Mixed dynamics in planar reversible maps

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Outline

1. Newhouse regions and mixed dynamics in Reversible systems
2. Two words on Reversible systems
3. Statement of the problem
4. Some remarks about the proof: key points
   - Construction of the single-round map $\mathcal{T}_{km}$
   - Local expression around the saddle points
   - Asymptotic behaviour and bifurcations
5. An example

Newhouse regions

- Existence of regions of structural instability was a crucial discovery for Bifurcation Theory.
- Pioneering works of Newhouse 1970, 1974, 1979: there exist open domains in the space of dynamical systems filled densely by systems with saddle fixed points having tangencies of their stable and unstable manifolds along homoclinic orbits
- Moreover, such regions appear in one-parameter families unfolding the initial homoclinic tangency, Gonchenko, Turaev and Shil'nikov 1993.

- Both regions, in the space of dynamical systems and in the parameter space, are commonly called **Newhouse Regions.**

Newhouse regions

- Extension of existence of Newhouse regions to higher dimension systems, dissipative and conservative settings were provided by the following (some of them) references:
  - Gavrilov, Shil'nikov, 1972, 1973.
  - Gonchenko, Shil'nikov, Turaev, 1993, 2007, 2008.
  - Palis, Viana, 1994.
  - Romero, 1995.
  - Duarte, 1994 (APM), 2000.

Mixed dynamics

- Phenomenon of Mixed dynamics, inspired on Newhouse 1979, Gavrilov and Shil'nikov, 1972, 1973 and discovered by Gonchenko, Turaev and Shil'nikov, 1997; Gonchenko, Sten'kin and Shil'nikov, 2002, 2006, is observed in some Newhouse regions.
- We say that a dynamical system exhibits mixed dynamics if:
  - (*i*) it has, simultaneously, infinitely many hyperbolic periodic orbits of all possible types (stable, completely unstable, saddle) and
  - (*ii*) these orbits are not *separated*, i.e., the closures of sets of p.o. of different types have non-empty intersections.

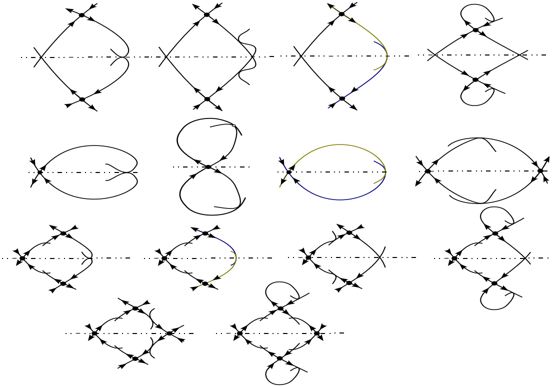
Mixed dynamics

- Newhouse regions arise near diffeos with a non-transversal heteroclinic cycle. Precisely, in Gonchenko, Turaev and Shilnikov 1997 it was shown that the property of mixed dynamics can be generic in sets of 2-dim. diffeos containing at least two saddle fix points with jacobian $J < 1$ and $J > 1$ and a simple non-transversal heteroclinic cycle.

  Heteroclinic cycle: set consisting on saddle fixed points $\mathcal{O}_i$ and heteroclinic orbits $\Gamma_{i,i+1} \in W^-(\mathcal{O}_i) \cap W^+(\mathcal{O}_{i+1})$ and closing. It is called non-transversal if at least one of the previous intersections is non-transversal. One says that is simple if it contains only one non-transversal heteroclinic orbit and this tangency is quadratic.
- Such type of heteroclinic cycles are rather usual in reversible maps.

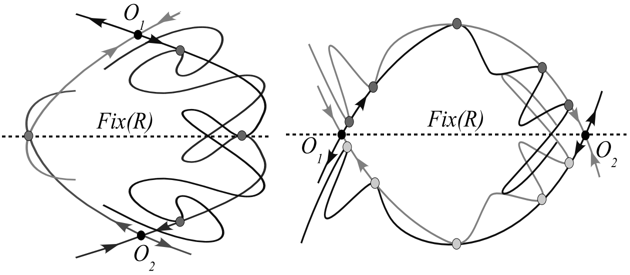
The Reversible mixed dynamics conjecture

Reversible Mixed Dynamics Conjecture (Gonchenko, Turaev, Lamb, Sten'kin, Delshams, L):
Two-dimensional reversible maps with mixed dynamics are generic in Newhouse regions where maps with symmetric homoclinic or/and heteroclinic tangencies are dense.



The Reversible mixed dynamics conjecture

In this Reversible setting, we have a first work of J. Lamb and O. Sten'kin, 2004, where mixed dynamics is proved in case (a): a couple of symmetric saddle points with quadratic heteroclinic tangency.



(a) (b)

Mixed dynamics for the second case has been proved by Delshams, S. Gonchenko, V. Gonchenko, L and Sten'kin 2013.

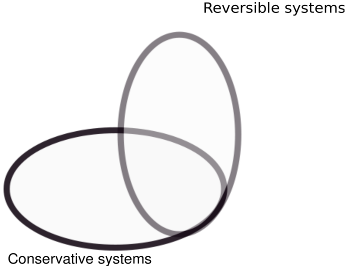
The Reversible mixed dynamics conjecture

Why considering a reversible setting?

- Reversibility has been considered during decades essentially a tool to reduce and simplify analysis and computations of conservative systems. From the works of Devaney, Moser, Bibikov, Arnol'd, Sevryuk and many others (70's and 80's) they started to be considered interesting by themselves.
- Reversible systems can exhibit simultaneously conservative and dissipative-like behaviour.

Reversible dynamical systems: definition

- **Continuous DS**: $\dot{x} = F(x)$ is (time)-reversible if it is invariant under the action of an involution in the spatial variables R and a change in time's arrow, that is $R_* F = -F$.
- **Discrete DS**: a diffeo $\bar{x} = f(x)$ is reversible if there exists an involution R such that $f^{-1} = R \circ f \circ R$.
- Remark: R involution ($R^2 = \mathrm{Id}$, $R \neq \mathrm{Id}$) non necessarily linear. If R is just a diffeo they are called weakly reversible.



Planar reversible diffeomorphisms: definition

- Def.: $f : \mathbb{R}^2 \to \mathbb{R}^2$ diffeo is reversible iff (equivalent definitions):
  - $\exists$ R involution s.t. $f \circ R \circ f = R$,
  - $\exists$ R involution s.t. $f^{-1} = R \circ f \circ R$,
  - $\exists$ H, G involutions s.t. $f = H \circ G$.

  A reversible map has infinitely many symmetries (for instance, $f^k \circ R$, $k \in \mathbb{Z}$). They are also called reversors of the map.
- Mainly in the 80's. Authors: Devaney, Moser, Bibikov, Webster, Arnol'd, Sevryuk, Vanderbauwhede, Teixeira, Lamb, Roberts, Quispel, Capel, Broer, Iooss, Gaeta, Cicogna, Walcher, Lombardi, Champneys, Tkhai, ... and many others.
- They can exhibit simultaneously conservative and dissipative structures.

Planar reversible systems: examples

- Example 1 (APM): Standard map $SM : \begin{cases} \bar{x} &= x + \bar{y}, \\ \bar{y} &= y + \sin x. \end{cases}$

  We have that $SM = H \circ G$ where H, G are the involutions:

$$H : \begin{cases} \bar{x} &= x - y, \\ \bar{y} &= -y, \end{cases} \qquad G : \begin{cases} \bar{x} &= x, \\ \bar{y} &= -y - \sin x \end{cases}$$

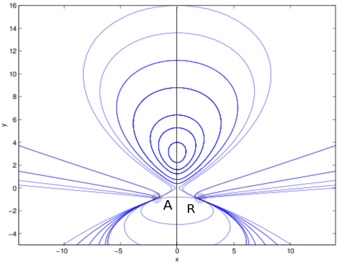
  H is linear but G is not.

Planar reversible systems: examples

- Example 2 (ode,non-hamilt): A. Politi, G.L. Oppo, R. Badii, 1986
  2-dim. simplified model of a class B injected laser:

$$\left\{\begin{array}{lcl}\dot{x} & = & f_1 y + f_2 y^2 + f_3 x^2, \qquad f_i, g_i \in \mathbb{R} \\ \dot{y} & = & g_1 x + g_2 xy\end{array}\right.$$

If $f_1 g_1 < 0$ and $f_1(f_1 g_2 - f_2 g_1) f_2 < 0$ then 4 eq. points: 2 symm. (Ellip + Hyper) and 2 non-symm. (Atract + Repell)

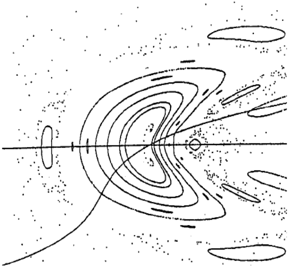


Planar reversible systems: examples

- Example 3 (non-conserv.diffeo): J. Roberts and G. Quispel, 1992

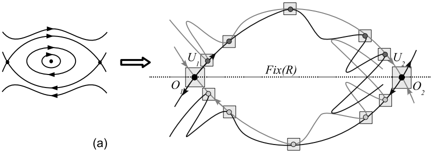
$$\begin{cases} \bar{x} &= (C-y)\left(1+(\bar{y}-1)^2\right), \\ \bar{y} &= \dfrac{x}{1+(C-y-1)^2}, \end{cases}$$

is G-reversible, where $G(x,y) = (x, C-y)$. For instance for $C = 2.87$ we have an attractor and a repeller:

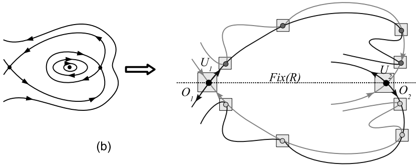


The problem

Aim: to find mixed dynamics for the cases of (a) “inner quadratic tangency” and (b) “outer quadratic tangency”.



(a)



(b)

Example (a): Poincaré map of $\dot{x} = y, \quad \dot{y} = -x + x^3 + \varepsilon(\alpha + \beta y \sin \omega t)$.
Example (b): Pikovsky-Topaj model (2002) of three coupled rotators on the 3d-torus.

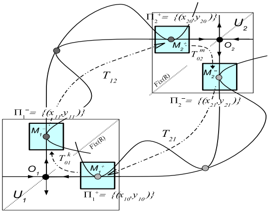
Hypotheses

Let consider $f_0 : \mathbb{R}^2 \to \mathbb{R}^2$, $f_0 \in \mathcal{C}^r$, $r \geq 4$, R-reversible, dim Fix$R = 1$.

[A] f_0 has two symmetric fixed points $\mathcal{O}_1$, O_2 (that is $\mathcal{O}_1, \mathcal{O}_2 \in$ Fix$R$).

- Multipliers $\{\lambda_i, \lambda_i^{-1}\}$, $0 < |\lambda_i| < 1$, $i = 1, 2$
- Reversibility implies: $W^-(\mathcal{O}_i) = RW^+(\mathcal{O}_i)$, $W^+(\mathcal{O}_i) = RW^-(\mathcal{O}_i)$.

[B] Symmetric non-transversal heteroclinic cycle C with quadratic tangency: $C = \{O_1, O_2, \Gamma_{12}, \Gamma_{21} = R(\Gamma_{12})\}$



[C] Non-degeneracy condition of T_{12}: $\partial_2 J(T_{12})|_{M_1^-} \neq 0$.
$J(T_{12})) > 0$ orientable case; $J(T_{12}) < 0$ non-orientable case.

Main result

Theorem

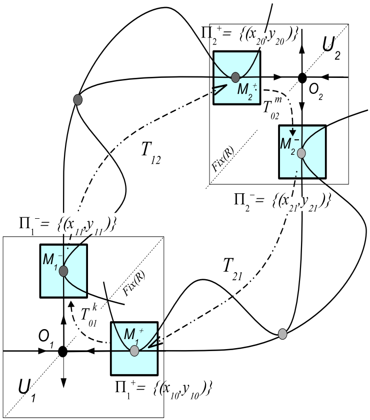
Let $\{f_\mu\}$ 1-parameter family of R-reversible diffeos s.t. f_0 satisfies hypotheses $[\mathrm{A}]$, $[\mathrm{B}]$ and $[\mathrm{C}]$. Then, any interval $[-\varepsilon,\varepsilon]$ contains infinitely many subintervals $\delta_{km}=[\mu^{(k,m)}_{fold},\mu^{(k,m)}_{pf}]$ (non-necessarily disjoints) with $\delta_{km}\to\{0\}$ as $k,m\to\infty$ and satisfying:

- *$\mu=\mu^{(k,m)}_{fold}$ corresponds to a non-degenerate conservative fold bifurcation: –¿ 2 single-round symmetric p.o. (saddle + elliptic).*
- *$\mu=\mu^{(k,m)}_{pf}$ corresponds to symmetric pitch-fork bifurcation:*
  - *Inner tangency case: symmetric couple of attracting and repelling p.o. which undergo later a simultaneous period-doubling bifurc.*
  - *Outer tangency case: symmetric couple of contracting-expanding p.o. No further bifurcations.*

Single-round map $\mathcal{T}_{km} = T_{21} \circ T_2^m \circ T_{12} \circ T_1^k$

- Points $M_i^- = RM_i^+$ and neighbourhoods $\Pi_i^- = R\Pi_i^+$, $i = 1, 2$.
- Local maps: $T_i := f_\mu|_{U_i}$ ⟳, $i = 1, 2$
- Global maps:
  $T_{12} := f_\mu^q : \Pi_1^- \to \Pi_2^+$
  $T_{21} := f_\mu^q : \Pi_2^- \to \Pi_1^+$

  R-reversibility
  $\Rightarrow T_{21} = R \circ T_{12}^{-1} \circ R$.



Local Maps

Lemma (S.V. Gonchenko and L.P. Shilnikov, 1990, 2000 + Bochner Thm)

T_0 $\mathcal{C}^r$, S-reversible diffeo, dimFixS=1
$\mathcal{O}$ a saddle point, $\mathcal{O} \in FixS$, multipliers λ, λ^{-1}, $|\lambda| < 1$.
Then, $\exists$ $\mathcal{C}^{r-1}$ local coordinates around $\mathcal{O}$ s.t.

$$T_0 : \begin{cases} \bar{\xi} & = & \lambda\xi + h_1(\xi,\eta)\xi^2\eta, \\ \bar{\eta} & = & \lambda^{-1}\eta + h_2(\xi,\eta)\xi\eta^2. \end{cases}$$

Moreover T_0 is L-reversible in these coordinates, with $L(\xi,\eta) = (\eta,\xi)$.

Shil'nikov cross-form for equations

- Shil'nikov cross-form:

$$T_0 : \left\{ \begin{array}{rcl} \bar{\xi} & = & \lambda\xi + h(\xi,\bar{\eta})\xi^2\bar{\eta}, \\ \eta & = & \lambda\bar{\eta} + h(\bar{\eta},\xi)\xi\bar{\eta}^2 \end{array} \right. \qquad |\lambda| < 1$$

  Very convenient for compositions and for fixed-point arguments.

- Shil'nikov form is particularly suitable for reversible maps. Indeed,
  - Any diffeo $F: \ (x,y) \mapsto (\bar{x},\bar{y})$ written in Shil'nikov form

    $$\left\{ \begin{array}{rcl} \bar{x} & = & f(x,\bar{y}) \\ y & = & f(\bar{y},x) \end{array} \right.$$

    is always reversible under the involution $(x,y) \mapsto (y,x)$.
  - If F is R-reversible map (R involution, non necessarily linear, orientation reversing with dim Fix $R = 1$), origin fixed point + IF Thm, then locally admits the cross-form above.

Construction of $\mathcal{T}_{km}$

- We define $\mathcal{T}_{km} = T_{21} \circ T_2^m \circ T_{12} \circ T_1^k$.
- $\mathcal{T}_{km} : \Pi_1^+ \longrightarrow \Pi_1^+$ and is R-reversible.
- Fixed points of $\mathcal{T}_{km}$ correspond to single-round p.o. of type (k, m), i.e., p.o. of period $k + m + 2q$.

We are concerned with the asymptotic behaviour of $\mathcal{T}_{km}$. Thus we assume that $\lambda_1^k \simeq \lambda_2^m$ as $k, m \to \infty$ (remind that $|\lambda_1|, |\lambda_2| < 1$).

A reescaling lemma

Lemma

Under hypotheses of Main Thm, there $\exists$ suitable coordinates s.t. $\mathcal{T}_{km}$ is defined implicitly by

$$g_M(x, y, \bar{x}, \bar{y}) = 0, \qquad g_M(\bar{y}, \bar{x}, y, x) = 0 \tag{1}$$

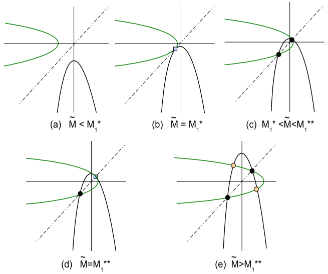
Equation (1) is reversible under $L(x, y) = (y, x)$.
$M = M(\lambda_1, \lambda_2, k, m, \mu)$ is a parameter.

Asymptotic behaviour of $\mathcal{T}_{km}$

$$\begin{cases} g_M(x, y, \bar{x}, \bar{y}) = 0, \\ g_M(\bar{y}, \bar{x}, y, x) = 0, \end{cases} + \quad \begin{array}{c} \lambda_1^k \simeq \lambda_2^m, \\ k, m \to \infty, \end{array} \longrightarrow \mathcal{H} : \begin{cases} \widetilde{M} + \widetilde{c}\bar{y} - \bar{x}^2 = y, \\ \widetilde{M} + \widetilde{c}x - y^2 = \bar{x}, \end{cases}$$

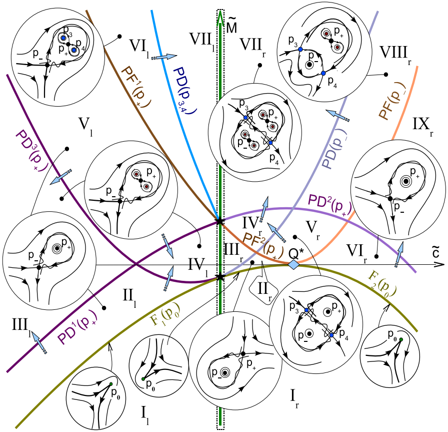
$\mathcal{T}_{km}$ is the composition of two Hénon maps. Fixed points come, for $\widetilde{c} \neq 1$, from the intersection of the two parabolas:

$$y(1 - \widetilde{c}) = \widetilde{M} - x^2, \qquad x(1 - \widetilde{c}) = \widetilde{M} - y^2.$$



- No intersection
- 1 point in Fix$R$.
- 2 points in Fix$R$ (quadr. tangency)
- 2 points in Fix$R$ (cubic tangency)
- 4 points (2 in Fix$R$ + 2 symm.)

Bifurcation diagram



where: (F) Fold, (PD) Period-doubling, (PF): Pitch-fork

Vertical Duffing

We look for values of α_j's and t_0 such that

$$M(t_0) = M'(t_0) = 0, \qquad M''(t_0) \neq 0$$

which provides a homoclinic point with quadratic tangency.

For instance, we can choose

$$N = 1, \quad t_0 = \pi/2, \quad \alpha_1 = -\frac{\sinh(\pi/2)}{21\sinh(3\pi/2)}\,\alpha_0, \quad \alpha_0 := 1,$$

that corresponds to the perturbation

$$\begin{cases} \dot{x} &= y - y^3 \quad +\varepsilon x\,(\sin t + \beta\sin 3t) \\ \dot{y} &= x \end{cases}$$

The corresponding Melnikov function is $M(t_0) = \frac{1}{3}\mathrm{e}^{-\pi/2}\sin t_0(4\cos^2 t_0)$.

Homoclinic points with quadratic tangency

Lemma

- $\dot{z} = F(z) + \varepsilon G(z,t)$, *with* G T*-periodic.*
- R*-reversible,* R *non necessary linear.*
- $Z_h(t)$ *homoclinic/heteroclinic curve, with* $z_h(0) \in \mathrm{Fix}\, R$.

Then,

1. $M(t_0)$ *is* T*-periodic.*
2. $M(-t_0) = -M(t_0)$, *that is* M *is odd. In particular:* $M(0) = 0$.

This implies that (Fourier):

$$M(t_0) = \sum_{k \geq 1} M_k \sin \frac{2\pi k}{T} t_0.$$

Homoclinic points with quadratic tangency

Have in mind that $\sin n\tau = \sin\tau\, U_{n-1}(\cos\tau)$, where $U_{n-1}(x)$ is Chebychev polynomial of 2nd type, which satisfy:

- $U_n(\cos\tau) = \dfrac{\sin((n+1)\tau)}{\sin\tau}$.
- $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$
- $\deg(U_n(x)) = n$, odd if n odd, even if n even.

Homoclinic points with quadratic tangency

Thus,

$$M(t_0) = \sin \frac{2\pi}{T} t_0 \sum_{k \geq 0} M_{k+1} U_k(\cos \frac{2\pi}{T} t_0)$$

Then:

- $\sin 2\pi t_0^*/T = 0 \Rightarrow t_0^* = \dfrac{T}{2} j$, $j \in \mathbb{Z}$. They cannot provide quadratic tangencies since $M'(t_0^*) = 0 \Rightarrow M''(t_0^*) = 0$
- We seek for t_0^{**} such that $\sum_{k \geq 0} M_{k+1} U_k(\cos \frac{2\pi}{T} t_0^{**}) = 0$ and $M'(t_0^{**}) = 0$, $M''(t_0^{**}) \neq 0$.

For instance, in our example $T = 2\pi$, $t_0^{**} = \pi/2$ and

$$M(t_0) = \sin t_0 \left(\frac{\mathrm{e}^{\pi/2}}{3} U_0(\cos t_0) + \frac{\mathrm{e}^{\pi/2}}{3} U_2(\cos t_0) \right).$$

Thanks

Thanks for your attention !