Lie Symmetries and non-integrability of maps: the Cohen map case.

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A general framework

In the last years we have been focusing on the dynamics of *integrable maps*, and how it can be studied using some *associated vector fields*, called *Lie Symmetries*.

A planar map F is *integrable* if there exists a non-constant function V such that

$$V(F(x,y)) = V(x,y),$$

i.e. the orbits of $(x_{n+1}, y_{n+1}) = F(x_n, y_n)$ lay on the curves

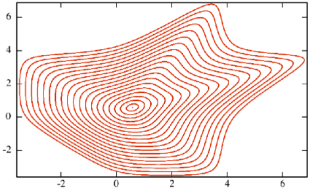
$$\{V(x,y) = h\}$$

The Cohen map case

Is the Cohen map

$$G(x,y) = \left(y, -x + \sqrt{y^2+1}\right)$$

integrable?



It seems that the non-integrability of the Cohen map was first conjectured by Cohen and communicated by C. de Verdière to Moser in 1993. In 1998 Rychlik and Torgesson shown that it *has not first integrals given by algebraic functions*.

This map is considered unlikely to be integrable since *numerical explorations* show that it has *isolated* hyperbolic periodic points and chains of islands of period 14 and 23 Petalas et al. 2009.

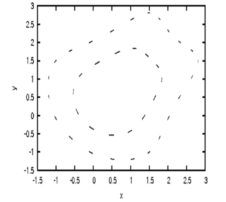


Figure from Petalas, Antonopoulos, Bountis, Vrahatis. Physics Letters A 373 (2009)

We face the problem inspired by a comment of G. Lowther at Mathoverflow given in:

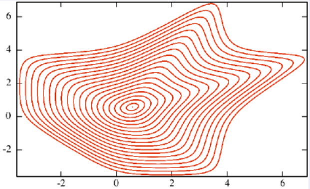
`http://mathoverflow.net/questions/93914/integrability-of-the-cohen-map`

Theorem (the Cohen map)

The Cohen map is NOT $\mathcal{C}^6$-locally integrable at its *elliptic* fixed point $\left(\sqrt{3}/3, \sqrt{3}/3\right)$.

A planar map F is *$\mathcal{C}^m$-locally integrable at an elliptic fixed point p* if $\exists$ a neighborhood $\mathcal{U}$ of p and a first integral $V \in \mathcal{C}^m(\mathcal{U})$ with $m \geq 2$, such that

- All the level curves $\{V = h\} \cap \mathcal{U}$ are closed curves surrounding p,
- p is an isolated non-degenerate critical point of V in $\mathcal{U}$.



The map is a *measure preserving map* if

$$m(F^{-1}(B)) = m(B)$$

for any measurable set B, where $m(B) = \int_B \nu(x,y)\,dxdy$, and $\nu|_{\mathcal{U}} \neq 0$ is called the *density*.

Birkhoff normal form at a k-resonant elliptic point

- A fixed point p of a $\mathcal{C}^1$-real planar map F is elliptic when the eigenvalues of $DF(p)$ have modulus one, i. e. they are $\lambda, \bar{\lambda} = 1/\lambda$.
- When the eigenvalues *are not roots of unity* of order ℓ for $0 < \ell \leq k$ we will say that p is **NOT** k-resonant.
- A $\mathcal{C}^{k+1}$-map, with a not k-resonant fixed point p, is locally conjugate to its *Birkhoff normal form*:

$$F_B(z) = \lambda z \left(1 + \sum_{j=1}^{[(k-1)/2]} B_j (z\bar{z})^j\right) + O(|z|^{k+1}), \tag{1}$$

where $z = x + iy$, and $[\cdot]$ denotes the integer part.

Recall our well-known normal form for weak-focus of planar vector fields

$$\dot{z} = iz \left(1 + \sum_{j=1}^{k-1} a_j (z\bar{z})^j + A_k (z\bar{z})^k\right) + O(|z|^{2k+3})$$

Proposition

For $1 \leq n \in \mathbb{N}$, consider a $\mathcal{C}^{2n+2}$-map F with an elliptic fixed point $p \in \mathcal{U}$, not $(2n+1)$-resonant. Let B_n be its first non-vanishing Birkhoff constant.

If $\mathrm{Re}(B_n) < 0$ (respectively $\mathrm{Re}(B_n) > 0$), the point p is a local attractor (respectively repeller point).

They are the discrete version of our Lyapunov constants.

In particular the map is not $\mathcal{C}^2$-locally integrable at p.

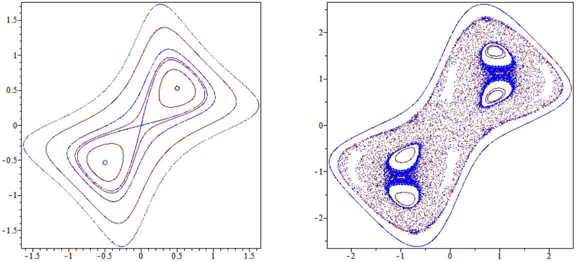
It suffices to prove that $V(z) = z\bar{z} = |z|^2$ is a strict Lyapunov function at the origin for the normal form map F_B of F.

For instance, when $\mathrm{Re}(B_n) < 0$,

$$V(F_B(z)) = |z|^2 \Big| 1 + 2\mathrm{Re}(B_n)(z\bar{z})^n + O(|z|^{2n+1}) \Big| < V(z),$$

for z in a small enough neighborhood of p, as we wanted to prove.

If $\mathrm{Re}(B_n) = 0$ and $\mathrm{Im}(B_n) \neq 0$ either can happen



Proposition B

If $\mathrm{Re}(B_n) = 0$ and $\mathrm{Im}(B_n) \neq 0$ and F is *measure preserving* $\mathcal{C}^{2n+4}$-locally integrable at p, with first integral V, then the *rotation number* $\theta(h)$ associated to each integral curve $\{V = h\} \cap \mathcal{U}$ is not constant.

In particular there exits $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ there is $M \in \mathbb{N}$ coprime with N such that $\theta(h_N) = M/N \in I$, with $\{V = h_N\} \cap \mathcal{U} \neq \emptyset$.

Hence the set $\{V = h_N\} \cap \mathcal{U}$ is full of N-periodic points of F.

We will focus on this point (period function, isochronous centers...).

Theorem A (non-integrability of maps)

Let F be a $\mathcal{C}^{2n+2}$-map defined on an open set $\mathcal{U} \subseteq \mathbb{R}^2$ with

- an elliptic fixed point p, *not $(2n+1)$-resonant*,
- s.t. its first non-vanishing Birkhoff constant is $B_n = i\, b_n$, for some $0 < n \in \mathbb{N}$ and $b_n \in \mathbb{R} \setminus \{0\}$.
- Moreover, assume that F is a *measure preserving* map with a density $\nu \in \mathcal{C}^{2n+3}$.

If, for an unbounded sequence of natural numbers $\{N_k\}_k$, F has *finitely many N_k-periodic points* in $\mathcal{U}$ then

it is **NOT** $\mathcal{C}^{2n+4}$-locally integrable at p.

Lie Symmetries

A *vector field* X is said to be a *Lie symmetry* of F if it satisfies

$$X(F(\mathbf{x})) = (DF(\mathbf{x}))\, X(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathcal{U}.$$

This implies that $\dot{\mathbf{x}} = X(\mathbf{x})$ is invariant under the change of variables given by F. From a dynamic viewpoint *F maps any orbit of $\dot{\mathbf{x}} = X(\mathbf{x})$, into another orbit of this system.*

Theorem (Cima, Gasull, M (2008))

Let X be a Lie Symmetry of a diffeomorphism $F : \mathcal{U} \to \mathcal{U}$. Let γ be an orbit of X invariant under F. Then $F|_\gamma$ is the $\tau-$time map of the flow of X, i.e.

$$F(\mathbf{p}) = \varphi(\tau, \mathbf{p}).$$

(a) If $\gamma \cong \{p\}$ (isolated) then p is a fixed point of F.

(b) If $\gamma \cong \mathbb{S}^1$, then $F|_\gamma$ is conjugated to a rotation, with rotation number $\theta = \tau/T$, where T is the period of γ.

(c) If $\gamma \cong \mathbb{R}$, then $F|_\gamma$ is conjugated to a translation of the line.

Corollary

Let F be a $\mathcal{C}^2(\mathcal{U})$ orientation preserving map with an *invariant measure* with density $0 \neq \nu \in \mathcal{C}^1(\mathcal{U})$ and with a first integral $V \in \mathcal{C}^2(\mathcal{U})$. Then

(a) The vector field

$$X = \frac{1}{\nu}\left(-V_y \frac{\partial}{\partial x} + V_x \frac{\partial}{\partial y}\right)$$

is a Lie Symmetry of F.

(b) If a connected component γ_h of $\{V(x) = h\}$ without fixed points is invariant by F and $\gamma_h \cong \mathbb{S}^1$, then

$$F(\mathbf{p})|_{\gamma_h} = \varphi(\tau(h), \mathbf{p})$$

is conjugate to a rotation with rotation number

$$\theta(h) = \frac{\tau(h)}{T(h)}.$$

(c) Therefore, if $F|_{\gamma_h}$ has rotation number $\theta(h) = q/p \in \mathbb{Q}$, with $\gcd(p, q) = 1$, then $\gamma_h \subset \mathcal{U}$ is a continuum of p-periodic points of F.

Theorem A

Consider F measure preserving with an elliptic fixed point p, *not* $(2n+1)$*-resonant*, with Birkhoff constant is $B_n = i\, b_n \in i\mathbb{R}$. Assume that $\{N_k\}_k$, F has *finitely many* N_k*-periodic points* in $\mathcal{U}$ then it is **NOT** $\mathcal{C}^{2n+4}$-locally integrable at **p**.

Suppose that F is locally integrable:

- Since F is *integrable* it preserves an *invariant measure* with a smooth density $\Rightarrow$ it posesses a smooth *Lie symmetry* X .
- Since **p** is a non-degenerate critical point of V (locally integrable) $\Rightarrow$ **p** is a non-degenerate center of X $\Rightarrow$ The dynamics of $F_{|\{V=h\}}$ is conjugate to a rotation and $\theta(h)$ is continuous.

  Since it is an elliptic fixed point of F, with non-zero purely imaginary *Birkhoff constant*, then the rotation number $\theta(h)$ is not constant.

  $$\Downarrow$$

  There should exist closed level sets s.t. on them F has rational rotation numbers with all denominators bigger that some N_0 (Proposition B).

  Since $\exists$ a Lie symmetry X, *on these levels we have continua* of such periodic points.
- But we are assuming that for an unbounded sequence of natural numbers $\{N_k\}_k$, F has *finitely many* N_k*-periodic points* in $\mathcal{U}$: a contradiction.

Proposition B

Measure preserving "+" $\mathcal{C}^{2n+4}$-locally integrable at p integrable "+" $B_n = i\,b_n \Rightarrow \theta(h)$ associated to each curve $\{V = h\}$ is not constant (continua of all periods $N \geq N_0$).

Supose that $\theta(h)$ is constant. We will prove that F is globally $\mathcal{C}^{2n+2}$-conjugate to the linear map $L(q) = DF(p)\,q$ (as it happens with isochronous centers).

- F posesses a smooth *Lie symmetry* $X = \frac{1}{\nu}\big(-V_y, V_x\big)$ of class $\mathcal{C}^{2n+3}$ with a *non-degenerate center* at p, in fact

$$DF(p) = e^{\tau_p\, DX(p)}, \text{ where } \tau_p = \lim_{h \to h_p} \theta(h) T(h).$$

- The new vector field $Y(x, y) = T(x, y)\, X(x, y)$, is also a Lie Symmetry of F of class $\mathcal{C}^{2n+2}(\mathcal{U})$, having an *isochronous* center at p with period function $T(h) \equiv 1$. Hence,

$$F(q) = \varphi_Y(\tau, q).$$

  with τ constant (not depending on h).

- The isochronous center Y linearizes. Since $DF(p) = e^{\tau\, DY(p)}$, we prove that the "Bochner"-type map

$$\Phi(q) = \int_0^1 \mathrm{e}^{-DY(p)\,s}\, \varphi_Y(s, q)\, ds,$$

  is a $\mathcal{C}^{2n+2}$-conjugation between F and the linear map $L(q) = DF(p)\,q$.

But F is $\mathcal{C}^{2n+2}$-conjugated to the Birkhoff normal form, which is non linear, a contradiction.

Mind the gap!

$X = \frac{1}{\nu}\left(-V_y, V_x\right)$ is $\mathcal{C}^{2n+3}(\mathcal{U})$ but $Y(x,y) = T(x,y)\,X(x,y)$, is $\mathcal{C}^{2n+2}(\mathcal{U})$.

Theorem C (regularity of the period function)

For non-degenerate centers the period function T is:

- Of class $\mathcal{C}^\omega$ (resp. $\mathcal{C}^\infty$) if the vector field is $\mathcal{C}^\omega$ (resp. $\mathcal{C}^\infty$), Villarini (1992).
- Of class $\mathcal{C}^{k-1}$ if the vector field is $\mathcal{C}^k$, $k \in \mathbb{N}$.

The regularity of T at p can not be improved.

For $a \in \mathbb{R}$ consider the vector field, with associated differential system,

$$\begin{cases} \dot{x} = -y\left(1 + (x^2+y^2)^a\right), \\ \dot{y} = x\left(1 + (x^2+y^2)^a\right), \end{cases}$$

Its period function is

$$T(x,y) = \frac{2\pi}{1 + (x^2+y^2)^a}.$$

Taking $a = k/2$ when k is odd, or $a = k/(k+1)$ when k is even, we obtain $\mathcal{C}^k$-vector fields such that its corresponding period function is of class $\mathcal{C}^{k-1}$, and not of class $\mathcal{C}^k$ at the origin.

We also need to check whether a map $F : \mathbb{R}^M \to \mathbb{R}^M$ has finitely many K-periodic points, i.e. when the system

$$\left\{\begin{array}{l} \mathbf{x}_1 - F(\mathbf{x}_0) = \mathbf{0} \\ \mathbf{x}_2 - F(\mathbf{x}_1) = \mathbf{0} \\ \qquad\vdots \\ \mathbf{x}_{K-1} - F(\mathbf{x}_{K-2}) = \mathbf{0} \\ \mathbf{x}_0 - F(\mathbf{x}_{K-1}) = \mathbf{0} \end{array}\right. \tag{2}$$

has finitely many real solutions.

"Our" result states:

Theorem D

Let $G : \mathbb{C}^N \to \mathbb{C}^N$ be a polynomial map. Let G_d denote the homogenous map corresponding to the maximum degree d terms of G. If $\mathbf{y} = \mathbf{0}$ is the unique solution in $\mathbb{C}^N$ of the homogeneous system $G_d(\mathbf{y}) = \mathbf{0}$, then $G(\mathbf{y}) = \mathbf{0}$ has finitely many solutions.

To make Thm A. effective, we try to apply Thm D. to the system (2).

The Cohen map writes as the difference equation

$$x_{n+2} = -x_n + \sqrt{x_{n+1}^2 + 1}.$$

Their solutions are contained in

$$(x_n + x_{n+2})^2 - x_{n+1}^2 - 1 = 0.$$

Therefore the N-periodic orbits satisfy the system

$$\left\{ \begin{array}{c} (x_1 + x_3)^2 - x_2^2 - 1 = 0, \\ (x_2 + x_4)^2 - x_3^2 - 1 = 0, \\ \vdots \\ (x_{N-1} + x_1)^2 - x_N^2 - 1 = 0, \\ (x_N + x_2)^2 - x_1^2 - 1 = 0. \end{array} \right.$$

Applying our result, to prove the non-existence of continua of N-periodic solutions it suffices to prove that $\mathbf{0}$ is the only solution of each of the systems:

$$\begin{cases} x_1 + x_3 = \pm x_2, \\ x_2 + x_4 = \pm x_3, \\ \vdots \\ x_{N-1} + x_1 = \pm x_N, \\ x_N + x_2 = \pm x_1, \end{cases}$$

for infinitely many N, or equivalently: We will prove that, for some unbounded sequence of values of N, $\mathbf{x} = \mathbf{0}$ is the unique solution of the linear systems

$$A_N \mathbf{x} = \mathbf{0},$$

where A_N are the $N \times N$ matrices,

$$A_N(\varepsilon_1, \ldots, \varepsilon_N) = \begin{pmatrix} 1 & \varepsilon_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \varepsilon_2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \varepsilon_1 & 1 & \cdots & 0 \\ & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & \varepsilon_{N-2} & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & \varepsilon_{N-1} \\ \varepsilon_N & 1 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix},$$

with $\varepsilon_j \in \{-1, 1\}$, for each $j = 1, \ldots, N$.

Lemma

For every choice of $\varepsilon_j \in \{-1, 1\}$, with $j = 1, \ldots, n$, and for all $N \neq \dot{3}$,

$$\det(A_N(\varepsilon_1, \ldots, \varepsilon_N)) \equiv F_N \mod 2,$$

where F_N are the celebrated Fibonacci numbers.

Recall that F_N are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

and modulus 2,

$$1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$$

Corollary

For all $N \neq \dot{3}$, the Cohen map has finitely many N-periodic points.

The computation of the Birkhoff normal form is a well-known issue. In particular, for a map of the form

$$F(x,y)=\left(\lambda x+\sum_{i+j\geq 2} f_{i,j}x^iy^j,\frac{1}{\lambda}y+\sum_{i+j\geq 2} g_{i,j}x^iy^j\right),$$

is:

$$B_1=\frac{\mathcal{P}_1(F)}{\lambda^2\left(\lambda-1\right)\left(\lambda^2+\lambda+1\right)},$$

where

$$\mathcal{P}_1(F)=$$
$$\left(f_{11}g_{11}+f_{21}\right)\lambda^4-f_{11}\left(2f_{20}-g_{11}\right)\lambda^3+\left(2f_{02}g_{20}-f_{11}f_{20}+f_{11}g_{11}\right)\lambda^2-\left(f_{11}f_{20}+f_{21}\right)\lambda+f_{11}f_{20}.$$

Lemma

The first Birkhoff's coefficient of the Cohen Map at the elliptic fixed point $(\sqrt{3}/3,\sqrt{3}/3)$ is $B_1=135/256i$.

Theorem (the Cohen map)

The Cohen map is NOT $\mathcal{C}^6$-locally integrable at its *elliptic* fixed point $\left(\sqrt{3}/3, \sqrt{3}/3\right)$.

Suppose that G is locally integrable:

- G posesses a *Lie symmetry* because it is *measure preserving*.
- At the fixed point $\left(\sqrt{3}/3, \sqrt{3}/3\right)$, $B_1 = 135/256i \neq 0$.
- Since $\exists$ a Lie symmetry, there should exists level sets with continua periodic points for all $N \geq N_0$.
- But we have proved that $\forall N \neq \dot{3}$, G has finitely many N-periodic points, a contradiction.

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THANK YOU!