0$

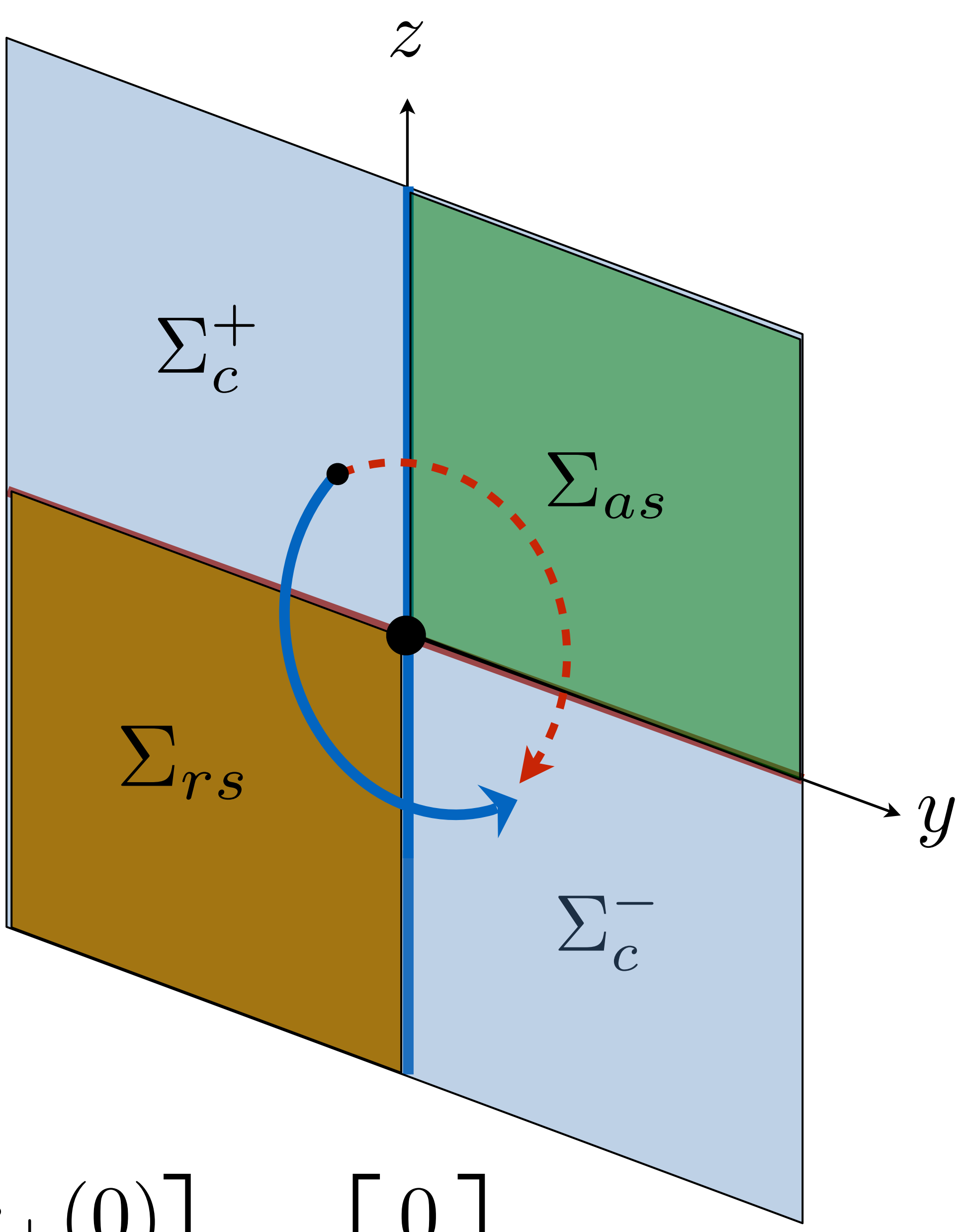
$$\mathbf{F}^s(\mathbf{x}) = \frac{1}{y+z} \begin{bmatrix} 0 \\ z - ay \\ -2z + y(1-y) \end{bmatrix}$$

Pseudo-equilibrium point at

$$\mathbf{x}(a) = (0, y(a), z(a)) = (0, 1-2a, a(1-2a))$$



The crossing dynamics bifurcation (periodic orbits?)



$$P_+ \begin{pmatrix} y \\ z \end{pmatrix} = P_-^{-1} \begin{pmatrix} y \\ z \end{pmatrix}$$

$$\begin{cases} \dot{x} & = & -y \\ \dot{y} & = & 1 \\ \dot{z} & = & -2 \end{cases}$$

The $(+)$ vector field gives

$$\begin{bmatrix} x_+(t) \\ y_+(t) \\ z_+(t) \end{bmatrix} = \begin{bmatrix} -t(t+2y_0)/2 \\ t + y_0 \\ -2t + z_0 \end{bmatrix} \text{ with } \begin{bmatrix} x_+(0) \\ y_+(0) \\ z_+(0) \end{bmatrix} = \begin{bmatrix} 0 \\ y_0 \\ z_0 \end{bmatrix}$$

$$\text{so that } P_+ \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} -y_0 \\ -4y_0 + z_0 \end{pmatrix}$$

The crossing dynamics

The integration of the (-) vector field is much more involved but the choice of stepped eigenvalues allows to express solutions in an algebraic way.

$$A^- = \begin{pmatrix} -6a & 0 & 1 \\ 6a^3 & 0 & 0 \\ -11a^2 & -1 & 0 \end{pmatrix} \qquad \mathrm{Spec}(A^-) = \{-a, -2a, -3a\}$$

$$e^{-A^- t} = \frac{e^{at}}{2} \begin{pmatrix} 1 - 8e^{at} + 9e^{2at} & -\frac{\left(-1+e^{at}\right)^2}{a^2} & -\frac{\left(-1+e^{at}\right)\left(-1+3e^{at}\right)}{a} \\ -6a^2\left(-1+e^{at}\right)\left(-1+3e^{at}\right) & 2\left(3-3e^{at}+e^{2at}\right) & 6a\left(-1+e^{at}\right)^2 \\ a\left(-1+e^{at}\right)\left(-5+27e^{at}\right) & -\frac{\left(-1+e^{at}\right)\left(-5+3e^{at}\right)}{a} & -5+16e^{at}-9e^{2at} \end{pmatrix} =$$

$$= \frac{1}{2u^3} \begin{pmatrix} u^2 - 8u + 9 & -\frac{(u-1)^2}{a^2} & -\frac{(u-3)(u-1)}{a} \\ -6a^2(u-3)(u-1) & 2\left(3u^2-3u+1\right) & 6a(u-1)^2 \\ a(u-1)(5u-27) & -\frac{(u-1)(5u-3)}{a} & -5u^2+16u-9 \end{pmatrix}$$

$$0 < u = e^{-at} < 1, \quad a > 0$$

Looking for $P_-^{-1}(y,z)$ we write

$$\begin{bmatrix} x_-^{-1}(t) \\ y_-^{-1}(t) \\ z_-^{-1}(t) \end{bmatrix} = \frac{1}{6a^2}\begin{bmatrix} 1 \\ -5a^2 \\ 6a \end{bmatrix} + e^{-A^- t}\left(\begin{bmatrix} 0 \\ y_0 \\ z_0 \end{bmatrix} - \frac{1}{6a^2}\begin{bmatrix} 1 \\ -5a^2 \\ 6a \end{bmatrix}\right)$$

The condition $x_-^{-1}(t) = 0$ gives

$$0 < u = e^{-at} < 1, \quad a > 0$$

$$a = \frac{(-1+u)(2+u-3y_0)}{3(-3+u)z_0}, \quad \text{and then}$$

$$y_-^{-1}(u) = \frac{-1-9u+9u^2+u^3+6(1-3u)y_0}{6(-3+u)u^2},$$

$$z_-^{-1}(u) = \frac{(-1-6u+u^2+6y_0)z_0}{2u^2(2+u-3y_0)}$$

Finally, imposing

$$P_+ \binom{y_0}{z_0} = \begin{bmatrix} -2y_0 \\ -4y_0 + z_0 \end{bmatrix} = P_-^{-1} \binom{y_0}{z_0} = \begin{bmatrix} y_-^{-1}(u) \\ z_-^{-1}(u) \end{bmatrix}$$

we get

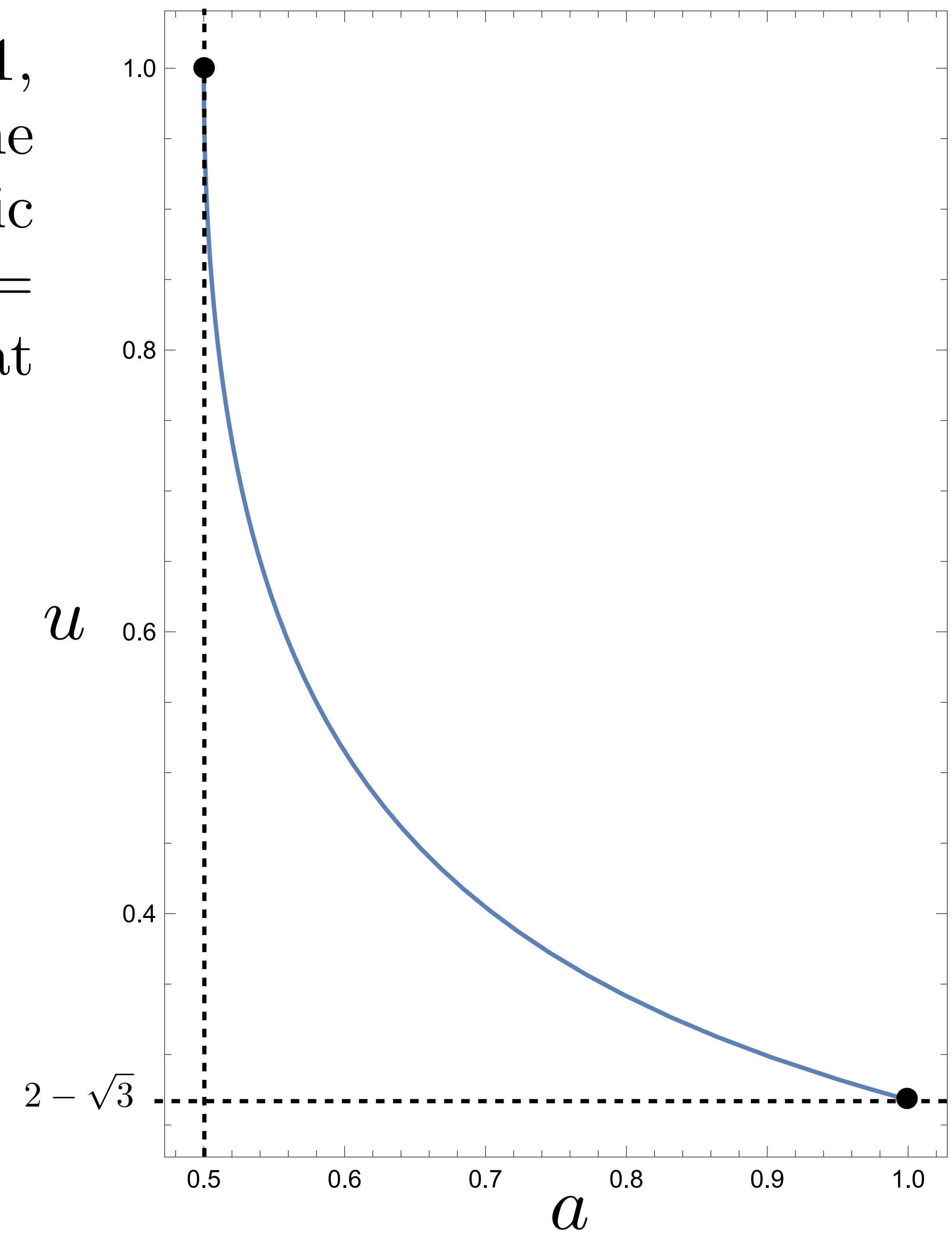
$$a = \frac{1 + 2u + 6u^2 + 2u^3 + u^4}{2u(1 + 10u + u^2)}, \quad \text{and then}$$

$$y_0 = -\frac{(-1 + u)(1 + 10u + u^2)}{6(1 + u)(1 - 4u + u^2)},$$

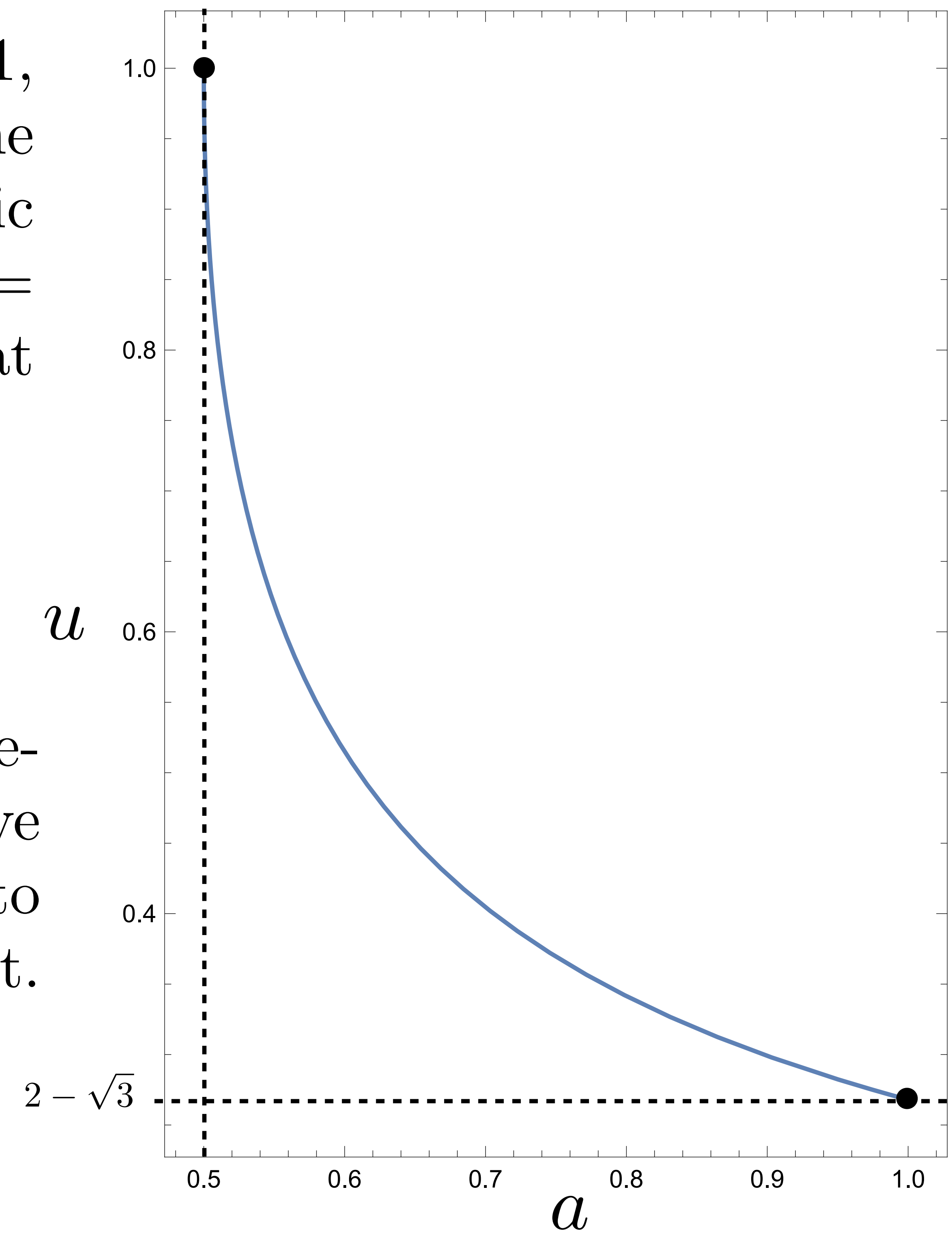
$$z_0 = \frac{(-1 + u)(-1 + 6u + 5u^2 + 2u^3)}{6a(1 + u)(1 - 4u + u^2)}$$

Note that $1 - 4u + u^2 = (u - 2)^2 - 3$, so that the above expressions are only valid for $2 - \sqrt{3} < u < 1$. Also, $\lim_{u \to 1^-} a = 1/2$, $\lim_{u \to (2-\sqrt{3})^+} a = \infty$.

A periodic orbit exists for $0.5 < a < 1$, corresponding to a decreasing of u in the interval $2-\sqrt{3} < u < 1$. The periodic orbit is born from the TS-point at $a = 0.5$ and disappears in a bifurcation at infinity for $a = 1$.

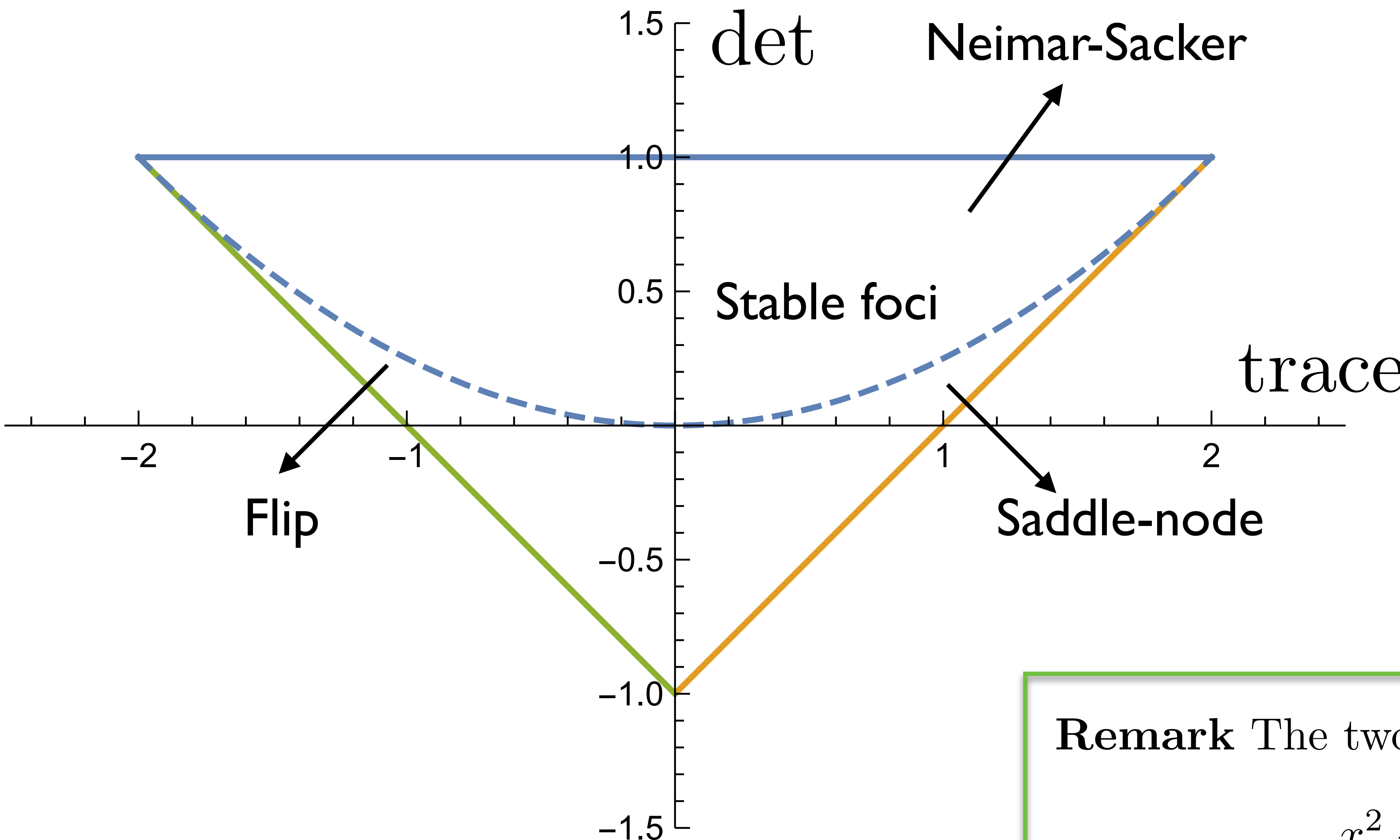


A periodic orbit exists for $0.5 < a < 1$, corresponding to a decreasing of u in the interval $2-\sqrt{3} < u < 1$. The periodic orbit is born from the TS-point at $a = 0.5$ and disappears in a bifurcation at infinity for $a = 1$.



We can compute in terms of u, the determinant and trace of the derivative $DP(y_0, z_0) = DP_-^{-1} \circ DP_+(y_0, z_0)$ to check the stability of the periodic orbit.

The stability triangle in the (trace,det)-plane



Remark The two roots of the quadratic equation

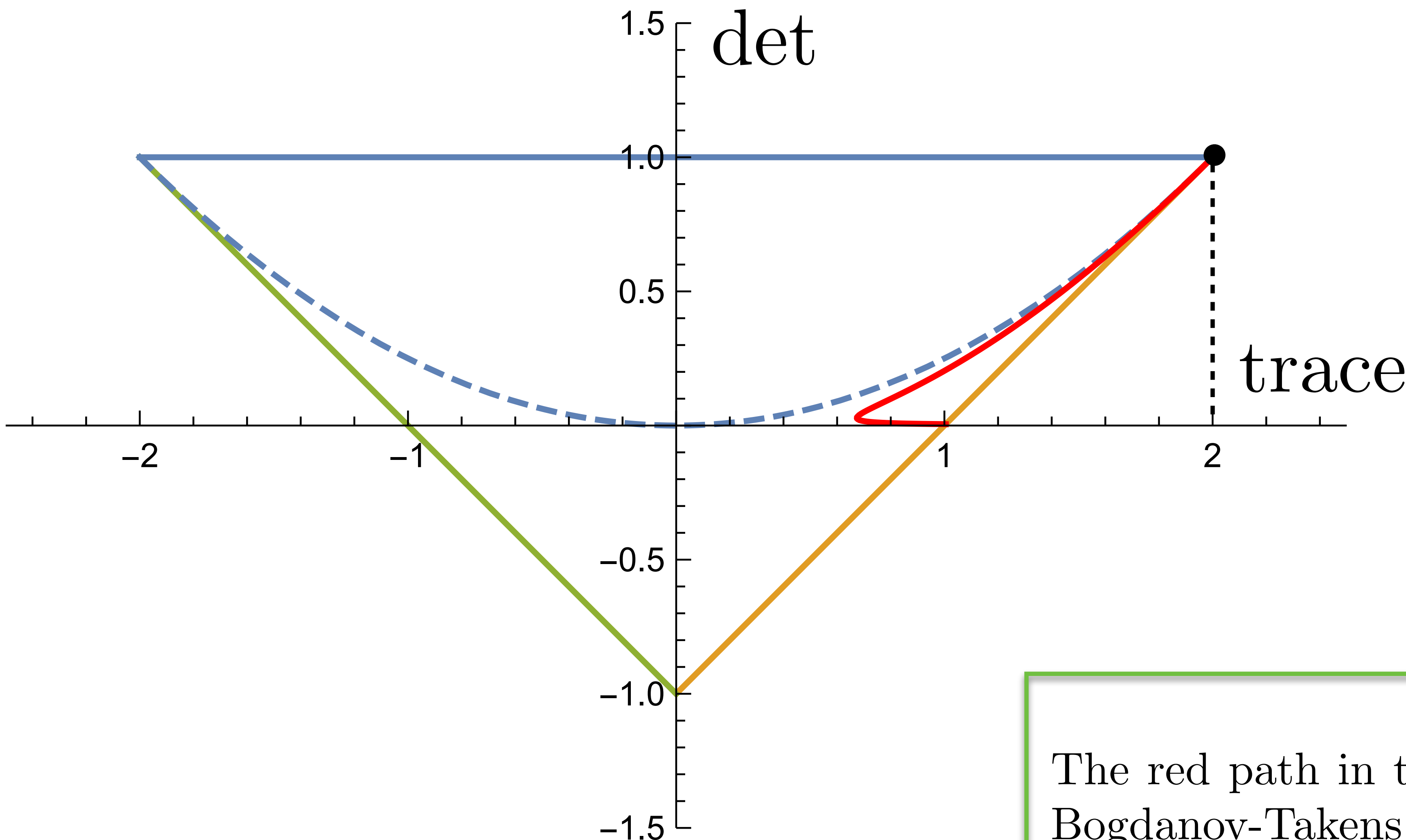
$$x^2 - \text{trace} \cdot x + \det = 0$$

are in the interior of the unit disk of the complex plane if and only if the two inequalities

$$|\det| < 1$$
$$|\text{trace}| < \det + 1$$

hold.

The followed path for 0.5< a <1 in the (trace,det)-plane



The red path in the (trace,det)-plane starts at the Bogdanov-Takens point (2,1). Note that the final point in the path is over the horizontal axis, namely

$$(98-56\sqrt{3}, 97-56\sqrt{3}).$$

For this example, the crossing periodic orbit which is born from the TS-point is always of **stable node** type.

Such a global control of the compound bifurcation for the Teixeira singularity is rather unusual: in what follows, we show how it is possible to characterize locally this bifurcation for discontinuous linear systems.

References

(*) A. F. Filippov, Differential Equations with Discontinuous Righthand Sides, Kluwer Academic Publishers, Dordrecht, 1988.

(*) M. A. Teixeira, Stability conditions for discontinuous vector fields, Journal of Differential Equations 88 (1990) 15–29.

(*) M. A. Teixeira, Generic bifurcation of sliding vector fields, J. Math. Anal. Appl. 176 (1993) 436–457.

(*) M. R. Jeffrey, A. Colombo, The two-fold singularity of discontinuous vector fields, SIAM J. Applied Dynamical Systems 8 (2) (2009) 624–640.

(*) A. Colombo, M. di Bernardo, E. Fossas, M. R. Jeffrey, Teixeira singularities in 3d switched feedback control systems, Systems & Control Letters 59 (2010) 615–622.

(*) A. Colombo, M. R. Jeffrey, Nondeterministic chaos, and the two-fold singularity in piece- wise smooth flows, SIAM J. Applied Dynamical Systems 10 (2) (2011) 423–451.

(*) R. Cristiano, D. J. Pagano, L. Benadero, and E. Ponce, Bifurcation analysis of a DC-DC bidirectional power converter operating with constant power load, International Journal of Bifurcation and Chaos, 26 (2016), p. 1630010

(*) E, Freire, R. Cristiano, D. Pagano and E. Ponce, Bifurcation analysis of the Texeira singularity in three-dimensional piecewise linear dynamical systems, preprint 2016.

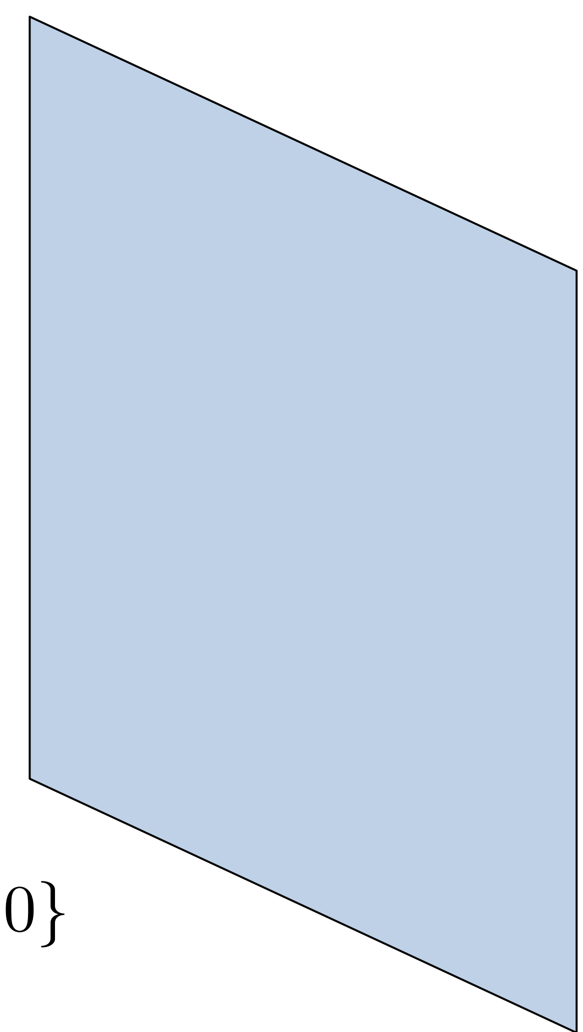
The general DPWL case

There exist three vectors $\mathbf{w}^-$, $\mathbf{w}^+$, $\mathbf{v} \in \mathbb{R}^3$, with $\mathbf{v} \neq \mathbf{0}$, two 3×3 square matrices A^- and A^+ and a scalar $\delta \in \mathbb{R}$, such that

$$\text{(a)} \quad \mathbf{F}(\mathbf{x}) = \begin{cases} \mathbf{F}^-(\mathbf{x}) = A^-\mathbf{x} + \mathbf{w}^-, & \text{if } \mathbf{v}^T\mathbf{x} + \delta < 0 \\ \mathbf{F}^+(\mathbf{x}) = A^+\mathbf{x} + \mathbf{w}^+, & \text{if } \mathbf{v}^T\mathbf{x} + \delta > 0 \end{cases}$$

(b) $A^-\mathbf{x} + \mathbf{w}^- \neq A^+\mathbf{x} + \mathbf{w}^+$ generically, when $\mathbf{v}^T\mathbf{x} + \delta = 0$.

$$\dot{\mathbf{x}} = \mathbf{F}^+(\mathbf{x})$$



$$\dot{\mathbf{x}} = \mathbf{F}^-(\mathbf{x})$$

The switching manifold is the plane

$$\Sigma = \{\mathbf{x} \in \mathbb{R}^3 : \ \mathbf{v}^T\mathbf{x} + \delta = 0\}$$

Hypotheses for having a TS-point

(H1) The tangency lines

$$T^- = \{\mathbf{x} \in \Sigma : \langle \mathbf{v}, \mathbf{F}^-(\mathbf{x}) \rangle = 0\}$$
$$T^+ = \{\mathbf{x} \in \Sigma : \langle \mathbf{v}, \mathbf{F}^+(\mathbf{x}) \rangle = 0\}$$

intersect transversally at a point $\hat{\mathbf{x}}$.

(H2) At the two-fold point $\widehat{\mathbf{x}}$ both tangencies are invisible: $\mathbf{v}^T A^- \mathbf{F}^-(\widehat{\mathbf{x}}) > 0$ and $\mathbf{v}^T A^+ \mathbf{F}^+(\widehat{\mathbf{x}}) < 0$.

The canonical form for DPWLS with a TS-point

Proposition. Under hypotheses **(H1)** and **(H2)**, it is possible through an invertible linear change of coordinates and a rescaling of time, to rewrite the system in the canonical form

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{F}^-(\mathbf{x}), & \text{if } x < 0, \\ \mathbf{F}^+(\mathbf{x}), & \text{if } x > 0, \end{cases} \tag{1}$$

with $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, where the linear vector fields $\mathbf{F}^{\pm} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ are

$$\mathbf{F}^+(\mathbf{x}) = \begin{bmatrix} f_1^+ & -1 & 0 \\ g_1^+ & g_2^+ & g_3^+ \\ h_1^+ & h_2^+ & h_3^+ \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ v_+ \end{bmatrix}, \quad \mathbf{F}^-(\mathbf{x}) = \begin{bmatrix} f_1^- & 0 & 1 \\ g_1^- & g_2^- & g_3^- \\ h_1^- & h_2^- & h_3^- \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ v_- \\ 1 \end{bmatrix},$$

for some constants $f_1^{\pm}$, $g_1^{\pm}$, $g_2^{\pm}$, $g_3^{\pm}$, $h_1^{\pm}$, $h_2^{\pm}$, $h_3^{\pm}$ and $v_{\pm}$.

The canonical form for DPWLS with a TS-point

Proposition. Under hypotheses **(H1)** and **(H2)**, it is possible through an invertible linear change of coordinates and a rescaling of time, to rewrite the system in the canonical form

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{F}^-(\mathbf{x}), & \text{if } x < 0, \\ \mathbf{F}^+(\mathbf{x}), & \text{if } x > 0, \end{cases} \qquad (1)$$

with $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, where the linear vector fields $\mathbf{F}^\pm : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ are

$$\mathbf{F}^+(\mathbf{x}) = \begin{bmatrix} f_1^+ & -1 & 0 \\ g_1^+ & g_2^+ & g_3^+ \\ h_1^+ & h_2^+ & h_3^+ \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ v_+ \end{bmatrix}, \quad \mathbf{F}^-(\mathbf{x}) = \begin{bmatrix} f_1^- & 0 & 1 \\ g_1^- & g_2^- & g_3^- \\ h_1^- & h_2^- & h_3^- \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ v_- \\ 1 \end{bmatrix},$$

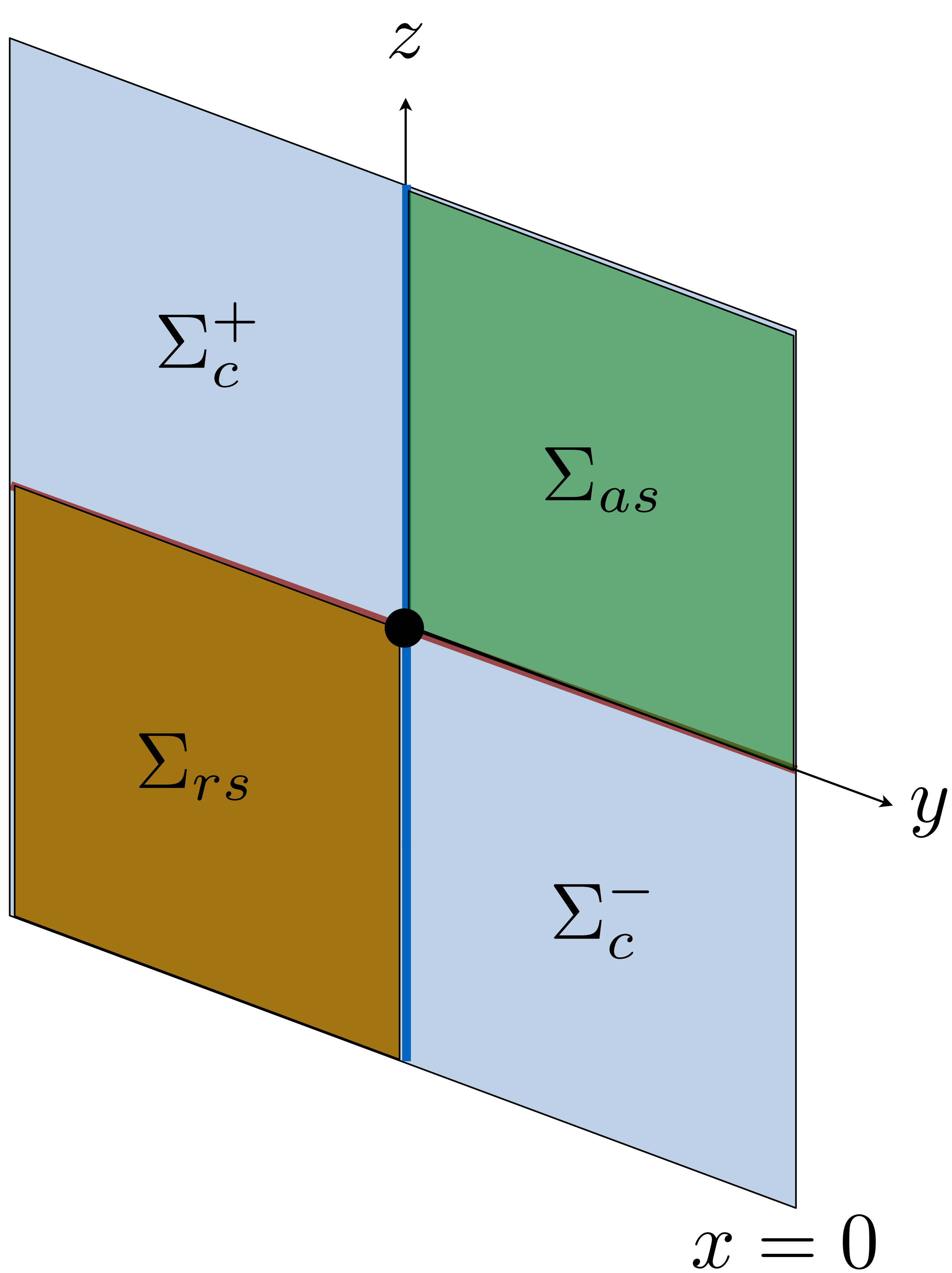
for some constants $f_1^\pm$, $g_1^\pm$, $g_2^\pm$, $g_3^\pm$, $h_1^\pm$, $h_2^\pm$, $h_3^\pm$ and $v_\pm$.

$T_+ = \{(x, y, z) \in \mathbb{R}^3 : \; x = y = 0\}$ $\qquad$ $T_- = \{(x, y, z) \in \mathbb{R}^3 : \; x = z = 0\}$

$\Sigma = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 : x = 0\}$ $\qquad$ TS-point at the origin!

The canonical form for DPWLS with a TS-point

$$\Sigma_{as} = \{(x,y,z) \in \mathbb{R}^3 : \; x = 0, \; y > 0 \text{ and } z > 0\}$$
$$\Sigma_{rs} = \{(x,y,z) \in \mathbb{R}^3 : \; x = 0, \; y < 0 \text{ and } z < 0\}$$
$$\Sigma_c^- = \{(x,y,z) \in \mathbb{R}^3 : \; x = 0, \; y > 0 \text{ and } z < 0\}$$
$$\Sigma_c^+ = \{(x,y,z) \in \mathbb{R}^3 : \; x = 0, \; y < 0 \text{ and } z > 0\}$$



The sliding dynamics bifurcation

The sliding vector field associated to system (1) is

$$\mathbf{F}^s(0,y,z) = \frac{1}{y+z}\begin{bmatrix} 0 \\ v_- y + z + g_2^- y^2 + (g_2^+ + g_3^-)yz + g_3^+ z^2 \\ y + v_+ z + h_2^- y^2 + (h_2^+ + h_3^-)yz + h_3^+ z^2 \end{bmatrix}.$$

We work with the desingularized system

$$\dot{\mathbf{x}} = \mathbf{F}_d^s(0,y,z) := \begin{bmatrix} 0 \\ v_- y + z + g_2^- y^2 + (g_2^+ + g_3^-)yz + g_3^+ z^2 \\ y + v_+ z + h_2^- y^2 + (h_2^+ + h_3^-)yz + h_3^+ z^2 \end{bmatrix} \qquad (2)$$

The sliding dynamics bifurcation

Pseudo-equilibrium points come from solving the two equations

$$v_- y + z + \begin{bmatrix} y & z \end{bmatrix} G \begin{bmatrix} y \\ z \end{bmatrix} = 0,$$

$$y + v_+ z + \begin{bmatrix} y & z \end{bmatrix} H \begin{bmatrix} y \\ z \end{bmatrix} = 0,$$

where

$$G = \begin{bmatrix} g_2^- & g_3^- \\ g_2^+ & g_3^+ \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} h_2^- & h_3^- \\ h_2^+ & h_3^+ \end{bmatrix}.$$

The sliding dynamics bifurcation

Pseudo-equilibrium points come from solving the two equations

$$J(0,0) = \begin{bmatrix} v_- & 1 \\ 1 & v_+ \end{bmatrix} \leftarrow$$

$$v_- y + z + \begin{bmatrix} y & z \end{bmatrix} G \begin{bmatrix} y \\ z \end{bmatrix} = 0,$$

$$y + v_+ z + \begin{bmatrix} y & z \end{bmatrix} H \begin{bmatrix} y \\ z \end{bmatrix} = 0,$$

where

$$G = \begin{bmatrix} g_2^- & g_3^- \\ g_2^+ & g_3^+ \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} h_2^- & h_3^- \\ h_2^+ & h_3^+ \end{bmatrix}.$$

The sliding dynamics bifurcation

Pseudo-equilibrium points come from solving the two equations

$$J(0,0) = \begin{bmatrix} v_- & 1 \\ 1 & v_+ \end{bmatrix} \leftarrow \quad \begin{aligned} v_- y + z + \begin{bmatrix} y & z \end{bmatrix} G \begin{bmatrix} y \\ z \end{bmatrix} &= 0, \\ y + v_+ z + \begin{bmatrix} y & z \end{bmatrix} H \begin{bmatrix} y \\ z \end{bmatrix} &= 0, \end{aligned}$$

where

$$G = \begin{bmatrix} g_2^- & g_3^- \\ g_2^+ & g_3^+ \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} h_2^- & h_3^- \\ h_2^+ & h_3^+ \end{bmatrix}.$$

Apart from the origin, we can have another non-trivial emanating branch for $v_- v_+ = 1$ (transcritical bifurcation). We assume all parameters fixed excepting

$$v_- = v_-(\varepsilon) := \frac{1+\varepsilon}{v_+}$$

The sliding dynamics bifurcation

Pseudo-equilibrium points come from solving the two equations

$$J(0,0) = \begin{bmatrix} v_- & 1 \\ 1 & v_+ \end{bmatrix} \leftarrow$$

$$v_- y + z + \begin{bmatrix} y & z \end{bmatrix} G \begin{bmatrix} y \\ z \end{bmatrix} = 0,$$

$$y + v_+ z + \begin{bmatrix} y & z \end{bmatrix} H \begin{bmatrix} y \\ z \end{bmatrix} = 0,$$

where

$$G = \begin{bmatrix} g_2^- & g_3^- \\ g_2^+ & g_3^+ \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} h_2^- & h_3^- \\ h_2^+ & h_3^+ \end{bmatrix}.$$

$$\underbrace{(v_+ v_- - 1)}_{\varepsilon} y + \begin{bmatrix} y & z \end{bmatrix} (v_+ G - H) \begin{bmatrix} y \\ z \end{bmatrix} = 0,$$

$$y + v_+ z + \begin{bmatrix} y & z \end{bmatrix} H \begin{bmatrix} y \\ z \end{bmatrix} = 0.$$

The sliding dynamics bifurcation

We can use the equivalent system of equations

$$\varepsilon y + \begin{bmatrix} y & z \end{bmatrix} (v_+ G - H) \begin{bmatrix} y \\ z \end{bmatrix} = 0,$$

$$y + v_+ z + \begin{bmatrix} y & z \end{bmatrix} H \begin{bmatrix} y \\ z \end{bmatrix} = 0.$$

Using the implicit function theorem for the second equation at $(y, z) = (0, 0)$ we get that, for z small, solutions must satisfy

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -v_+ \\ 1 \end{bmatrix} z + O(z^2).$$

We need $v_+ < 0$

Substituting this expansion in the first equation, desingularizing it and assuming

$$\kappa_S = \begin{bmatrix} -v_+ & 1 \end{bmatrix} (v_+ G - H) \begin{bmatrix} -v_+ \\ 1 \end{bmatrix} \neq 0,$$

we get

$$z = \frac{v_+}{\kappa_S} \varepsilon + O(\varepsilon^2).$$

The sliding dynamics bifurcation

Proposition (Pseudo-equilibrium transition) Assuming that the criticality coefficient

$$\kappa_S = \begin{bmatrix} -v_+ & 1 \end{bmatrix} (v_+ G - H) \begin{bmatrix} -v_+ \\ 1 \end{bmatrix} \neq 0$$

the following statements hold.

(a) System (2) undergoes a transcritical bifurcation for $\varepsilon = 0$, so that there exists a branch of equilibria $(\tilde{y}(\varepsilon), \tilde{z}(\varepsilon))$ with $(\tilde{y}(0), \tilde{z}(0)) = (0, 0)$ and $(\tilde{y}'(0), \tilde{z}'(0)) = \left(\frac{-v_+^2}{\kappa_S}, \frac{v_+}{\kappa_S} \right)$.

(b) For the particular case where $v_+ < 0$, the emanating branch is located at the quadrants with $yz > 0$. If $\kappa_S > 0$ ($\kappa_S < 0$) then in passing from $\varepsilon < 0$ to $\varepsilon > 0$ the origin passes from being a saddle to a stable node, while the nontrivial equilibrium passes from being a stable node in the first (third) quadrant to be a saddle in the third (first) quadrant.

The sliding dynamics bifurcation

Theorem S Assuming $v_+ < 0$, $v_- < 0$ and $\kappa_S \neq 0$, system (1) has for $\varepsilon = v_- v_+ - 1$ with $|\varepsilon| > 0$ small, one pseudo-equilibrium point $\widetilde{\mathbf{x}}(\varepsilon) = (0, \tilde{y}(\varepsilon), \tilde{z}(\varepsilon))$, such that

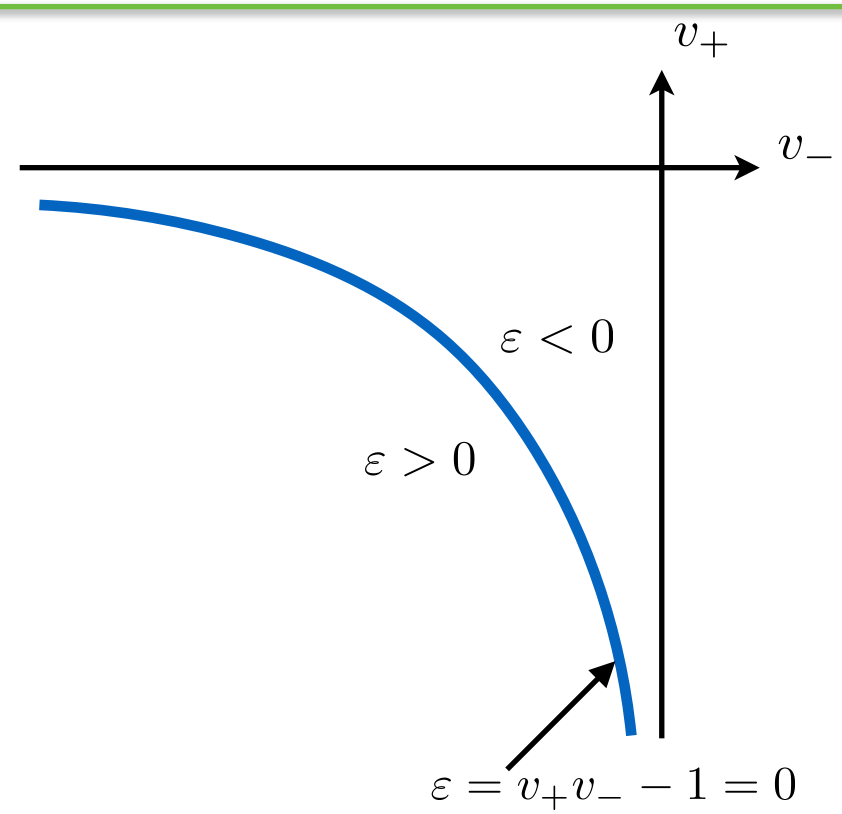
$$(\tilde{y}(\varepsilon), \tilde{z}(\varepsilon)) = \left(-\frac{v_+^2}{\kappa_S}, \frac{v_+}{\kappa_S}\right)\varepsilon + O(\varepsilon^2),$$

and the following statements hold.

(a) **(Supercritical case)** If $\kappa_S > 0$, then $\widetilde{\mathbf{x}}(\varepsilon) \in \Sigma_{as}$ is a stable pseudo-node for $\varepsilon < 0$, being $\widetilde{\mathbf{x}}(\varepsilon) \in \Sigma_{rs}$ a pseudo-saddle for $\varepsilon > 0$.

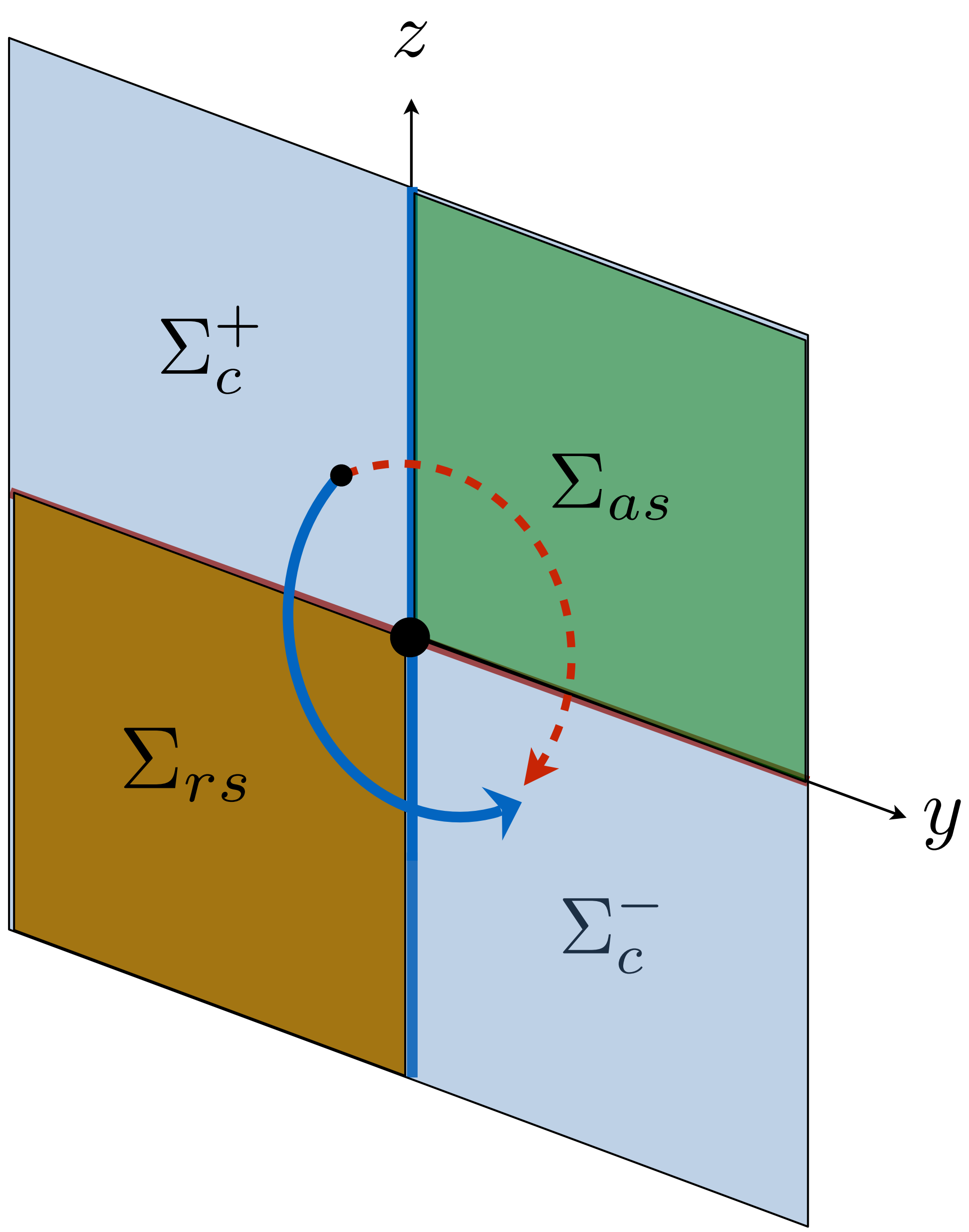
(b) **(Subcritical case)** If $\kappa_S < 0$, then $\widetilde{\mathbf{x}}(\varepsilon) \in \Sigma_{rs}$ is an unstable pseudo-node for $\varepsilon < 0$, being $\widetilde{\mathbf{x}}(\varepsilon) \in \Sigma_{as}$ a pseudo-saddle for $\varepsilon > 0$.

$$\kappa_S = \begin{bmatrix} -v_+ & 1 \end{bmatrix} (v_+ G - H) \begin{bmatrix} -v_+ \\ 1 \end{bmatrix}$$



The crossing dynamics bifurcation (periodic orbits?)

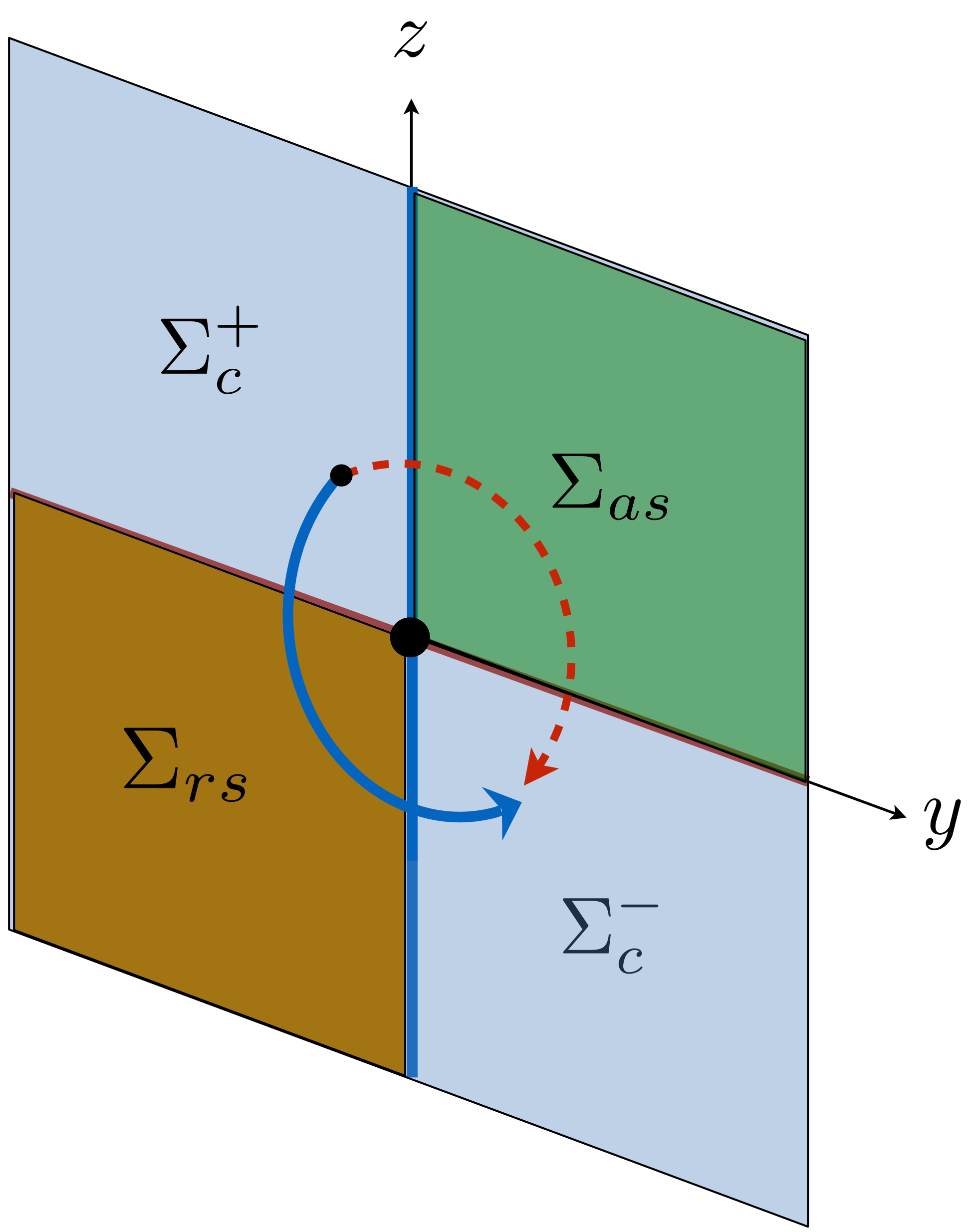
$$P_+ \begin{pmatrix} y \\ z \end{pmatrix} = P_-^{-1} \begin{pmatrix} y \\ z \end{pmatrix}$$



The crossing dynamics bifurcation (periodic orbits?)

$$P_+ \begin{pmatrix} y \\ z \end{pmatrix} = P_-^{-1} \begin{pmatrix} y \\ z \end{pmatrix} = P_- \begin{pmatrix} y \\ z \end{pmatrix}$$

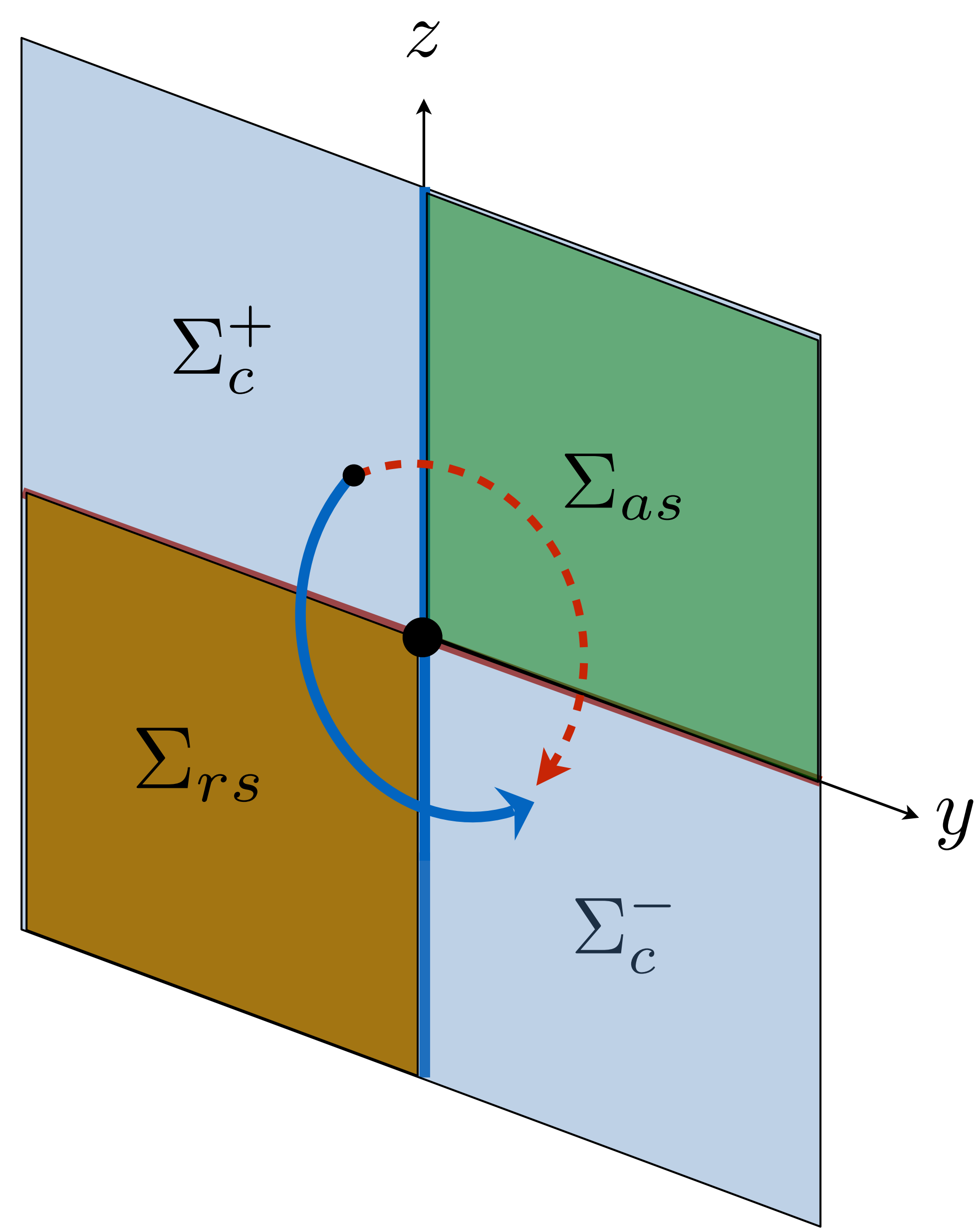
Involution property



The crossing dynamics bifurcation (periodic orbits?)

$$P_+ \begin{pmatrix} y \\ z \end{pmatrix} = P_-^{-1} \begin{pmatrix} y \\ z \end{pmatrix} = P_- \begin{pmatrix} y \\ z \end{pmatrix}$$

Involution property



$$P_+ \left(\begin{bmatrix} 0 \\ z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ z \end{bmatrix} ; \quad P_- \left(\begin{bmatrix} y \\ 0 \end{bmatrix} \right) = \begin{bmatrix} y \\ 0 \end{bmatrix}$$

Invariance of the tangency lines

The structure of return maps

$$P_+ \begin{pmatrix} y \\ z \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ -2v_+ & 1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + y \begin{bmatrix} q_{11}^+ y + q_{12}^+ z \\ q_{21}^+ y + q_{22}^+ z \end{bmatrix} + y \begin{bmatrix} c_{11}^+ y^2 + c_{12}^+ yz + c_{13}^+ z^2 \\ c_{21}^+ y^2 + c_{22}^+ yz + c_{23}^+ z^2 \end{bmatrix} + O(4)$$

$$P_- \begin{pmatrix} y \\ z \end{pmatrix} = \begin{bmatrix} 1 & -2v_- \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + z \begin{bmatrix} q_{11}^- y + q_{12}^- z \\ q_{21}^- y + q_{22}^- z \end{bmatrix} + z \begin{bmatrix} c_{11}^- y^2 + c_{12}^- yz + c_{13}^- z^2 \\ c_{21}^- y^2 + c_{22}^- yz + c_{23}^- z^2 \end{bmatrix} + O(4)$$

The structure of return maps

$$P_+\begin{pmatrix} y \\ z \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ -2v_+ & 1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} + y\begin{bmatrix} q^+_{11}y + q^+_{12}z \\ q^+_{21}y + q^+_{22}z \end{bmatrix} + y\begin{bmatrix} c^+_{11}y^2 + c^+_{12}yz + c^+_{13}z^2 \\ c^+_{21}y^2 + c^+_{22}yz + c^+_{23}z^2 \end{bmatrix} + O(4)$$

$$\begin{aligned} q^+_{12} &= 0; \\ q^+_{21} &= v_+(q^+_{11} - q^+_{22}); \\ c^+_{11} &= -v_+c^+_{12} - (q^+_{11})^2; \\ c^+_{13} &= 0; \\ c^+_{22} &= v_+(c^+_{12} - 2c^+_{23}) + q^+_{22}(q^+_{22} - q^+_{11})/2. \end{aligned}$$

Involution property

$$P_-\begin{pmatrix} y \\ z \end{pmatrix} = \begin{bmatrix} 1 & -2v_- \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} + z\begin{bmatrix} q^-_{11}y + q^-_{12}z \\ q^-_{21}y + q^-_{22}z \end{bmatrix} + z\begin{bmatrix} c^-_{11}y^2 + c^-_{12}yz + c^-_{13}z^2 \\ c^-_{21}y^2 + c^-_{22}yz + c^-_{23}z^2 \end{bmatrix} + O(4)$$

The structure of return maps

$$P_+\begin{pmatrix} y \\ z \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ -2v_+ & 1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} + y\begin{bmatrix} q_{11}^+ y + \cancel{q_{12}^+} z \\ q_{21}^+ y + q_{22}^+ z \end{bmatrix} + y\begin{bmatrix} c_{11}^+ y^2 + c_{12}^+ yz + \cancel{c_{13}^+} z^2 \\ c_{21}^+ y^2 + c_{22}^+ yz + c_{23}^+ z^2 \end{bmatrix} + O(4)$$

$$\begin{aligned} q_{12}^+ &= 0; \\ q_{21}^+ &= v_+(q_{11}^+ - q_{22}^+); \\ c_{11}^+ &= -v_+ c_{12}^+ - (q_{11}^+)^2; \\ c_{13}^+ &= 0; \\ c_{22}^+ &= v_+(c_{12}^+ - 2c_{23}^+) + q_{22}^+(q_{22}^+ - q_{11}^+)/2. \end{aligned}$$

Involution property

$$P_-\begin{pmatrix} y \\ z \end{pmatrix} = \begin{bmatrix} 1 & -2v_- \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} + z\begin{bmatrix} q_{11}^- y + q_{12}^- z \\ \cancel{q_{21}^-} y + q_{22}^- z \end{bmatrix} + z\begin{bmatrix} c_{11}^- y^2 + c_{12}^- yz + c_{13}^- z^2 \\ \cancel{c_{21}^-} y^2 + c_{22}^- yz + c_{23}^- z^2 \end{bmatrix} + O(4)$$

$$\begin{aligned} q_{21}^- &= 0; \\ q_{12}^- &= v_-(q_{22}^- - q_{11}^-); \\ c_{23}^- &= -v_- c_{22}^- - (q_{22}^-)^2; \\ c_{21}^- &= 0; \\ c_{12}^- &= v_-(c_{22}^- - 2c_{11}^-) + q_{11}^-(q_{11}^- - q_{22}^-)/2. \end{aligned}$$

Involution property

Looking for non-trivial fixed points

We impose the equality $P_+(y,z) = P_-^{-1}(y,z)$, to obtain

$$0 = -2y + 2v_- z + q_{11}^+ y^2 - q_{11}^- yz - q_{12}^- z^2 + c_{11}^+ y^3 + (c_{12}^+ - c_{11}^-) y^2 z - c_{12}^- yz^2 - c_{13}^- z^3 + O(4),$$
$$0 = -2v_+ y + 2z + q_{21}^+ y^2 + q_{22}^+ yz - q_{22}^- z^2 + c_{21}^+ y^3 + c_{22}^+ y^2 z + (c_{23}^+ - c_{22}^-) yz^2 - c_{23}^- z^3 + O(4).$$

Again, we can have a bifurcation at $v_+ v_- = 1$ so that we do a new bifurcation analysis by assuming all parameters fixed, excepting v_-, and take

$$v_- = \frac{1+\varepsilon}{v_+}$$

Note that, from the second equation and the implicit function theorem, we can assume the existence of a function $p(y,\varepsilon)$ such that

$$z = y \cdot p(y,\varepsilon), \quad \text{with } p(0,0) = v_+.$$

We need $v_+ < 0$

Looking for non-trivial fixed points

We get $z = y \cdot p(y, \varepsilon)$ with

$$p(y,\varepsilon) = v_+ - \left(q_{11}^+ - q_{22}^-(\varepsilon)v_+\right)\frac{y}{2} + \big[-2c_{21}^+ + q_{22}^+(2q_{11}^+ - q_{22}^+)v_+ +$$
$$+ \left(2c_{22}^-(\varepsilon) - 2c_{12}^+ + 2c_{23}^+ - 2q_{22}^-(\varepsilon)q_{11}^+ - q_{22}^-(\varepsilon)q_{22}^+\right)v_+^2 - 2c_{22}^-(\varepsilon)v_-(\varepsilon)v_+^3\big]\frac{y^2}{4} + \dots$$

After substituting z in the first equation, we can desingularize it to get

$$0 = 2\varepsilon - \left(q_{11}^+ - q_{11}^-(\varepsilon)v_+\right)\varepsilon y + P(\varepsilon)y^2 + \dots$$

with

$$P(\varepsilon) = -\frac{1}{2v_+}\Big[2c_{21}^+ + \left(2(q_{11}^+)^2 - 2q_{11}^+q_{22}^+ + (q_{22}^+)^2\right)v_+ +$$
$$+ \left(-2c_{11}^-(\varepsilon) + 2c_{22}^-(\varepsilon) + 2c_{12}^+ - 2c_{23}^+ + q_{11}^-(\varepsilon)q_{11}^+ + q_{22}^-(\varepsilon)q_{22}^+\right)v_+^2 +$$
$$+ \left(q_{11}^-(\varepsilon)^2 - 2q_{11}^-(\varepsilon)q_{22}^-(\varepsilon) + 2q_{22}^-(\varepsilon)^2\right)v_+^3 + 2c_{13}^-(\varepsilon)v_+^4 + O(\varepsilon)\Big]$$

Looking for non-trivial fixed points

Clearly, it is more convenient to parameterize the emanating branch in terms of y, so that we get the expansion

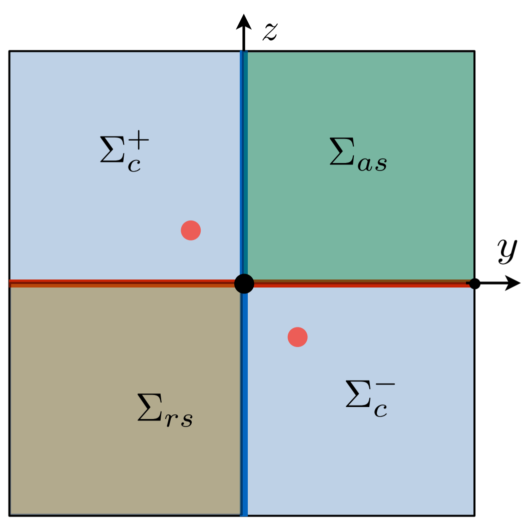
$$\varepsilon = \kappa_C \cdot y^2 + O(y^3),$$

where **the criticality coefficient is**

$$\kappa_C = \frac{1}{4v_+}\Big[2c_{21}^+ + \big(2(q_{11}^+)^2 - 2q_{11}^+ q_{22}^+ + (q_{22}^+)^2\big)\, v_+ + \\ + \big(-2c_{11}^-(0) + 2c_{22}^-(0) + 2c_{12}^+ - 2c_{23}^+ + q_{11}^-(0)q_{11}^+ + q_{22}^-(0)q_{22}^+\big)\, v_+^2 + \\ + \big(q_{11}^-(0)^2 - 2q_{11}^-(0)q_{22}^-(0) + 2q_{22}^-(0)^2\big)\, v_+^3 + 2c_{13}^-(0)v_+^4\Big]$$

In short, depending on the sign of κ_C we have a subcritical or supercritical bifurcation of non-trivial fixed points with

$$z(y) = v_+ \cdot y + O(y^2),$$
$$\varepsilon(y) = \kappa_C \cdot y^2 + O(y^3).$$



Topological type of non-trivial fixed points

Just computing the derivatives of the Poincaré half-return maps, and evaluating them at the branch of non-trivial fixed points, we get $DP(y) = D\left(P_{-} \circ P_{+}\right)(y, z(y))$ and the expansions for its determinant ant trace, namely

$$\det DP(y) = 1 + d_1 y + d_2 y^2 + O(y^3),$$
$$\operatorname{trace} DP(y) = 2 + t_1 y + t_2 y^2 + O(y^3).$$

It turns out that $t_1 = d_1$. When this common value vanishes, we have a degeneracy that should require much more long computations. We define $\sigma := t_1 = d_1$, and the computations give

$$\sigma = -2q_{11}^{+} + q_{22}^{+} + \left(2q_{22}^{-}(0) - q_{11}^{-}(0)\right) v_{+},$$
$$d_2 = \frac{1}{2}\left[6(q_{11}^{+})^2 - 5q_{11}^{+}\left(q_{22}^{+} - q_{11}^{-}(0)v_{+} + 2q_{22}^{-}(0)v_{+}\right) + \left(q_{22}^{+} - q_{11}^{-}(0)v_{+} + 2q_{22}^{-}(0)v_{+}\right)^2\right],$$
$$t_2 = -\frac{1}{2v_{+}}\left[8c_{21}^{+} + \left(2(q_{11}^{+})^2 - 3q_{11}^{+}q_{22}^{+} + 3(q_{22}^{+})^2\right)v_{+} + \right.$$
$$+ \left(-8c_{11}^{-}(0) + 8c_{22}^{-}(0) + 8c_{12}^{+} - 8c_{23}^{+} - q_{11}^{-}(0)q_{11}^{+} + 10q_{22}^{-}(0)q_{11}^{+} + 2q_{11}^{-}(0)q_{22}^{+}\right)v_{+}^2 +$$
$$\left. + \left(3q_{11}^{-}(0)^2 - 4q_{11}^{-}(0)q_{22}^{-}(0) + 4q_{22}^{-}(0)^2\right)v_{+}^3 + 8c_{13}^{-}(0)v_{+}^4\right]$$

Topological type of non-trivial fixed points

We have at the branch of non-trivial fixed points

$$\det DP(y) = 1 + \sigma y + d_2 y^2 + O(y^3),$$
$$\operatorname{trace} DP(y) = 2 + \sigma y + t_2 y^2 + O(y^3).$$

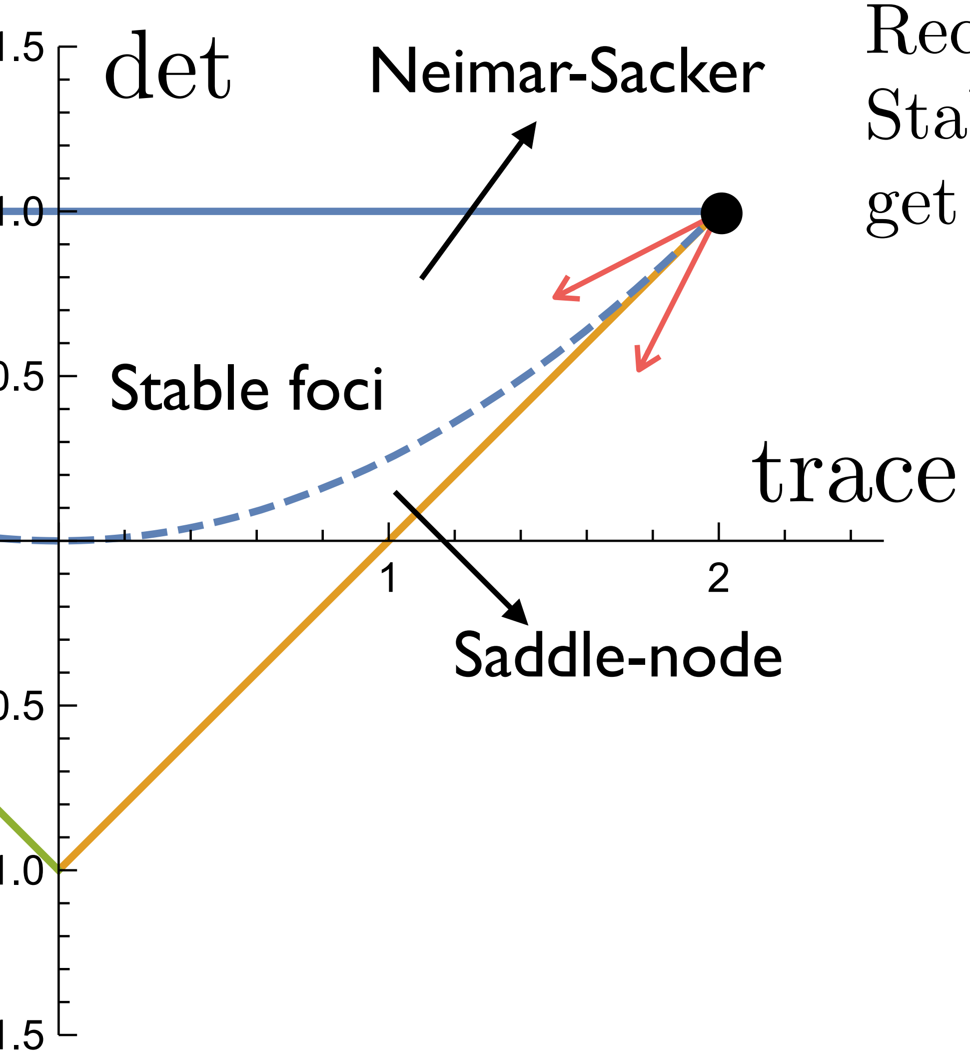
and, surprisingly, we get that $\boxed{d_2 - t_2 = 8\kappa_C.}$

Topological type of non-trivial fixed points

We have at the branch of non-trivial fixed points

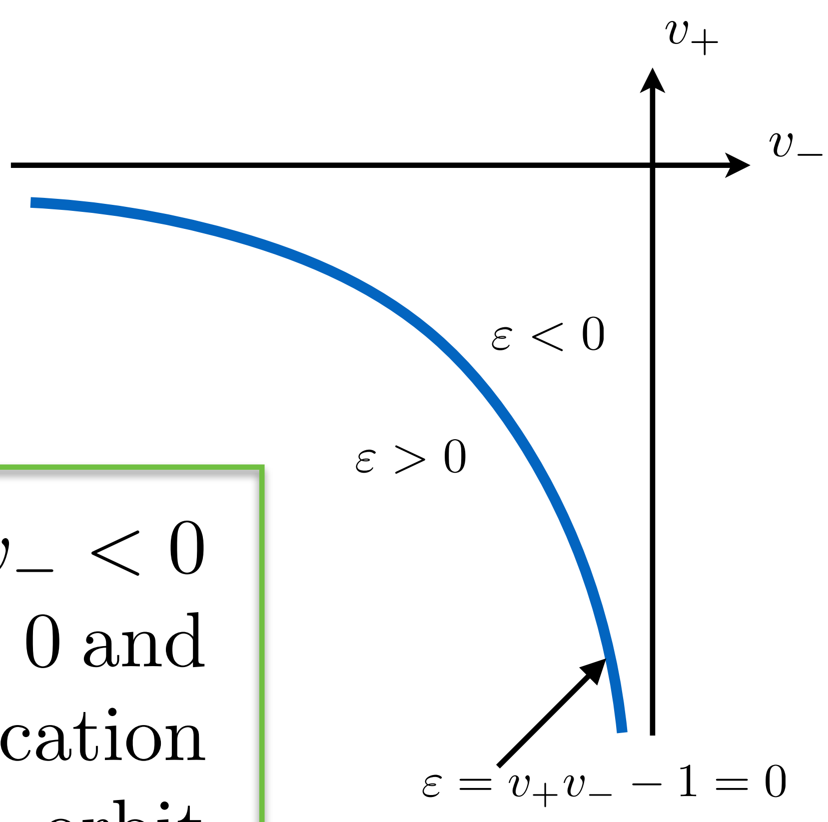
$$\det DP(y) = 1 + \sigma y + d_2 y^2 + O(y^3),$$
$$\operatorname{trace} DP(y) = 2 + \sigma y + t_2 y^2 + O(y^3).$$

and, surprisingly, we get that $\boxed{d_2 - t_2 = 8\kappa_C.}$



Recall that we have $y < 0$ and small in absolute value. Stability requires $\sigma > 0$ and $\kappa_C > 0$. Furthermore, we get

$$\kappa_C < 0 \longrightarrow \text{SADDLE}$$
$$0 < \kappa_C < \frac{\sigma^2}{32} \longrightarrow \text{NODE}$$
$$\kappa_C > \frac{\sigma^2}{32} \longrightarrow \text{FOCUS}$$



Theorem C Assume in system (1) that $v_+ < 0$, $v_- < 0$ with $v_- v_+ - 1 = 0$ and that the two conditions $\sigma \neq 0$ and $\kappa_S \neq 0$ hold. By moving v_- and using the bifurcation parameter $\varepsilon = v_- v_+ - 1$, one crossing periodic orbit bifurcates from the origin for $\kappa_S \cdot \varepsilon > 0$ small.
The bifurcating periodic orbit is stable whenever $\sigma > 0$ and $\kappa_S > 0$.
The topological type of the corresponding fixed point for the Poincaré map is saddle, node or focus depending on whether $\kappa_S < 0$, $0 < \kappa_S < \sigma^2/32$ or $\kappa_S > \sigma^2/32$, respectively.

Computing the return maps

To compute the return map P_+ we write

$$\mathbf{x}(\tau) = e^{A^+\tau}\mathbf{x}_0 + \int_0^\tau e^{A^+(\tau-s)}\mathbf{v}^+ ds,$$

where

$$A^+ = \begin{bmatrix} f_x^+ & -1 & 0 \\ g_x^+ & g_y^+ & g_z^+ \\ h_x^+ & h_y^+ & h_z^+ \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 0 \\ y_0 \\ z_0 \end{bmatrix}, \quad \mathbf{v}^+ = \begin{bmatrix} 0 \\ 1 \\ v_+ \end{bmatrix},$$

being $y_0 < 0$. In practice, we take

$$\mathbf{x}(\tau) = \mathbf{x}_0 + \left(\tau I + \frac{\tau^2}{2}A^+ + \frac{\tau^3}{6}(A^+)^2 + \frac{\tau^4}{24}(A^+)^3\right) M + \mathcal{O}(\tau^5)$$

where $M = A^+\mathbf{x}_0 + \mathbf{v}^+$ and I is the identity matrix of order 3.

From the first component we can determine an expression for the time $\tau_+ = \tau_+(y_0, z_0)$ such that $x(\tau_+) = 0$. The third order polynomial approximation is given by

$$\tau_+(y_0, z_0) = a_{10}y_0 + a_{20}y_0^2 + a_{11}y_0z_0 + a_{30}y_0^3 + a_{21}y_0^2z_0 + a_{12}y_0z_0^2,$$

with

$$\begin{aligned}
a_{10} &= -2, \\
a_{20} &= \frac{2}{3}(f_1^+ + g_2^+ - 2g_3^+v_+), \\
a_{11} &= 2g_3^+, \\
a_{20} &= \frac{2}{3}(f_1^+ + g_2^+ - 2g_3^+v_+), \\
a_{30} &= -\frac{2}{9}\left(2(f_1^+ + g_2^+)^2 - 3(f_1^+g_2^+ - g_3^+h_2^+ + g_1^+) - \right. \\
&\quad \left. -(5f_1^+ + 5g_2^+ + 3h_3^+)g_3^+v_+ + 8(g_3^+v_+)^2\right), \\
a_{12} &= -2(g_3^+)^2, \\
a_{21} &= \frac{4}{3}g_3^+(3g_3^+v_+ - f_1^+ - g_2^+ - h_3^+).
\end{aligned}$$

The half-return map $(y_1, z_1) = P_+(y_0, z_0)$ satisfies

$$\begin{aligned} y_1 &= -y_0 + q_{11}^+ y_0^2 + y_0(c_{11}^+ y_0^2 + c_{12}^+ y_0 z_0) + O(4), \\ z_1 &= -2v_+ y_0 + z_0 + y_0(q_{21}^+ y_0 + q_{22}^+ z_0) + y_0(c_{21}^+ y_0^2 + c_{22}^+ y_0 z_0 + c_{23}^+ z_0^2) + O(4), \end{aligned}$$

where

$$\begin{aligned} \rightarrow \quad q_{11}^+ &= \frac{2}{3}\left(f_1^+ + g_2^+ + g_3^+ v_+\right), \\ \rightarrow \quad q_{22}^+ &= -2(h_3^+ - g_3^+ v_+), \\ q_{21}^+ &= v_+(q_{11}^+ - q_{22}^+) = \frac{2}{3}\left(f_1^+ + g_2^+ - 2g_3^+ v_+ + 3h_3^+\right) v_+, \\ c_{11}^+ &= -v_+ c_{12}^+ - (q_{11}^+)^2 = -\frac{2}{9}\left[2(f_1^+ + g_2^+)^2 + (f_1^+ + g_2^+ + 3h_3^+) g_3^+ v_+ - 4(g_3^+ v_+)^2\right], \\ \rightarrow \quad c_{12}^+ &= -\frac{2}{3}\left(f_1^+ + g_2^+ + 2g_3^+ v_+ - h_3^+\right) g_3^+, \\ \rightarrow \quad c_{21}^+ &= -\frac{2}{9}\left\{3(h_1^+ + f_1^+ h_2^+ - h_2^+ h_3^+) + \left[2(f_1^+ + g_2^+)^2 - 3(f_1^+ g_2^+ + g_1^+ - g_3^+ h_2^+) + \right.\right. \\ &\quad \left.\left. + 6h_3^+(f_1^+ + g_2^+ + h_3^+)\right] v_+ + 5g_3^+(f_1^+ + g_2^+ + 3h_3^+) v_+^2 + 8(g_3^+)^2 v_+^3\right\}, \\ c_{22}^+ &= v_+(c_{12}^+ - 2c_{23}^+) + q_{22}^+(q_{22}^+ - q_{11}^+)/2 = \\ &= \frac{2}{3}\left[h_3^+(f_1^+ + g_2^+ + 3h_3^+) - 2g_3^+(f_1^+ + g_2^+ + 5h_3^+) v_+\right] + 4(g_3^+ v_+)^2, \\ \rightarrow \quad c_{23}^+ &= 2g_3^+(h_3^+ - g_3^+ v_+) = -g_3^+ q_{22}^+. \end{aligned}$$

Analogously, we can determine the half-return map $(y_2, z_2) = P_-^{-1}(y_0, z_0)$, getting

$$y_2 = y_0 - 2v_- z_0 + z_0(q_{11}^- y_0 + q_{12}^- z_0) + z_0(c_{11}^- y_0^2 + c_{12}^- y_0 z_0 + c_{13}^- z_0^2) + O(4)$$
$$z_2 = -z_0 + q_{22}^- z_0^2 + z_0(c_{22}^- y_0 z_0 + c_{23}^- z_0^2) + O(4),$$

where

$$q_{11}^- = -2(g_2^- - h_2^- v_-),$$
$$q_{12}^- = v_-(q_{22}^- - q_{11}^-) = \frac{2}{3}\left(f_1^- + h_3^- - 2h_2^- v_- + 3g_2^-\right) v_-,$$
$$q_{22}^- = \frac{2}{3}\left(f_1^- + h_3^- + h_2^- v_-\right),$$
$$c_{11}^- = 2h_2^-(g_2^- - h_2^- v_-) = -h_2^- q_{11}^-,$$
$$c_{12}^- = v_-(c_{22}^- - 2c_{11}^-) + q_{11}^-(q_{11}^- - q_{22}^-)/2 =$$
$$= \frac{2}{3}\left[g_2^-(f_1^- + h_3^- + 3g_2^-) - 2h_2^-(f_1^- + h_3^- + 5g_2^-)v_-\right] + 4(h_2^- v_-)^2,$$
$$c_{13}^- = \frac{2}{9}\Big\{3(g_1^- - f_1^- g_3^- + g_2^- g_3^-) - \big[2(f_1^- + h_3^-)^2 + 3(h_1^- + h_2^- g_3^- - f_1^- h_3^-) +$$
$$+ 6g_2^-(f_1^- + g_2^- + h_3^-)\big] + 5h_2^-(f_1^- + 3g_2^- + h_3^-)v_-^2 - 8(h_2^-)^2 v_-^3\Big\},$$
$$c_{22}^- = -\frac{2}{3}\left(f_1^- - g_2^- + 2h_2^- v_- + h_3^-\right) h_2^-,$$
$$c_{23}^- = -v_- c_{22}^- - (q_{22}^-)^2 = -\frac{2}{9}\left[2(f_1^- + h_3^-)^2 + h_2^-(f_1^- + 3g_2^- + h_3^-)v_- - 4(h_2^- v_-)^2\right].$$

Examples & Applications

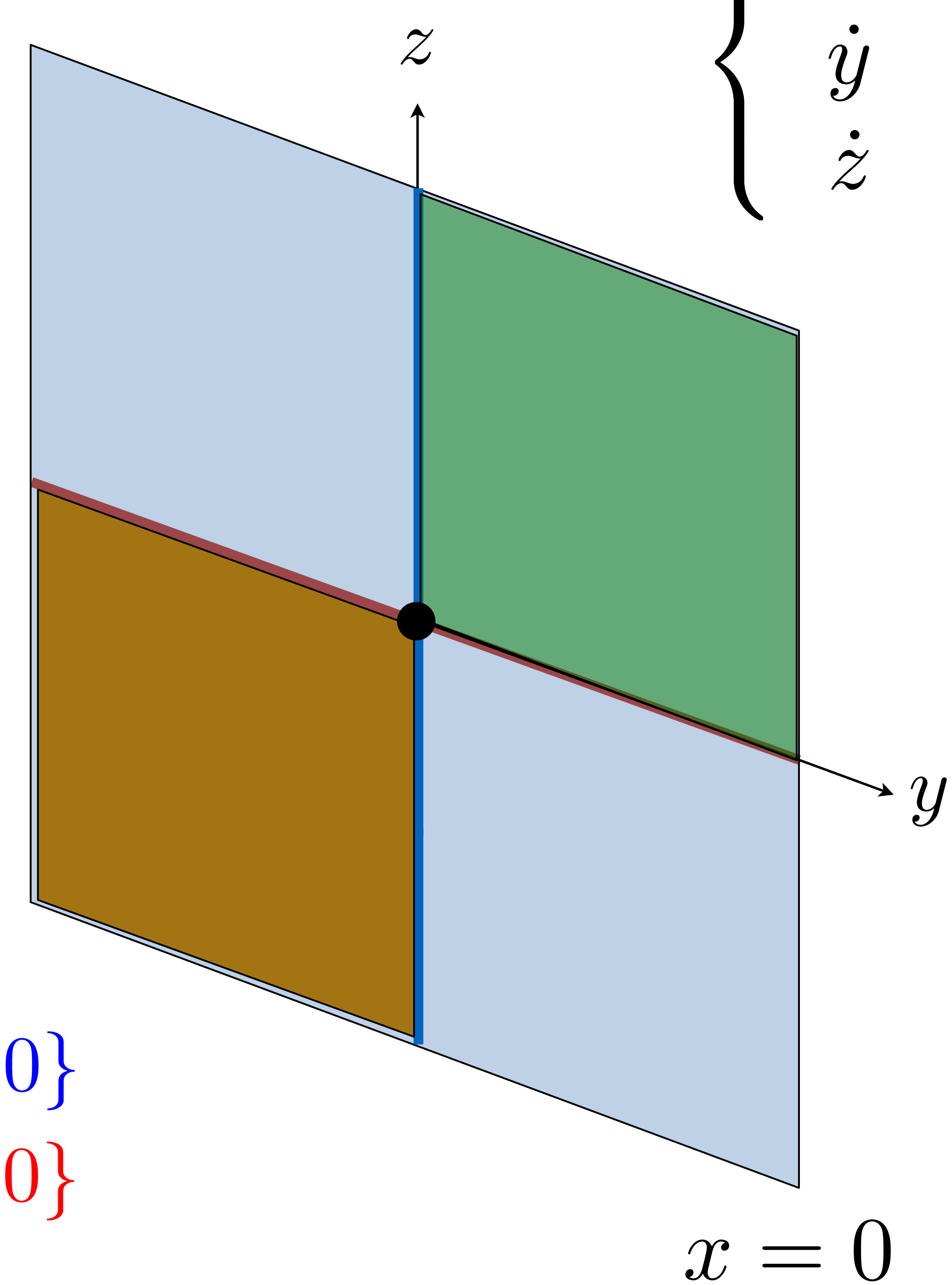
A multi-parametric example

$$\begin{cases} \dot{x} & = & z \\ \dot{y} & = & v_- \\ \dot{z} & = & 1 \end{cases}$$

$$\begin{cases} \dot{x} & = & -y \\ \dot{y} & = & 1 \\ \dot{z} & = & ay + bz + v_+ \end{cases}$$

$$a < 0$$

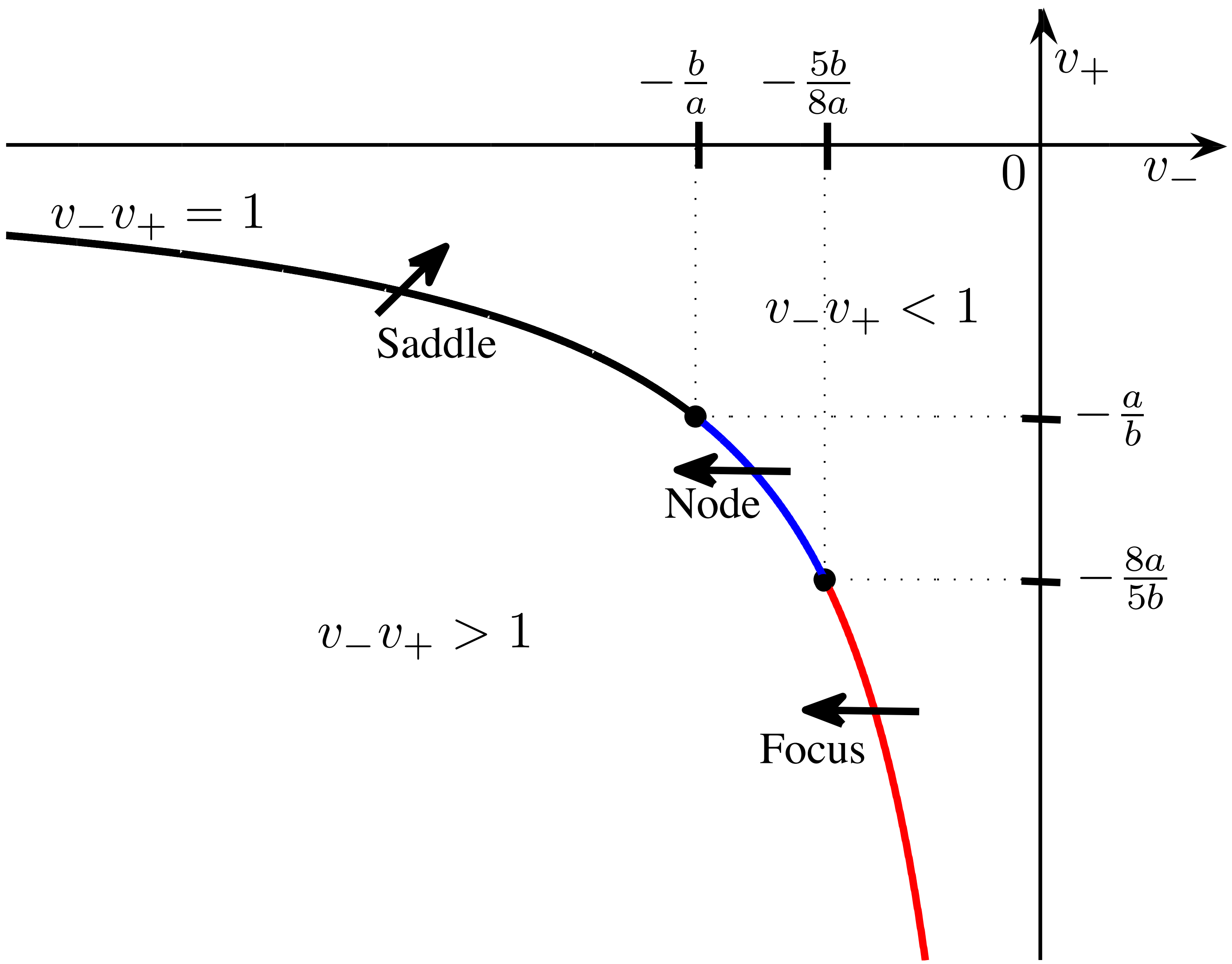
$$b < 0$$



$$T_+ = \{(x, y, z) \in \mathbb{R}^3 : \; x = y = 0\}$$

$$T_- = \{(x, y, z) \in \mathbb{R}^3 : \; x = z = 0\}$$

A multi-parametric example



We have for this example

$$\kappa_S = av_+ - b,$$
$$\sigma = -2b,$$
$$\kappa_C = b\frac{a + bv_+}{3v_+}.$$

$$a < 0$$
$$b < 0$$

Bifurcation set to be completed.
See the preprint by A. Algaba, E. Freire, E. Gamero and C. García, Bifurcation analysis of planar nilpotent reversible systems.

Electronic converters

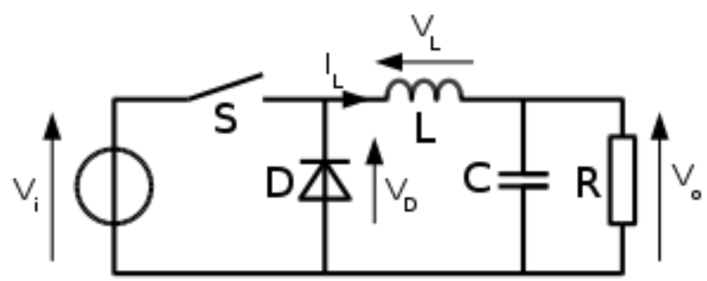
- Electronic Power converters are Switching-Mode Power Supplies (SMPS).

- Basically, they are built using semiconductor switches (diodes, transistors) and energy storage elements (inductors, capacitors)

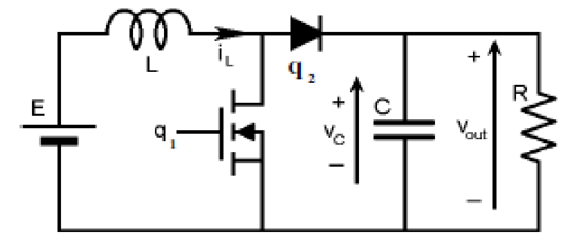
- Examples: Rectifiers AC-DC, inverters DC-AC, DC-DC converters (buck, boost, buck-boost)

- Their mathematical models commonly lead to non-smooth dynamical systems

DC-DC converters

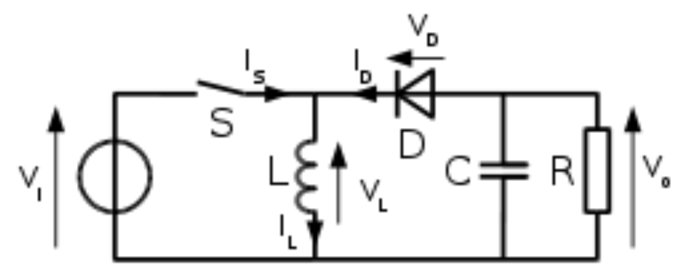


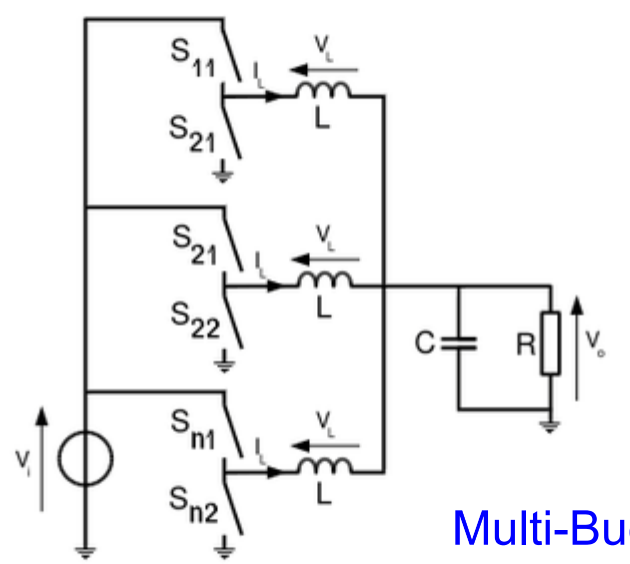
Buck (step-down)

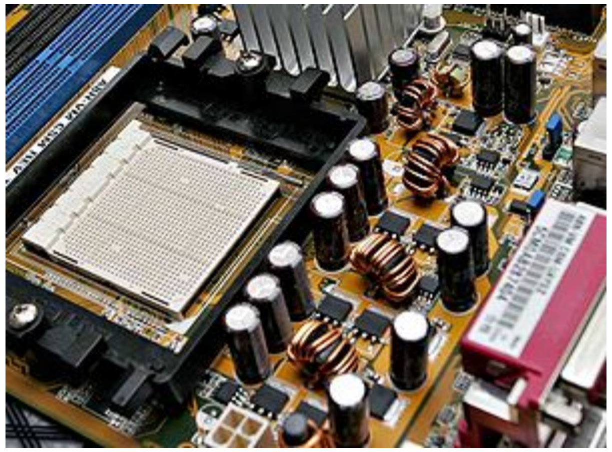


Boost (step-up)

Buck-Boost (step-up or step-down)

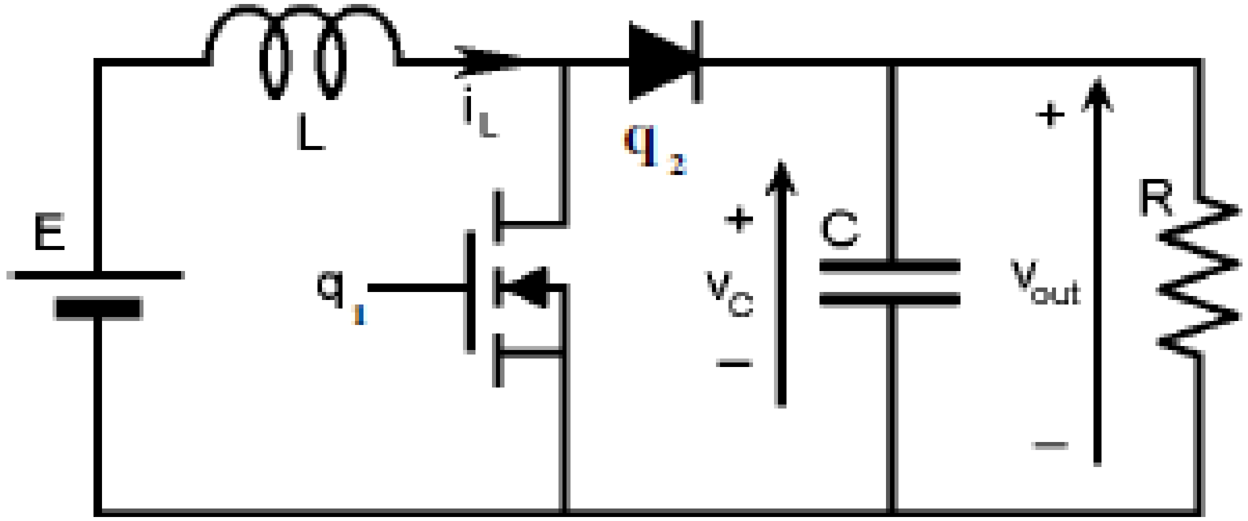


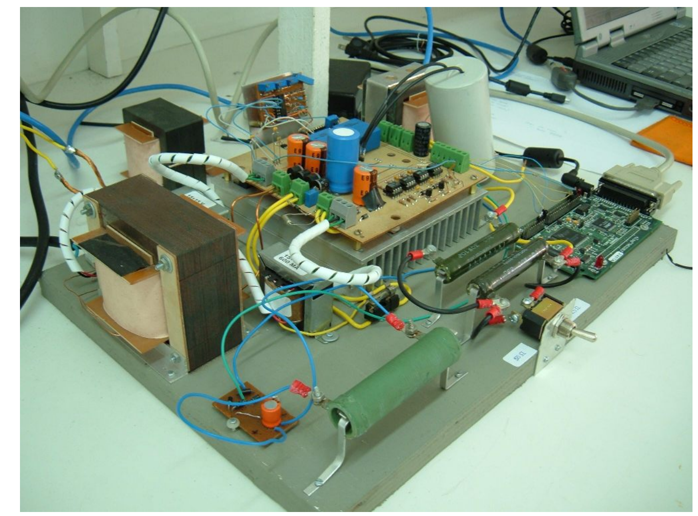




Multi-Buck (CPU)

The BOOST converter

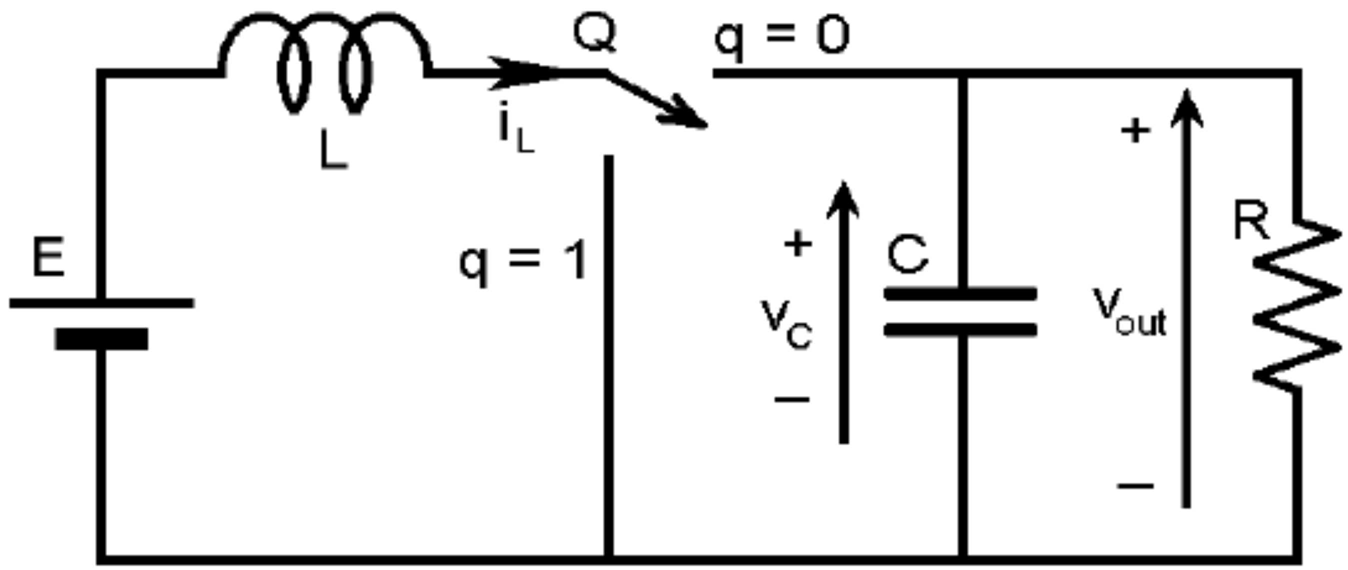




- The value of q_1 stands for a controlled switch
- when q_1 is turned **ON** → current in L increases and energy is stored in it
- when q_1 is turned **OFF** → the stored energy in L is dropped and the polarity of the L voltage changes so that it adds to the input voltage

> The goal is to get $V_{out} > E$

The BOOST converter



The model of the system is given by

$$L\frac{di_L}{dt} = V_{in} - r_L i_L - u \cdot v_c$$
$$C\frac{dv_c}{dt} = u \cdot i_L - \frac{v_c}{R} \qquad (V_{in} = \mathsf{E})$$

where $v_c > 0$ is the capacitor voltage, $i_L > 0$ is the inductor current and $u \in \{0, 1\}$ is the control action. Input voltage is assigned as V_{in}, r_L is the equivalent series resistance of the inductor, R is the resistive load, C and L are the capacitor and inductor, respectively.

The BOOST converter

To analyze the model, the system is normalized

$$(i_L, v_c) = \left(V_{in}\sqrt{\frac{C}{L}}x, V_{in}y \right) \quad \text{and} \quad t = \tau\sqrt{LC};$$

and new parameters are taken, namely

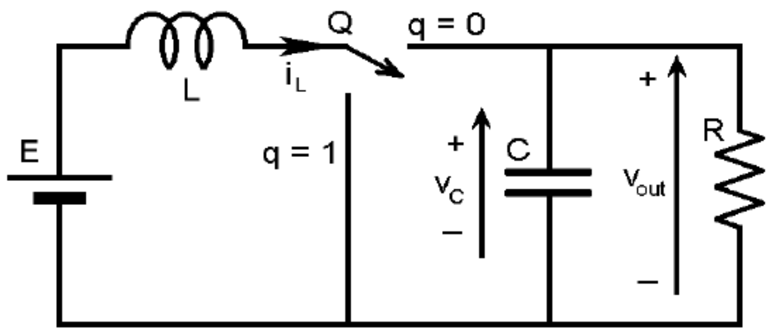
$$b = r_L\sqrt{\frac{C}{L}} \quad \text{and} \quad a = \frac{1}{R}\sqrt{\frac{L}{C}};$$

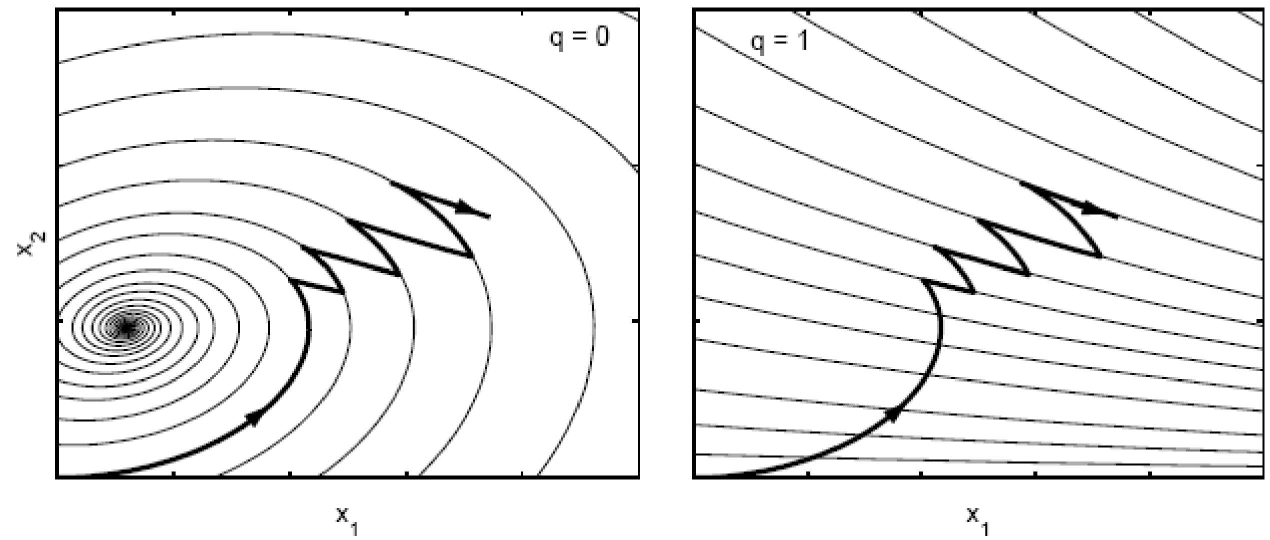
so that the boost converter model in dimensionless normal form is

$$\dot{x} = 1 - bx - u \cdot y$$
$$\dot{y} = u \cdot x - ay,$$

for $x, y > 0$, $u = \{0, 1\}$, $b \geq 0$, and $a > 0$.

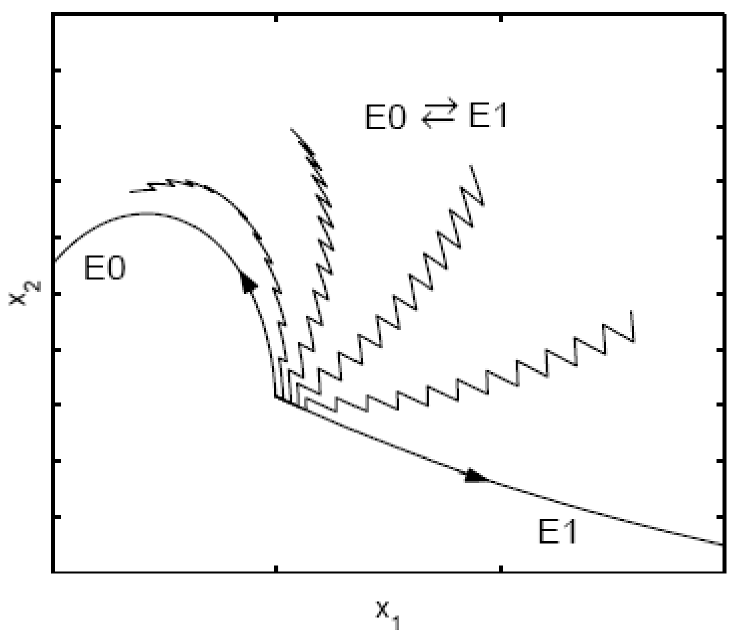
The strategy at the BOOST converter





$$E_0 \begin{cases} \dot{x} = 1 - b\,x - y \\ \dot{y} = x - a\,y \end{cases}$$

$$E_1 \begin{cases} \dot{x} = 1 - b\,x \\ \dot{y} = -a\,y \end{cases}$$



-Frequency variable Control
-Frequency constant Control
using PWM (Pulse-Width Modulation)

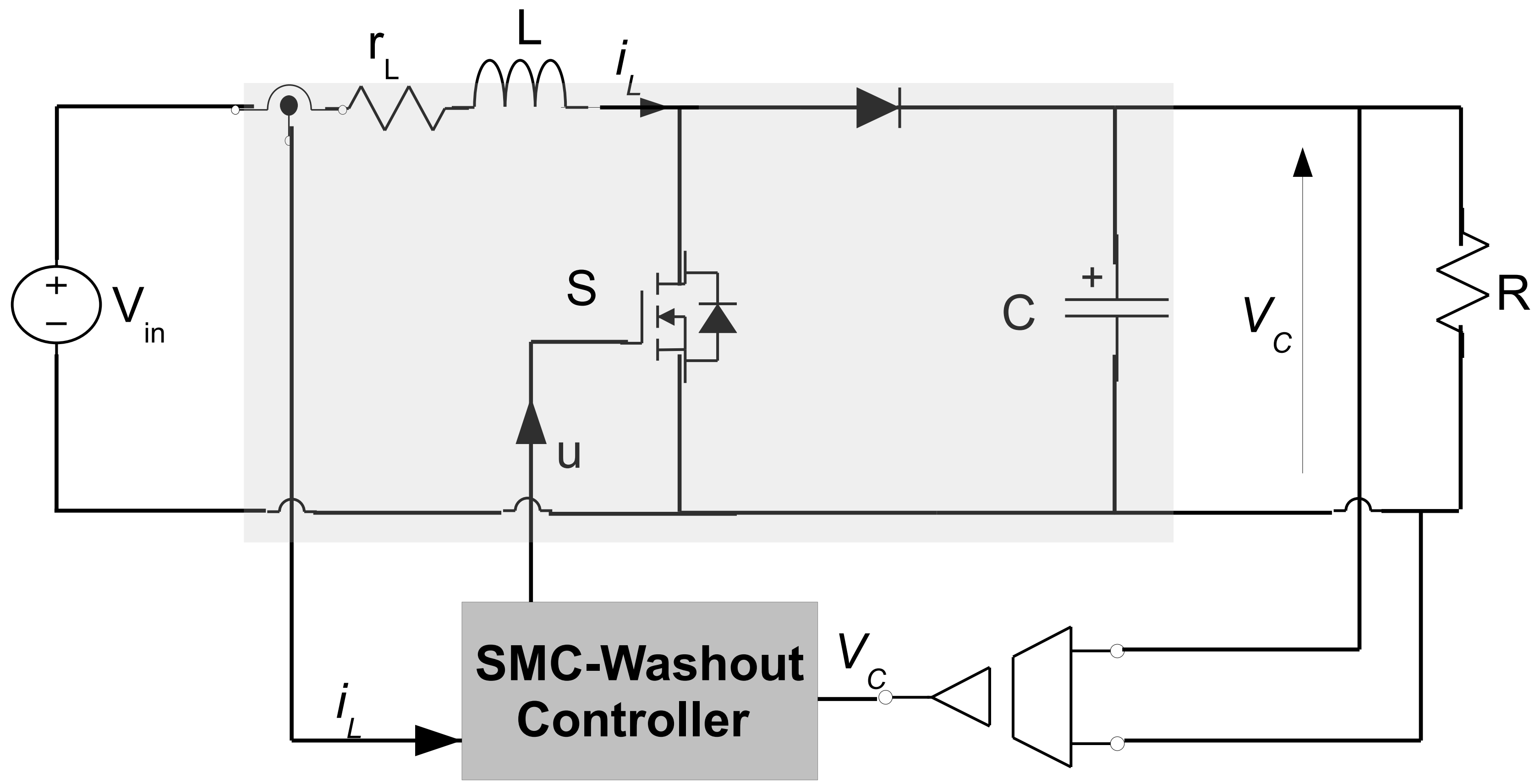
Control of the Boost converter

We want to regulate the output voltage to a desired value $y = v_c/V_{in} = y_r > 1$, ensuring robustness under parameter variation of a, produced by load changes in R.
A washout filter is used: the inductor current x can be filtered to get a new signal x_F by means of a washout filter given by the transfer function

$$G_F(s) = \frac{X_F(s)}{X(s)} = \frac{s}{s+w} = 1 - \frac{w}{s+w},$$

where w is the reciprocal of the filter constant and x_F is the filter output.
A differential equation is added, $\dot{z} = w(x - z)$, where z is a new state satisfying the output equation $x_F = x - z$.

Control of the Boost converter



Control of the Boost converter

The SMC strategy consists in the choice of

$$\Sigma = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 : h(\mathbf{x}) = y - y_r + k(x - z) = 0\}$$

as the switching boundary, where we want to be located the pseudo-equilibrium point.

Control parameter

We use the vector fields defined by

$$\mathbf{F}^+(\mathbf{x}) = \begin{bmatrix} 1 - bx - y \\ x - ay \\ w(x - z) \end{bmatrix} \quad \text{and} \quad \mathbf{F}^-(\mathbf{x}) = \begin{bmatrix} 1 - bx \\ -ay \\ w(x - z) \end{bmatrix},$$

for $h(\mathbf{x}) > 0$ and $h(\mathbf{x}) < 0$, respectively.

In what follows, we assume for simplicity $b = 0$ (ideal inductance), $w = 1$ and $y_r = 2$ (doubling the voltage).

Control of the Boost converter

Therefore, we work with

$$\Sigma = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 : h(\mathbf{x}) = y - 2 + k(x - z) = 0\}$$

Control parameter

and

$$\mathbf{F}^+(\mathbf{x}) = \begin{bmatrix} 1 - y \\ x - ay \\ x - z \end{bmatrix} \quad \text{and} \quad \mathbf{F}^-(\mathbf{x}) = \begin{bmatrix} 1 \\ -ay \\ x - z \end{bmatrix},$$

Load parameter

for $h(\mathbf{x}) > 0$ and $h(\mathbf{x}) < 0$, respectively.

We need to compute $\nabla h \cdot \mathbf{F}^{\pm}$ on Σ, and look for Σ_{as}, namely

$$\nabla h \cdot \mathbf{F}^+(\mathbf{x})\big|_{\Sigma} = [k, 1, -k] \begin{bmatrix} 1 - y \\ x - ay \\ x - z \end{bmatrix} = k(1-y) + x - ay - k(x-z) = x + (1-a-k)y + k - 2 < 0,$$

and

$$\nabla h \cdot \mathbf{F}^-(\mathbf{x})\big|_{\Sigma} = [k, 1, -k] \begin{bmatrix} 1 \\ -ay \\ x - z \end{bmatrix} = k - ay - k(x-z) = (1-a)y + k - 2 > 0.$$

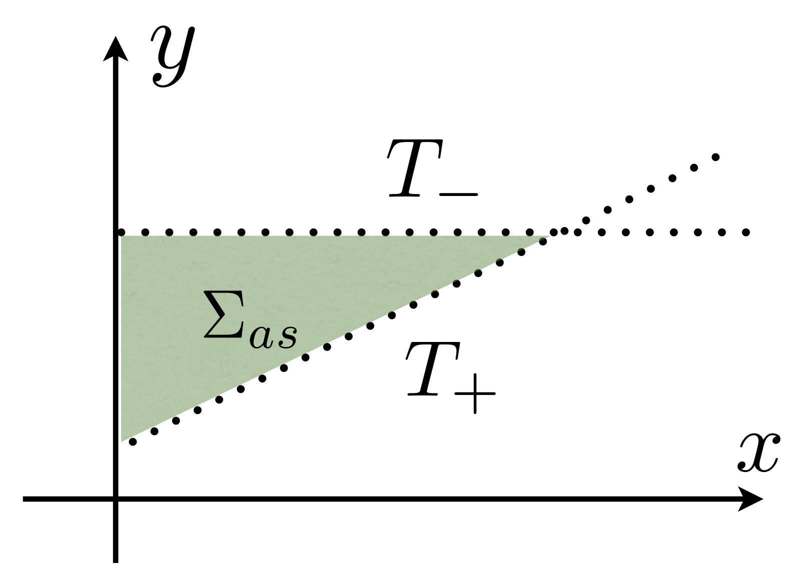
TS-point in a DC-DC Boost converter

The tangency lines are

$$T_- = \{\mathbf{x} \in \Sigma : \ (1-a)y + k - 2 = 0\},$$
$$T_+ = \{\mathbf{x} \in \Sigma : \ x + (1-a-k)y + k - 2 = 0\}.$$

The double tangency point occurs where the tangency lines intersect transversally, i.e., at the point $\hat{\mathbf{x}} = (k\hat{y}, \hat{y}, \hat{z})$, with

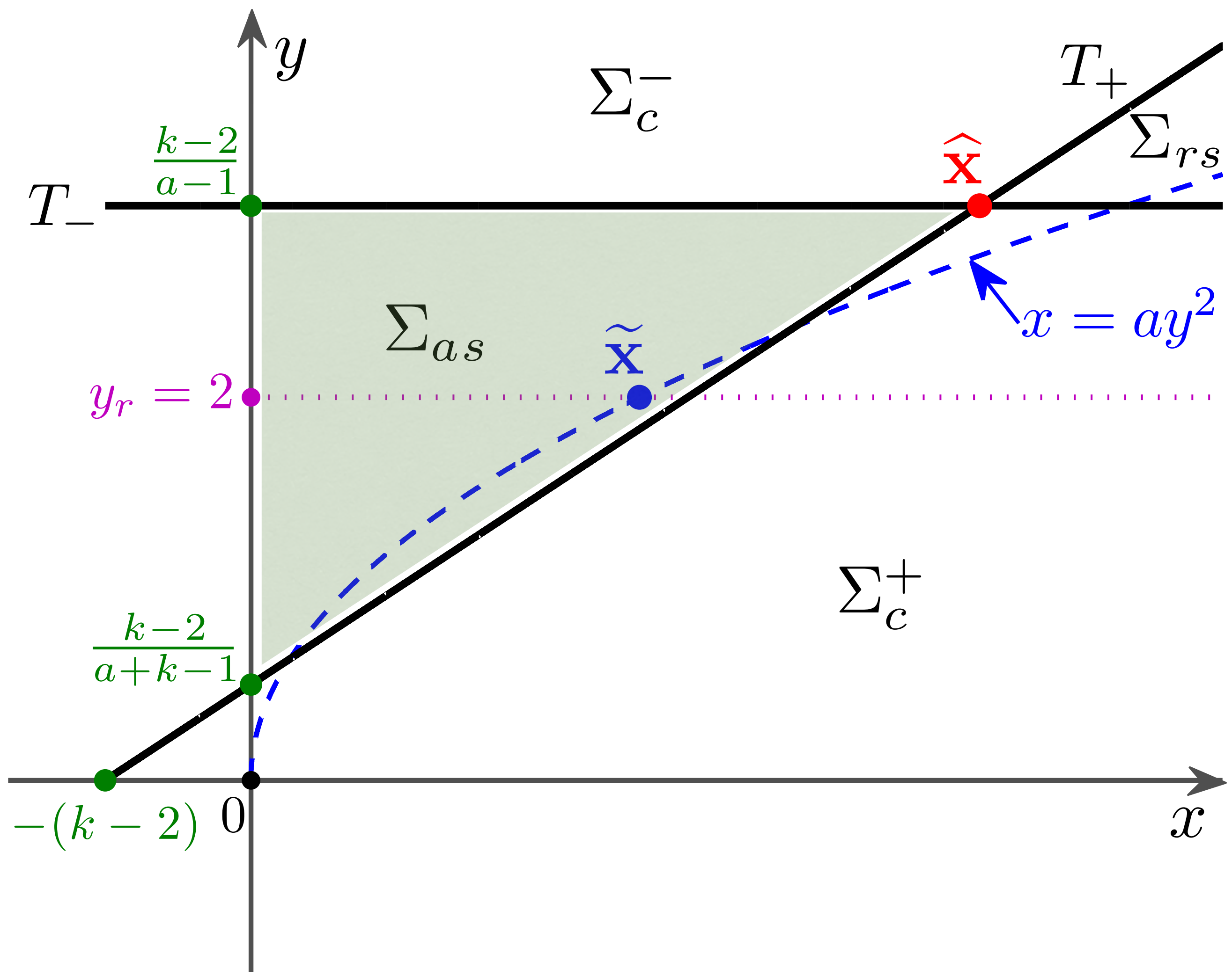
$$\hat{y} = \frac{k-2}{a-1},$$
$$\hat{z} = \frac{(1+k^2)\hat{y} - 2}{k}.$$



where we assume $a \neq 1$.

To have $\hat{y} > 0$, we will consider $\text{sign}(k-2) = \text{sign}(a-1)$.

TS-point in a DC-DC Boost converter



TS-point in a DC-DC Boost converter

The pseudo-equilibrium point is at $(\tilde{x}, \tilde{y}, \tilde{z}) = (4a, 2, 4a)$ to be in Σ_{as} only if $k > 2a$. Thus, we must expect to have the compound bifurcation for $k = 2a$.

First, we need to check the conditions **(H2)** to have a TS-point. Computations lead to the two conditions

$$k > 2, \quad 1 < a < \frac{k - 3 + \sqrt{17 - 34k + 37k^2 - 20k^3 + 4k^4}}{2(k-2)}.$$

Assuming these inequalities, we put the system in the canonical form and compute the critical coefficients.

We obtain for the new coefficients

$$v_- = \frac{1 - a^2(k-2) - a(3 - 3k + k^2)}{(a-1)\omega_+\omega_-}, \quad v_+ = \frac{(a-k)(k-2)}{\omega_+\omega_-},$$

along with

$$\begin{aligned}
g_2^- &= h_1^- = h_2^- = 0, & g_1^- &= -\frac{1}{k\omega_-\omega_+}, & f_1^- &= -\frac{1}{\omega_-}, \\
g_3^- &= \frac{a^2k + a(1-k+k^2) - 1}{(a-1)k\omega_+}, & h_3^- &= -\frac{a}{\omega_-}, & f_1^+ &= -\frac{k-1}{k\omega_+}, \\
g_1^+ &= \frac{k^2 + (a-2)k + 1}{k^2\omega_+^2}, & g_2^+ &= -\frac{k^2 + (a-1)k + 1}{k\omega_+}, & h_1^+ &= -\frac{a-1}{k\omega_-\omega_+}, \\
g_3^+ &= -\frac{(k^2 + (a-2)k + 2 - a)\omega_-}{(a-1)\omega_+^2}, & h_2^+ &= \frac{a-1}{\omega_-}, & h_3^+ &= \frac{k-1}{\omega_+},
\end{aligned}$$

where the two conditions

$$\omega_-^2 = a(k-2) > 0, \quad \omega_+^2 = \frac{(k-1)^3 + (k-3)a - (k-2)a^2}{a-1} > 0,$$

are guaranteed from Hypothesis (H2).

TS-point in a DC-DC Boost converter

The bifurcation curve in the (a, k)-plane of parameters turns out to be

$$v_- v_+ - 1 = (2a - k)\hat{y} = 0,$$

i.e.,

$$k = 2a.$$

Moreover, we have

$$v_+\Big|_{k=2a} = \frac{2a(1-a)}{\omega_+\omega_-} < 0 \quad \text{and} \quad v_-\Big|_{k=2a} = -\frac{(1-a)^2 + 5a^2}{\omega_+\omega_-} < 0$$

for all $a > 1$. The sliding bifurcation is supercritical, since

$$\kappa_S\Big|_{k=2a} = \frac{2}{((1-a)^2 + 5a^2)^{\frac{3}{2}}} > 0.$$

TS-point in a DC-DC Boost converter

Regarding the crossing bifurcation, we obtain

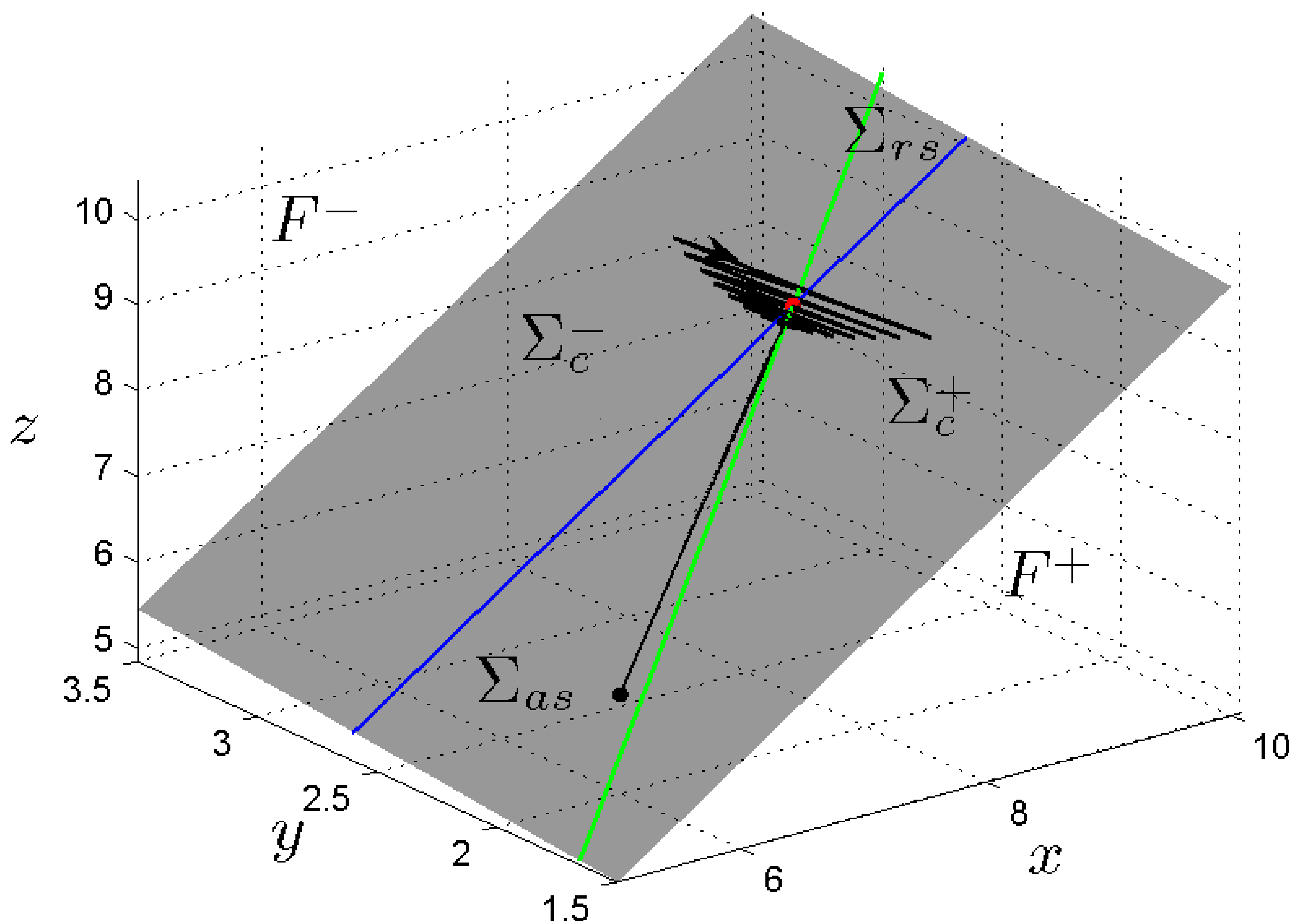
$$\sigma\Big|_{k=2a} = \frac{2(5 - 4a + 16a^2 + 24a^3)}{3(1 - 2a + 6a^2)^{\frac{3}{2}}} > 0,$$

and

$$\kappa_C = \frac{2}{3((1-a)^2 + 5a^2)^2} > 0,$$

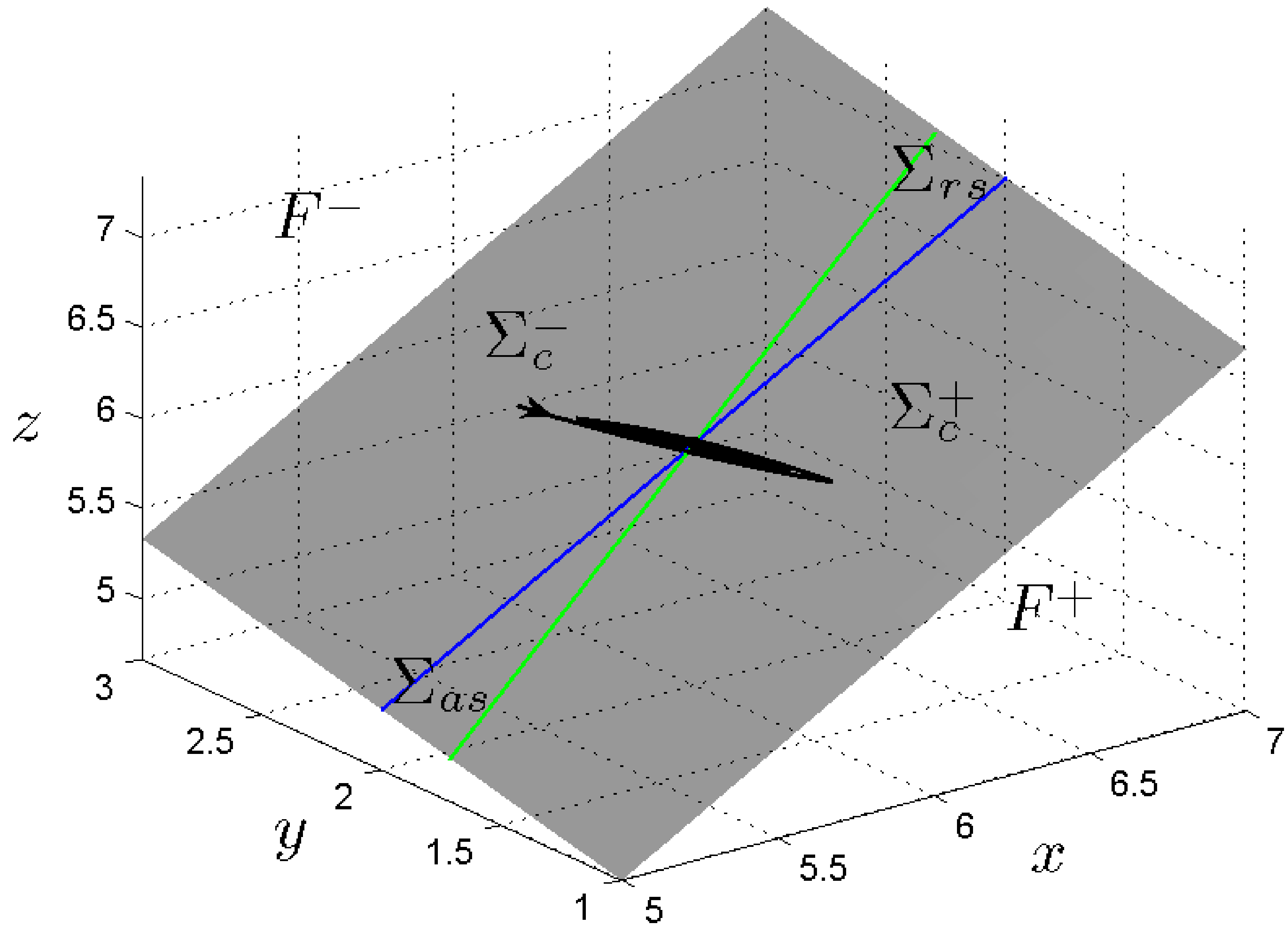
so that the bifurcating periodic orbit for $k < 2a$ is of **stable node** type.

Simulation results



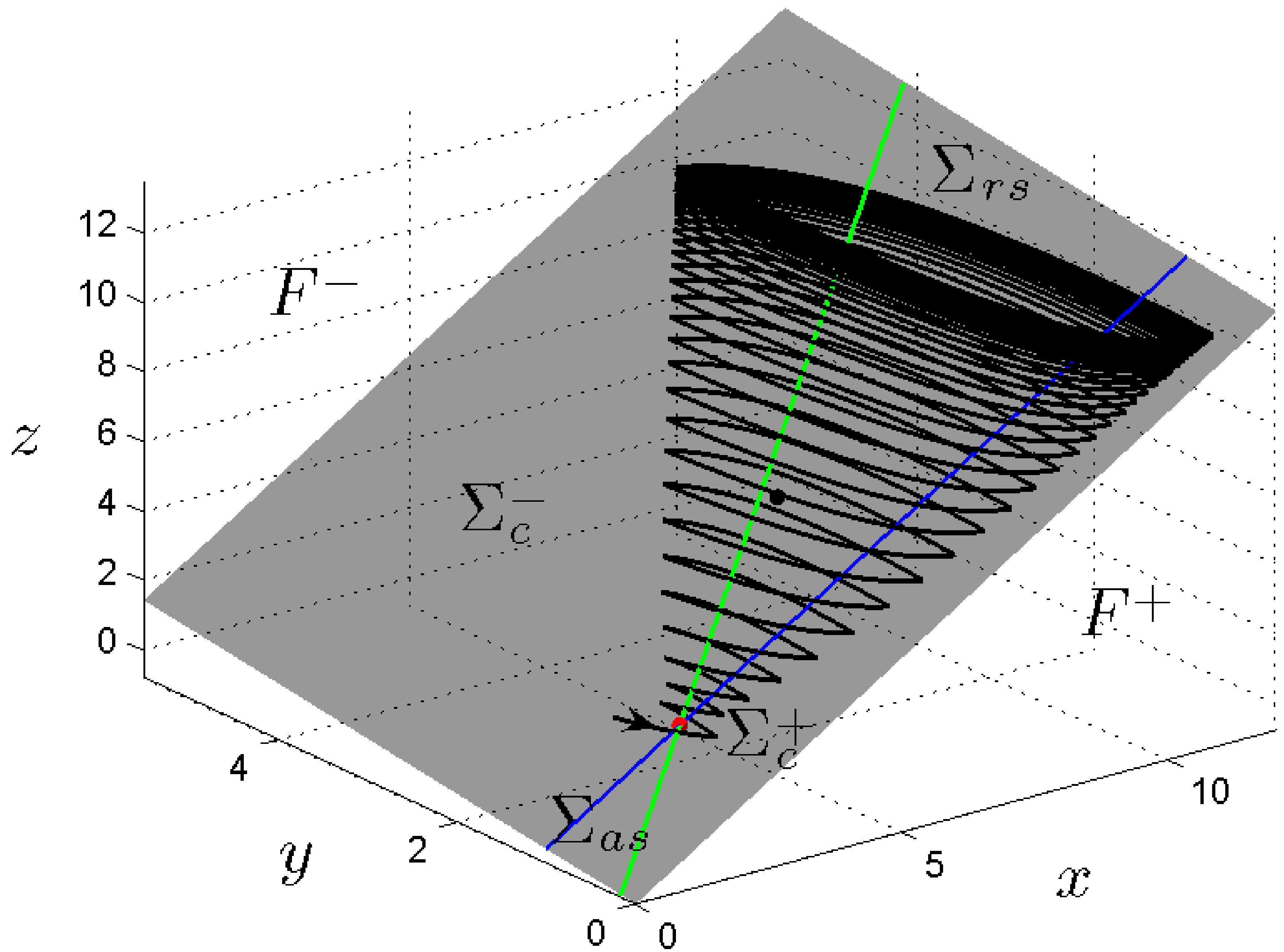
$$3.3 = k > 2a = 3$$

Simulation results



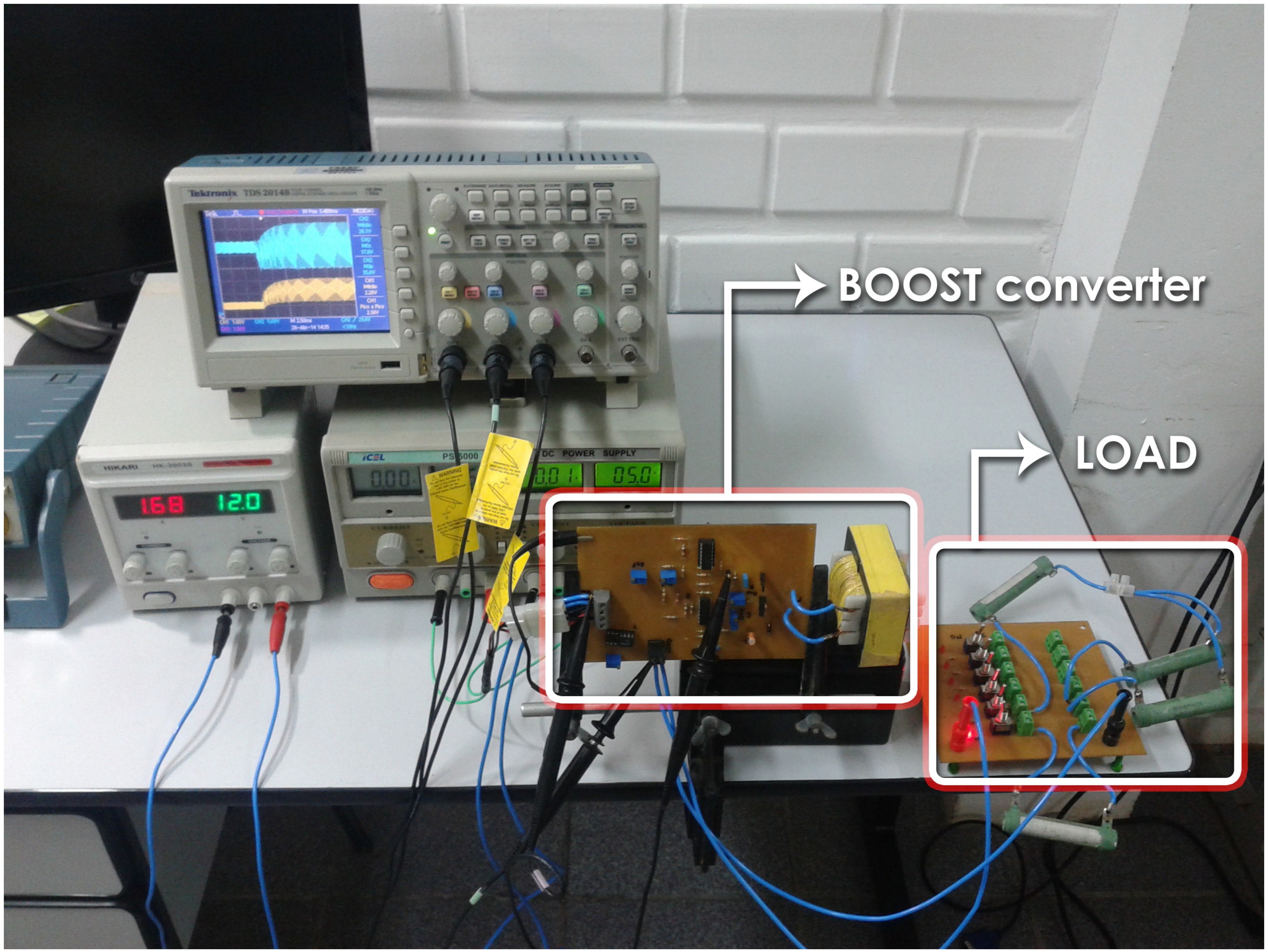
$$k = 2a = 3$$

Simulation results



$$2.5 = k < 2a = 3$$

Undesirable oscillation observed in laboratory due to the TS-point in the DC-DC Boost converter



Conclusions

- The TS compound bifurcation has been characterized for piecewise linear systems, and a procedure for computing the essential coefficients has been provided.

- Several examples have been shown, and in particular the bifurcation is detected in DC-DC converters under SMC strategy with a washout filter.

- A codimension-two unfolding is still needed, gaining information on secondary bifurcation curves.

- A question to solve: Can this study with linear pieces serve as a normal form for nonlinear vector fields?

Thank you for your attention!