Slow-fast Bogdanov-Takens bifurcations in an application

Peter De Maesschalck

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This talk is about slow-fast systems

\[
\begin{align*}
\dot{x} &= f(x, y, \varepsilon, \lambda) \\
\dot{y} &= \varepsilon g(x, y, \varepsilon, \lambda)
\end{align*}
\]

where \(\varepsilon > 0\) is small, \(\lambda\) is some parameter.
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Mathematical foundations:

- Geometric singular perturbation Theory by Fenichel, Jones
- Desingularization by Dumortier, Roussarie
- Canards by Benoit et al
- Asymptotics by Eckhaus, Wasow, Ramis et al
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Motivation is 2-fold:
- Study of periodic orbits (Hilbert 16th problem)
- Applications to natural rhythms in biology, neurology, ecology, …
The Fitzhugh-Nagumo model and slow-fast Hopf bifurcations

\[
\begin{cases}
\dot{v} &= v - \frac{1}{3}v^3 - w + I \\
\dot{w} &= \varepsilon(v + a - bw)
\end{cases}
\]
The Fitzhugh-Nagumo model and slow-fast Hopf bifurcations

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\end{aligned}
\]
Type I vs type II excitation

Type I

Type II

![Graphs showing Type I and Type II excitation with SNIC, SN, and HB bifurcations.](#)
Changing the $w$-nullcline

**Type I**

- $v_{fold}$
- $I > I_{bif}$
- $I < I_{bif}$
- $I_{bif}$

**Type II**

- $v_{fold}$
- $I > I_{bif}$
- $I < I_{bif}$
- $I_{bif}$

Stable

Unstable

$v_{fold}$
Slow-fast equations

\[
\begin{align*}
    w' &= \epsilon g(w, v, \epsilon, \lambda) \\
    v' &= f(w, v, \epsilon, I)
\end{align*}
\]  

(1)

A more specific neuronal model:

\[
\begin{align*}
    w' &= \epsilon (G(v) - w) \\
    v' &= v^2(d - v) - w + I
\end{align*}
\]  

(2)

with

\[
G(v) = \begin{cases} 
    cv, & v \leq v_{th} \\
    cv + e(v - v_{th})^2, & v > v_{th}
\end{cases}
\]  

(3)
Slow-fast analysis

\[
\begin{align*}
\begin{cases}
  w' &= 0 \\
  v' &= v^2(d - v) - w + l
\end{cases}
\quad
\begin{cases}
  w' &= G(v) - w \\
  0 &= v^2(d - v) - w + l
\end{cases}
\end{align*}
\]

The critical manifold \( S \) is cubic shaped and given as a graph \( \{ w = \phi_I(v) \} \), i.e.

\[
S = S_a^- \cup F^- \cup S_r \cup F^+ \cup S_a^+ ,
\]
Slow-fast analysis

\[
\begin{aligned}
\left\{ \begin{array}{c}
w' = 0 \\
v' = v^2(d - v) - w + I
\end{array} \right. \\
\left\{ \begin{array}{c}
w' = G(v) - w \\
0 = v^2(d - v) - w + I
\end{array} \right.
\end{aligned}
\]

Along the \( w \)-nullcline \( g(w, v, 0, \lambda) = 0 \):

\[
\frac{\partial g}{\partial w} \neq 0, \quad \frac{\partial g}{\partial v} \cdot \frac{\partial g}{\partial w} \leq 0.
\]
Slow-fast analysis

\[
\begin{align*}
\begin{cases}
    w' &= 0 \\
    v' &= v^2(d - v) - w + I
\end{cases}
\quad \begin{cases}
    w' &= G(v) - w \\
    0 &= v^2(d - v) - w + I
\end{cases}
\end{align*}
\]

The system can have one, two or three equilibria on \( w = \phi_I(v) \), all of them located either on \( S_r \) or on \( S_{a^+} \).
The fold point $F^+ = (w^+, v^+)$ is a regular jump point.
Assumption

Consider

\[
\begin{align*}
  w' &= \varepsilon g(w, v, \varepsilon, \lambda) \\
  v' &= f(w, v, \varepsilon, I)
\end{align*}
\]

and define

\[
G(v, I, \lambda) = g(\phi_I(v), v, 0, \lambda).
\]

For fixed \((I, \lambda) = (I_{bif}, \lambda_{bif})\), the fold point \(F^- = (w^-, v^-)\) is a singular contact point that undergoes a singular Bogdanov-Takens bifurcation with respect to the parameters \((I, \lambda)\):

\[
G(v^-, I_{bif}, \lambda_{bif}) = 0, \quad \frac{\partial G}{\partial v}(v^-, I_{bif}, \lambda_{bif}) = 0, \quad \frac{\partial^2 G}{\partial v^2}(v^-, I_{bif}, \lambda_{bif}) > 0, \quad \frac{\partial G}{\partial I}(v^-, I_{bif}, \lambda_{bif}) \neq 0, \quad \frac{\partial G}{\partial \lambda}(v^-, I_{bif}, \lambda_{bif}) = 0, \quad \frac{\partial^2 G}{\partial \lambda \partial v}(v^-, I_{bif}, \lambda_{bif}) \neq 0.
\]

Besides the possible singular points near \(F^-\) occurring in this bifurcation, there are no other singular points on \(S^-\).
Proposition

Under these assumptions, the family of vector fields can be locally transformed in the following normal form near $F^{-}$,

\begin{align*}
x' &= \varepsilon \left( cy - \sigma x - a + O(x^2, y^3, xy, \varepsilon y^2) \right) \\
y' &= y^2 - x + \beta y^3 + O(y^4),
\end{align*}

(4)

where $\sigma = \pm 1$, $\sigma c \geq 0$ and $\beta \neq 0$. The coefficients $a$, $c$ and $\beta$ can be computed explicitly in terms of $(I, \lambda, \varepsilon)$.

For system (2), the coefficients in the normal form are given by

$c \geq 0$ and

\begin{align*}
a &= dl, \\
\beta &= -1/d^2, \\
\sigma &= 1,
\end{align*}

(5)
(a) $a = c = 0$: singular SNIC
(b) $a = 0$, $c > 0$ below $c_{cusp}$: slow-fast Hopf (truncated)
(c) $a = 0$, $c = c_{cusp}$: slow-fast Hopf (truncated)
(d) $a = 0$, $c > c_{cusp}$: slow-fast Hopf
SNIC of canard type

singular BT/SNIC

AH

SN

(a) (b) (d)

(c)

SN

(cusp)

a

c

singular
Theorem

For fixed $0 < c < c_{\text{cusp}}$ and $0 < \varepsilon \ll 1$ there exists an unstable equilibrium on the middle branch $S_{r,\varepsilon}$ bounded away from the lower fold $F^-$. Furthermore, there exist functions

$$0 < a_{\text{snpo}}(\varepsilon) < a_\ell(\varepsilon) < a_s(\varepsilon) < a_c(\varepsilon) < a_h(\varepsilon) < a_{\text{sn}}^+(\varepsilon)$$

that all converge to zero in the singular limit $\varepsilon \to 0$ (except $a_{\text{sn}}^+$) and for which the following holds:

1. For $a_{\text{sn}}^+ < a$, the fold $F^-$ is of regular jump type and a large stable relaxation cycle exists.

2. At $a = a_{\text{sn}}^+$, a saddle-node bifurcation of singular points on the middle branch $S_{r,\varepsilon}$ in an $O(c)$-neighbourhood of $F^-$; the large relaxation cycle persists.

3. For $a_h < a < a_{\text{sn}}^+$, the system has a saddle $p_+$ and an unstable focus/node $p_-$ on the middle branch $S_{r,\varepsilon}$ surrounded by the large relaxation cycle. The unstable focus/node $p_-$ is closer to the fold $F^-$. 
4. At $a = a_h$, $p_-$ changes stability and a subcritical singular Andronov-Hopf bifurcation takes place; the large relaxation cycle persists.

5. For $a_c < a < a_h$, repelling small amplitude limit cycles appear around the stable focus $p_-$; the large relaxation cycle persists.

6. For $a_s < a < a_c$, small jump-back canard cycles appear that rapidly grow in amplitude (canard explosion); the large relaxation cycle perturbs to a large-amplitude jump-forward canard cycle.

7. At $a = a_s$, a small jump-back homoclinic loop of canard type, issued from the saddle $p_+$, appears together with a stable large-amplitude canard cycle.

8. For $a_{\ell} < a < a_s$, the small homoclinic loop breaks and only the stable large-amplitude canard cycle persists.

9. At $a = a_{\ell}$, a large-amplitude homoclinic loop of canard type, issued from the saddle $p_+$, appears together with the outer large-amplitude cycle.

10. As $a$ decreases from $a_{\ell}$, large amplitude canard cycles appear that grow in amplitude until it disappears in a saddle-node bifurcation of limit cycles at $a = a_{\text{snpo}}$. 
Heteroclinic connections of canard type undergo a transition from headless canard to canard with head, from the jump-back canard homoclinic to the jump-away canard homoclinic:
In order to get a hold on the parameters close to $c = 0$, we rescale the parameters and introduce

$$(c, a) = (\varepsilon C, \varepsilon^2 A), \quad (C, A) \in [0, M] \times [-M, M] \quad (6)$$

for some large $M > 0$. By doing this we in fact assume that $c = O(\varepsilon)$ and $a = O(\varepsilon^2)$. After the parameter rescaling (6), we study the system

$$x' = \varepsilon (-\varepsilon^2 A + \varepsilon Cy - x + O(x^2, y^3, xy, \varepsilon y^2))$$
$$y' = y^2 - x + O(y^3). \quad (7)$$

The singularity at $(x, y, \varepsilon) = (0, 0, 0)$ is a slow-fast Bogdanov-Takens point.
Near the fold, we study the system using blow-up [?, ?]. We write

$$(x, y, \varepsilon) = (r^2X, rY, rE), \quad r \geq 0, (X, Y, E) \in S^2_+$$

where $S^2_+$ denotes the half-sphere $X^2 + Y^2 + E^2 = 1$ with $E \geq 0$ (also known as Poincaré or blow-up sphere). The weights are chosen in a way that the higher order (big-oh) terms in (12) have also higher order in the rescaled equation.
\[ x' = \varepsilon \left( cy - \sigma x - a + O(x^2, y^3, xy, \varepsilon y^2) \right) \]
\[ y' = y^2 - x + \beta y^3 + O(y^4), \] \hspace{1cm} (8)

\[(x, y) = (\varepsilon^2 X, \varepsilon Y), \quad (X, Y) \in [-R, R]^2, \] \hspace{1cm} (9)

for some large \( R > 0 \). Applying this rescaling to (12), we can divide out a common factor \( \varepsilon \), thus transforming the system into a regular perturbation family

\[ \dot{X} = -A + CY - X + O(\varepsilon), \]
\[ \dot{Y} = Y^2 - X + O(\varepsilon). \] \hspace{1cm} (10)
$A = C^2/4$ : 

- $C < \frac{1}{2}$
- $C = \frac{1}{2}$
- $\frac{1}{2} < C < 1$
- $C = 1$
- $C > 1$
Theorem
There exists a parameter surface $A_{sn}^+(C, \varepsilon) = C^2/4 + O(\varepsilon)$ along which a saddle-node singularity $p_{\pm}$ exists. On this surface, there exists a curve $C = \frac{1}{2} + O(\varepsilon)$ along which a saddle-node homoclinic (SN-HOM$_\ell$) connection appears containing the hyperbolic separatrix of the saddle-node. For $C < \frac{1}{2} + O(\varepsilon)$ on this parameter surface, there is a SNIC connection containing a center separatrix of the saddle-node. For $C > \frac{1}{2} + O(\varepsilon)$, there is no SNIC connection.
Proof: SN-bifurcation is stable so there exists $A^+_{sn}$-curve which is perturbation of $A = C^2/4$. There exists a $C^k$- center outgoing separatrix $W$ and a $C^\infty$ incoming stable separatrix $V$. Both are $(C, \varepsilon)$-families of curves. Intersect $V$ with a transverse section parameterized by a coordinate $s$ so that

$$V: s = \psi(C, \varepsilon),$$

for some smooth $\psi$. Then integrate $W$ following the vector field until it reaches $V$. This gives

$$s = \phi(C, \varepsilon),$$

for some $C^k$-function $\phi$. Next $\phi(1/2, 0) = \psi(1/2, 0)$ and $\frac{\partial \phi}{\partial C}(1/2, 0) \neq \frac{\partial \psi}{\partial C}(1/2, 0)$ (Melnikov-like computation).
So there we can apply IFT: there exists $C = C(\varepsilon)$ along which a SN-HOM connection appears. Finally for $C < C(\varepsilon)$ we apply the technique of rotating vector fields to see that the SNIC connection is made. The same technique shows that $C > C(\varepsilon)$ shows a big relaxation cycle.
\[ A = -\frac{1}{16} + \frac{C}{4} : \]

- For \( C < \frac{1}{2} \):
  - Diagram with arrows pointing clockwise from \( p_n \) to \( p_r \) and from \( p_r \) to \( p_a \) and from \( p_a \) to \( p_s \) and from \( p_s \) to \( p_n \).

- For \( C = \frac{1}{2} \):
  - Diagram with arrows pointing clockwise from \( p_n \) to \( p_r \) and from \( p_r \) to \( p_a \) and from \( p_a \) to \( p_s \) and from \( p_s \) to \( p_n \).

- For \( C > \frac{1}{2} \):
  - Diagram with arrows pointing clockwise from \( p_n \) to \( p_r \) and from \( p_r \) to \( p_a \) and from \( p_a \) to \( p_s \) and from \( p_s \) to \( p_n \).
Theorem

Let $C_{\text{min}} > \frac{1}{2}$. There exists a parameter surface $A_\ell(C, \varepsilon) = -\frac{1}{16} + \frac{C}{4} + O(\varepsilon), \ C > C_{\text{min}}$ along which a large-amplitude saddle-homoclinic (HOM$_\ell$) connection exists. On this surface, there exists a curve $C = \frac{3}{4} + O(\varepsilon)$ along which the homoclinic changes stability (Resonant HOM$_\ell$): for lower values of $C$, the homoclinic is stable, for larger values it is unstable. From this curve emerges a surface $A = A_{\text{snpo}}(C, \varepsilon)$ along which a SNPO bifurcation takes place. The surfaces $A_{\text{snpo}}(C, \varepsilon)$ and $A_\ell(C, \varepsilon)$ are exponentially close.
Proof:
The existence is HOMOCLINIC is similar to that of SN-HOM. (A little attention must be paid gluing HOM to SN-HOM.)
Take \( C^1 \)-normal form around hyperbolic saddle (for equivalence)

\[
\begin{aligned}
\dot{X} &= -X \\
\dot{Y} &= \rho(C) Y, \\
\rho(C, \varepsilon) &:= 4C - 2 + O(\varepsilon) > 0.
\end{aligned}
\]

Take a section \( Y = 1 \) with coordinate \( x_0 \) and integrate backward in time until we reach the section \( X = 1 \) with coordinate \( y_0 \). This gives

\[
y_0 = x_0^{\rho(C, \varepsilon)}.
\]

Next integrate in positive time.

\[
y_0 = \phi(A, C, \varepsilon) + \exp(-\tilde{I}(\varepsilon^2, A, C, \varepsilon)/\varepsilon).
\]

(formula explained on blackboard)
Along \( A = -\frac{1}{16} + \frac{C}{4} \) we have a HOM connection given by \( x_0 = 0 \), so
\[
\phi(A, C, \varepsilon) + \exp(-\tilde{I}(0, A, C, \varepsilon)/\varepsilon) = 0.
\]
Applying again a Melnikov argument shows that this bifurcation line perturbs to \( \varepsilon > 0 \).

The Saddle-node of Periodic Orbits (SNPO):
\[
\Delta := \phi(A, C, \varepsilon) + \exp(-\tilde{I}(\varepsilon^2 x_0, A, C, \varepsilon)/\varepsilon) - x_0^{\rho(C, \varepsilon)}.
\]
We derive with respect to \( x_0 \):
\[
\Delta' = \exp(-\hat{I}(\varepsilon^2 x_0, A, C, \varepsilon)/\varepsilon) - \rho(C, \varepsilon)x_0^{\rho(C, \varepsilon) - 1}.
\]
It is clear that when \( \rho(C) < 1 \) this expression tends to \(-\infty\) as \( x_0 \to 0 \). When \( \rho(C) > 1 \) we find a critical point at a solution of the equation
\[
x_0 = \left(\frac{1}{\rho}\right)^{1-\rho} \exp\left(\frac{-\hat{I}(\varepsilon^2 x_0, A, C, \varepsilon)}{\varepsilon(1-\rho)}\right).
\]
Clearly, this solution is of exponentially-flat type (w.r.t. $\varepsilon$ and $1 - \rho$). Combine with
\[
\phi(A, C, \varepsilon) + \exp(-\tilde{I}(\varepsilon^2 x_0, A, C, \varepsilon) / \varepsilon) = x_0^\rho(C, \varepsilon)
\]
to find $A = A_{snpo}(C, \varepsilon)$. 
How to connect both diagrams:
Before we have used the rescaling

\[(c, a) = (\varepsilon C, \varepsilon^2 A), \quad (C, A) \in [0, M] \times [-M, M]\]

It is better to do

\[(c, a, \varepsilon) = (vC, v^2 A, vE), \quad (C, A, E) \in S^2, v \geq 0.\]

The previous rescaling amounts to looking on a chart of the sphere in the direction of \(E = 1\).

\[(c, a, \varepsilon) = (v, v^2 A, vE), \quad E \approx 0, A \in [-M, M], v \geq 0.\]

Since \(c = v\) we can simplify to

\[(a, \varepsilon) = (c^2 A, cE), \quad E \approx 0, A \in [-M, M], c \geq 0.\]

The vector field:

\[
x' = cE \left( -c^2 A + cy - x + O(x^2, y^3, xy, \varepsilon y^2) \right) \\
y' = y^2 - x + O(y^3). \tag{11}
\]
\begin{align*}
x' &= cE \left( -c^2 A + cy - x + O(x^2, y^3, xy, \varepsilon y^2) \right) \\
y' &= y^2 - x + O(y^3) \tag{12}
\end{align*}

\begin{align*}
(x, y) &= (c^2 X, cY), \quad (X, Y) \in [-R, R]^2, \tag{13}
\end{align*}

for some large $R > 0$. Applying this rescaling to (12), we can divide out a common factor $\varepsilon$, thus transforming the system into a regular perturbation family

\begin{align*}
\dot{X} &= E(-A + Y - X + O(c)), \\
\dot{Y} &= Y^2 - X + O(c).
\end{align*}

\implies \text{ in this parameter regime the Bogdanov-Takens contact point blows up to a slow-fast Hopf situation.}
\[
\dot{X} = E(-A + Y - X + O(c)),
\]
\[
\dot{Y} = Y^2 - X + O(c).
\]

We find at \( A = 1/4 + O(c) \) a saddle-node in the slow dynamics, at \( A = 0 + O(c) \) a slow-fast Hopf point. At the slow-fast Hopf point there is an extra singularity on the middle branch, giving rise to HOM-connections. Furthermore, as the canard cycle grows towards the HOM connection, somewhere in between there is a zero of the slow divergence integral, giving rise to the SNPO. 
\(
\implies \text{the bifurcation diagram is complete!}
\)