Separatrix bifurcations in some 3-parameter families of cubic vector fields

Magdalena Caubergh

Grup de Sistemes Dinàmics
Departament de Matemàtiques
Universitat Autònoma de Barcelona

Joint work with J. Llibre and J. Torregrosa

Lleida, September 7th, 2016
Fourth Symposium on Planar Vector Fields

Setting

Family of cubic vector fields $X = X_{(\delta,a,b,c,d,e,f,g)}$,

$$\begin{aligned}\dot{x} &= \delta x - y + M(x,y) + (dx - y)(x^2+y^2),\\ \dot{y} &= x + \delta y + N(x,y) + (x + dy)(x^2+y^2),\end{aligned}$$

where $M(x,y) = ax^2 + bxy + cy^2$ and $N(x,y) = ex^2 + fxy + gy^2$ for $a, b, c, e, f, g, \delta, d \in \mathbb{R}$,
such that its phase portrait presents a center at the origin as well as at infinity.
We say that infinity is a center for X if after transformation $x = \cos\theta/r$ and $y = \sin\theta/r$ the origin of the transformed vector field is a center. In this case, X has an unbounded period annulus.

Lemma

For X to present simultaneously a center at the origin as well as at infinity, it is necessary that $\delta = d = 0$.

Characterization for coexisting center at origin and at infinity

After rotation

- Hamiltonian class with Hamiltonian $H = H_{(g,c,e)}$:

$$X^H_{(g,c,e)} \leftrightarrow \begin{cases} \dot{x} = -y - 2gxy + cy^2 - y\left(x^2 + y^2\right), \\ \dot{y} = x + ex^2 + gy^2 + x\left(x^2 + y^2\right), \end{cases}$$

$$H(x, y) = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1}{3}ex^3 + gxy^2 - \frac{1}{3}cy^3 + \frac{1}{4}\left(x^2 + y^2\right)^2$$

- Reversible class (with respect to $(x, y, t) \to (x, -y, -t)$):

$$X^R_{(g,b,e)} \leftrightarrow \begin{cases} \dot{x} = -y + bxy - y\left(x^2 + y^2\right), \\ \dot{y} = x + ex^2 + gy^2 + x\left(x^2 + y^2\right), \end{cases}$$

where $e \geq 0$ is sufficient.

Characterization for coexisting center at origin and at infinity

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- Reversible class (with respect to $(x, y, t) \to (x, -y, -t)$), introducing new parameter f by $f = b + 2g$:

$$X^R_{(g,f,e)} \leftrightarrow \begin{cases} \dot{x} = -y + (f - 2g)xy - y\left(x^2 + y^2\right), \\ \dot{y} = x + ex^2 + gy^2 + x\left(x^2 + y^2\right), \end{cases}$$

where $e \geq 0$ is sufficient.

No limit cycles

$\mathrm{div}(X) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 2ax + by + fx + 2gy$

- Hamiltonian class: $b = -2g, a = f = 0$, hence divergence is identically 0. (good!)
- Reversible class: $a = c = f = 0$, hence divergence reduces to: $(b + 2g)y$. As a consequence, if $X^R_{(g,b,e)}$ not Hamiltonian, periodic orbits have to pass the x-axis.

Matters of study

- Global center
- Classification of global phase portraits by topological equivalence; use Markus-Neumann-Peixoto Theorem:
  - Assume that $(\mathbb{R}^2, \varphi_1)$ and $(\mathbb{R}^2, \varphi_2)$ are two continuous flows with only isolated singular points.
  - Then these flows are topologically equivalent if and only if their completed separatrix skeletons are equivalent.
- Separatrix bifurcations

References

Joint work with J. Llibre and J. Torregrosa

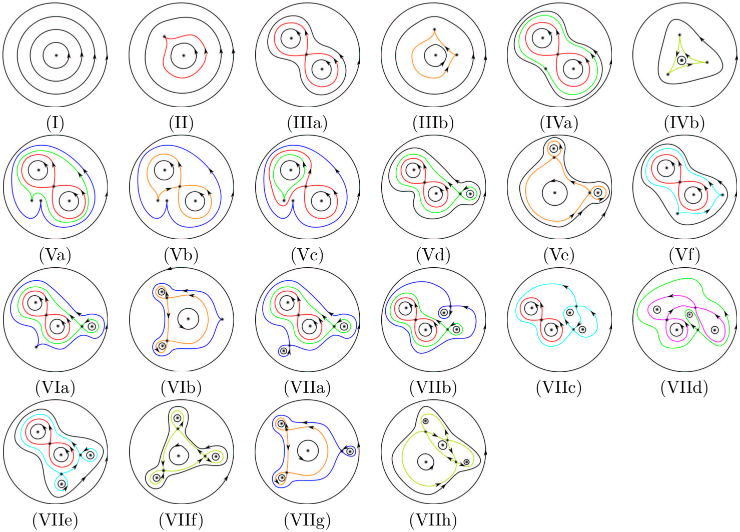
- Global Classification of a class of Cubic Vector Fields whose canonical regions are period annuli, in International Journal of Bifurcation and Chaos (CLT2011)
- Global phase portraits of some reversible cubic centers with collinear or infinitely many singularities (CLT2012)
- Global phase portraits of some reversible cubic centers with non-collinear singularities (CT2013)
- Relative movement of connections for some reversible cubic vector fields (C2016)

Classification results of global phase portraits

Up to topological equivalence,

- 61 different global phase portraits
  - At most 7 singularities or infinitely many singularities
  - No limit cycles
- 22 for the Hamiltonian Class [CLT2011]
  - Finitely many singularities
- 53 for the Reversible Class, of which
  - 14 also for Hamiltonian class
  - 8 with collinear singularities [CLT2012]
  - 44 with noncollinear singularities [CT2013,C2016]
  - 1 with infinitely many singularities

Result: Classification of $X^H_{(g,c,e)}$ by 22 Phase portraits



Polar coordinates (r, θ)

For $(x, y) = (r\cos\theta, r\sin\theta)$:

$$\begin{aligned} \dot{r} &= r^2 A(\theta), \\ \dot{\theta} &= 1 + rB(\theta) + r^2, \end{aligned}$$

where the trigonometric functions A, B are defined by

$$\begin{aligned} A(\theta) &= \cos\theta \cdot M(\cos\theta, \sin\theta) + \sin\theta \cdot N(\cos\theta, \sin\theta), \\ B(\theta) &= \cos\theta \cdot N(\cos\theta, \sin\theta) - \sin\theta \cdot M(\cos\theta, \sin\theta). \end{aligned}$$

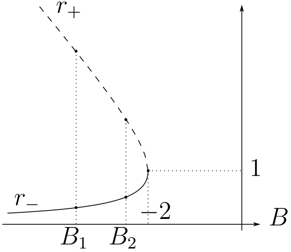
and satisfy $A(\theta + \pi) = -A(\theta)$, $B(\theta + \pi) = -B(\theta)$.

Singularities in polar coordinates (r, θ)

$(r, \theta) = (r^*, \theta^*)$ with $r^* = r^*(B(\theta^*))$:

- Along rays $\theta = \theta^*$ with $A(\theta^*) = 0$ with $B(\theta^*) \leq -2$
- radius r^* given by

$$r_\pm = \left(-B(\theta) \pm \sqrt{(B(\theta))^2 - 4}\right)/2.$$



Properties of the rays

Lemma

- *If for a certain* θ^* *holds* $A(\theta^*) = A'(\theta^*) = A''(\theta^*) = A^{(3)}(\theta^*) = 0$, *then* $A \equiv 0$
- *For the Hamiltonian class, we have* $B \equiv -3A'$.

Proposition

If $A \equiv 0$, then

- the Hamiltonian vector field $X^H_{(g,c,e)}$ reduces to the global center $\dot{x} = -y(1 + x^2 + y^2), \quad \dot{y} = x(1 + x^2 + y^2)$;
- the reversible vector field $X^R_{g,b,e}$ reduces to $\dot{x} = -y(1 + ex + x^2 + y^2), \quad \dot{y} = x(1 + ex + x^2 + y^2)$, which presents
  - for $0 \leq e < 2$: a global center;
  - for $e = 2$: two nested period annuli separated by a homoclinic loop;
  - for $e > 2$: two nested period annuli separated by a continuum of graphics defined by a circle of singularities.

Corollary

- *If $A \neq 0$, then A has finite order n at any zero θ^*, with $n \leq 3$: $A^{(j)}(\theta^*) = 0, \forall 0 \leq j < n$ and $A^{(n)}(\theta^*) = \gamma \neq 0$.*
- *If $A \neq 0$, then X has at most 7 singularities.*

Systematic approach of global classification of phase portraits

Case 1 or 'One triple ray' $e-2g=c=0.$

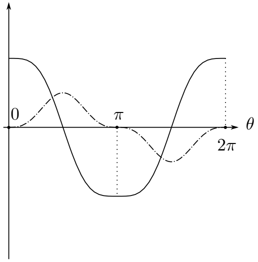
Case 2 or 'One simple ray - two complex rays' $e-2g\neq 0$ and
$c^2-4g\,(e-2g)<0.$

Case 3 or 'One double ray - one simple ray' $e-2g=0, c\neq 0.$

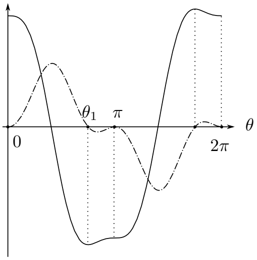
Case 4 or 'Three simple rays' $e-2g\neq 0$ and
$c^2-4g\,(e-2g)>0.$

(here with parameter values for Hamiltonian class)

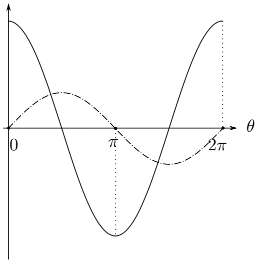
Geometric analysis based on A, B



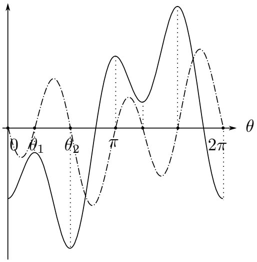
Case 1: One triple ray



Case 3: One double, one simple ray

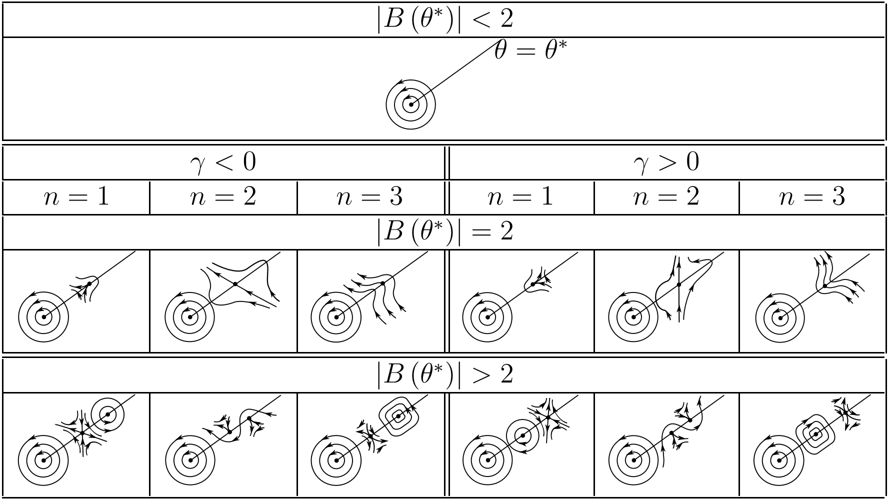


Case 2: One simple, two complex rays



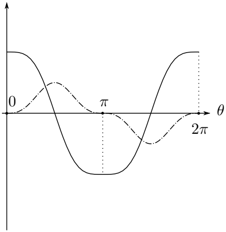
Case 4: Three simple rays

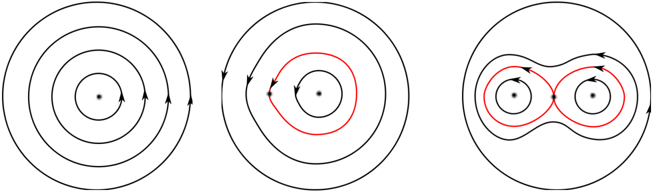
Local classification of the singularities along rays



Case 1: 1 triple ray $(g > 0, c = 0, e = 2g)$

$$A(\theta) = g\sin^3\theta \text{ and } B(\theta) = g\cos\theta\left(2 + \sin^2\theta\right).$$

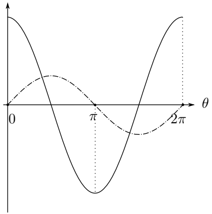


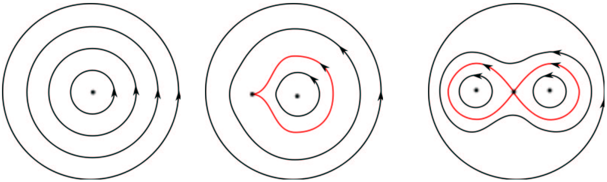


(a) $g < 1$ (b) $g = 1,\ h^0 = \frac{1}{12}$ (c) $g > 1,\ h^0_+ < h^0_- < \frac{1}{12}$

Case 2: 1 simple, 2 complex rays ($g > 0, e \geq 0$)

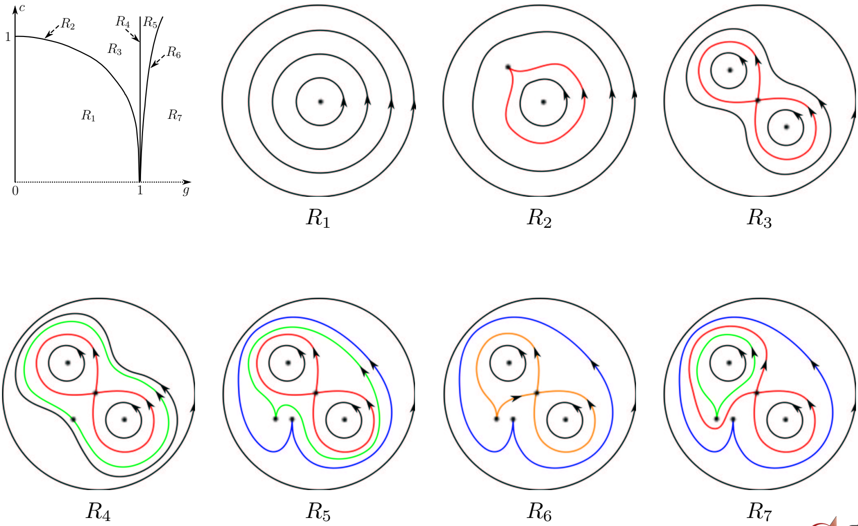
$$A(0) = 0, \quad A'(0) = e - 2g > 0, \quad B(0) = e.$$





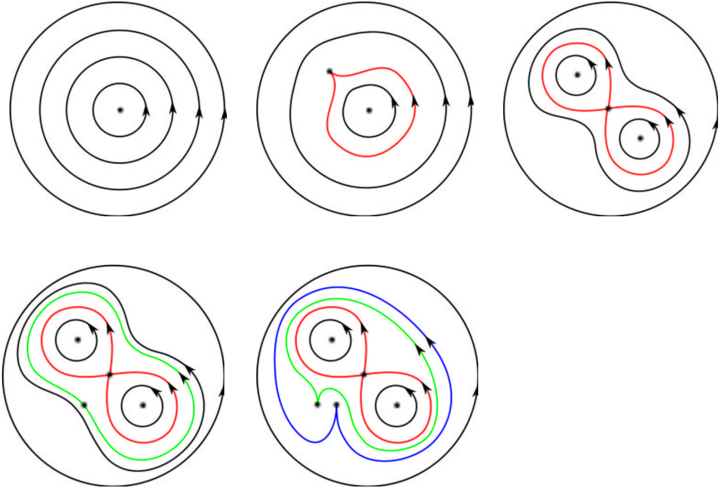
(a) $0 \leq e < 2$ (b) $e = 2, h^0 = \frac{1}{12}$ (c) $e > 2, h^0_+ < h^0_- < \frac{1}{12}$

Case 3: 1 double, 1 simple ray $(g, c > 0)$



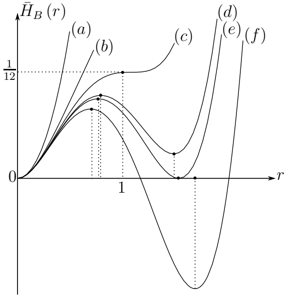
Appearance and splitting of singularities

- $B(\theta_1) < -B(0), \forall e, c, g$ in Case 3
- Bifurcation values $c_i = c_i\left(\alpha\right), i = 1, 2 : c_1 = -\frac{2}{B(\theta_1)} < c_2 = \frac{2}{B(0)}$



Analysis of the Hamiltonian - in terms of r

- For $r, B \in \mathbb{R} : \overline{H}_B(r) = r^2 \left(\frac{1}{2} + \frac{1}{3}Br + \frac{1}{4}r^2\right).$
- For $(x, y) = (r\cos\theta, r\sin\theta) : H(x, y) = \overline{H}_{B(\theta)}(r),$

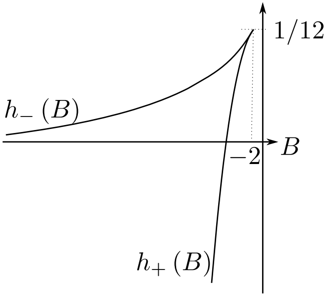


- $(a, b)\, B > -2$, $(c)\, B = -2$, $(d, e, f)\, B < -2$.

Analysis of the Hamiltonian - in terms of B

For $B < -2$ we define the functions $h_\pm : (-\infty, -2) \to \mathbb{R}$ by

$$h_\pm(B) \equiv \overline{H}_B(r_\pm)$$
$$= -\frac{1}{48}\left(-2 + B^2 \mp B\sqrt{B^2-4}\right)\left(-6 + B^2 \mp B\sqrt{B^2-4}\right),$$



Linear dependence of B on the parameter $c > 0$.

- Introduction of new parameter $\alpha = \cot\theta_1 = -\frac{g}{c}$

$$A(\theta) = c\sin^2\theta\,(\cos\theta - \alpha\sin\theta)\,,$$
$$B(\theta) = -c\left(2\alpha\cos^3\theta + 3\alpha\cos\theta\sin^2\theta + \sin^3\theta\right)\,,$$

- $B_1 < -B_0 < 0 \implies cB_1 < -cB_0 < 0, \forall c > 0.$
- Another bifurcation value $c_3 = c_3(\alpha) > c_2$:
  - $h_-^1(c) < h_+^0(c)$ for $c_2 < c < c_3$
  - $h_-^1(c_3) \equiv h_-(c_3 B_1) = h_+(-c_3 B_0) \equiv h_+^0(c_3)$
  - $h_-^1(c) > h_+^0(c)$ for $c > c_3$.

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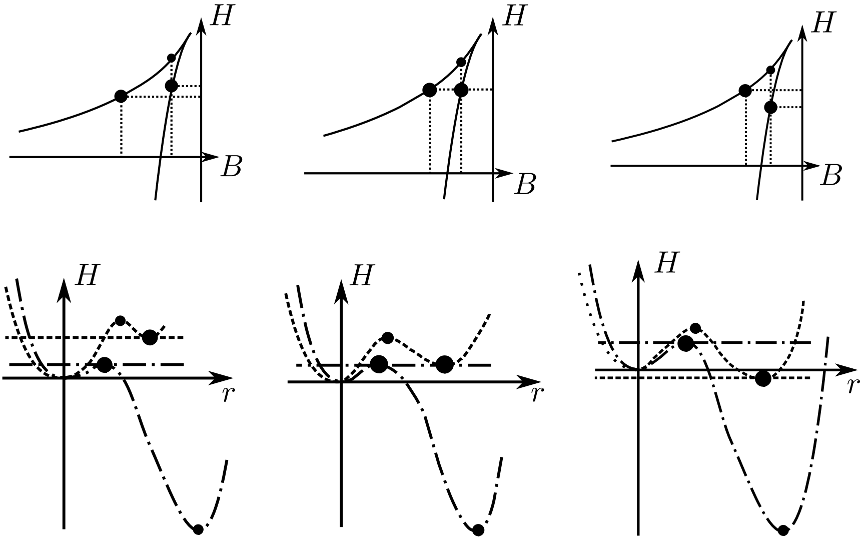
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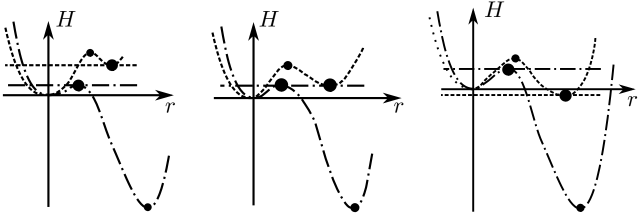
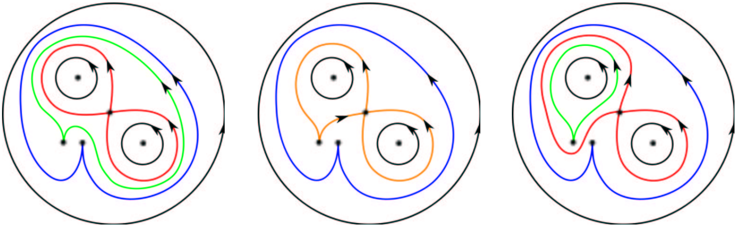
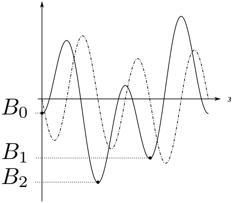
Bifurcation of crossing of connections

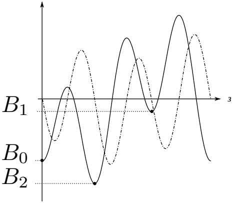




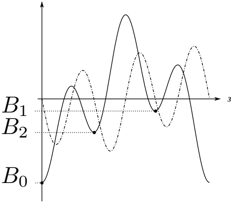
Case 4: 3 simple rays



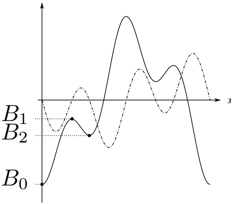
Case 4Ai



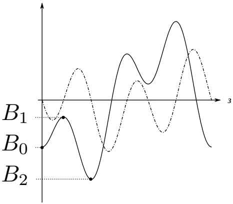
Case 4Aii



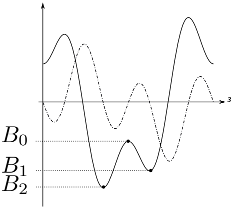
Case 4Aiii



Case 4Bi



Case 4Bii



Case 4Biii

Case $4^+ : \Pi > 0$, Case $4^0 : \Pi = 0$ and Case $4^- : \Pi < 0$

where $\Pi \equiv \overline{B}_0 \overline{B}_1 \overline{B}_2$

Linear dependence of B on the parameter $\lambda > 0$.

$$A(\theta) = -\lambda \sin\theta\,(\cos\theta - \alpha\sin\theta)\,(\cos\theta - \beta\sin\theta)\,,$$
$$B(\theta) = -\lambda\left((1+2\alpha\beta)\cos^3\theta + 3\alpha\beta\cos\theta\sin^2\theta + (\alpha+\beta)\sin^3\theta\right).$$

- Introduction of new parameters: (α, β, λ) :

$$\lambda = -(e-2g) = -A'(0) > 0 \text{ and } 0 \le -\beta \le \alpha :$$

$$\alpha = \frac{c+\sqrt{c^2+4g\lambda}}{2\lambda} > 0 \quad \text{and} \quad \beta = \frac{c-\sqrt{c^2+4g\lambda}}{2\lambda}.$$

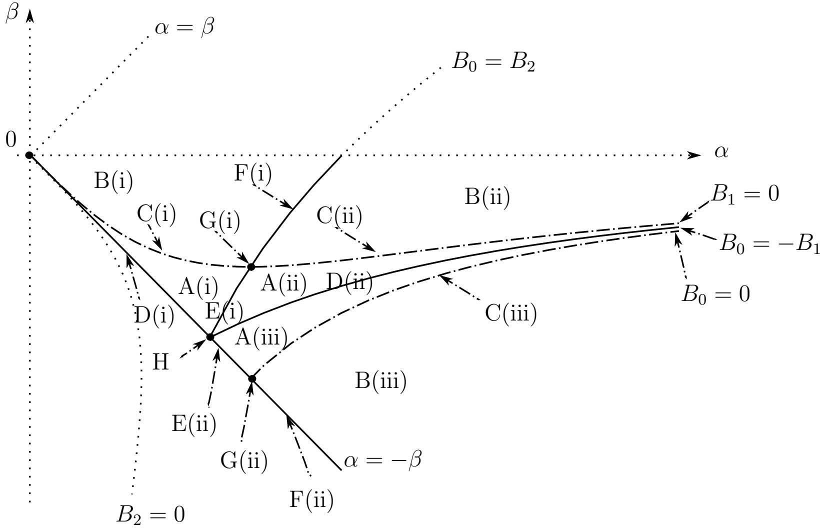
- $e = -\lambda\,(1+2\alpha\beta)\,, \quad g = -\lambda\alpha\beta, \quad c = \lambda\,(\alpha+\beta)\,.$

$$B(0) = -\lambda\,(1+2\alpha\beta) \equiv \lambda\overline{B}_0\,(\alpha,\beta) \equiv B_0\,(\lambda)\,,$$

$$B\,(\theta_1) = -\frac{\lambda\left(2\alpha^2\beta+\alpha+\beta\right)}{\sqrt{1+\alpha^2}} \equiv \lambda\overline{B}_1\,(\alpha,\beta) \equiv B_1\,(\lambda)\,,$$

$$B\,(\theta_2) = -\frac{\lambda\left(2\alpha\beta^2+\alpha+\beta\right)}{\sqrt{1+\beta^2}} \equiv \lambda\overline{B}_2\,(\alpha,\beta) \equiv B_2\,(\lambda)\,.$$

Bifurcation diagram in (α, β)-plane

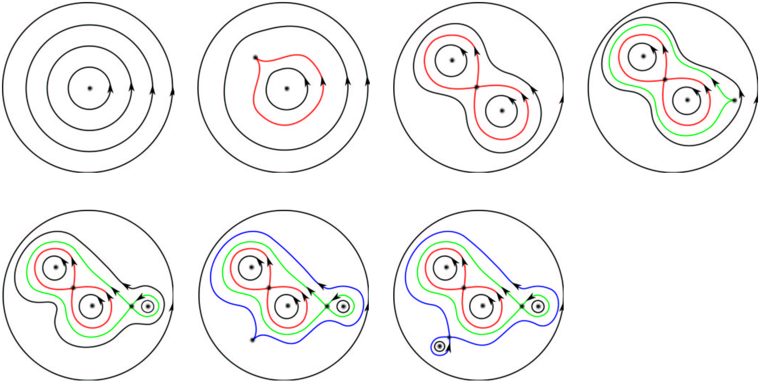


Generic bifurcations

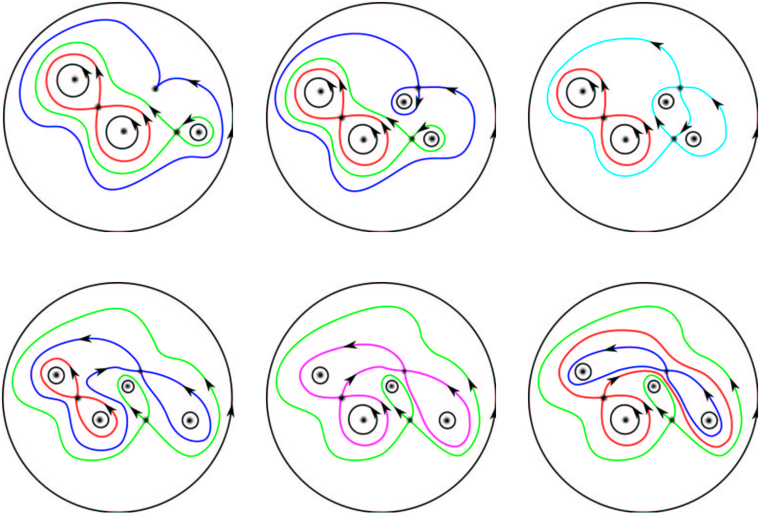
- Case 4A: $(\alpha, \beta) \in \mathcal{A}$ $(\Pi > 0)$
  - (i) $\overline{B}_0 < \overline{B}_2 < -\overline{B}_1 < 0.$
  - (ii) $\overline{B}_2 < \overline{B}_0 < -\overline{B}_1 < 0.$
  - (iii) $\overline{B}_2 < -\overline{B}_1 < \overline{B}_0 < 0.$

- Case 4B: $(\alpha, \beta) \in \mathcal{B}$ $(\Pi < 0)$
  - (i) $\overline{B}_0 < \overline{B}_2 < \overline{B}_1 < 0.$
  - (ii) $\overline{B}_2 < \overline{B}_0 < \overline{B}_1 < 0.$
  - (iii) $\overline{B}_2 < -\overline{B}_1 < -\overline{B}_0 < 0.$

Bifurcation in Case 4A



Bifurcation in Case 4B (after first 5 of Case 4A)



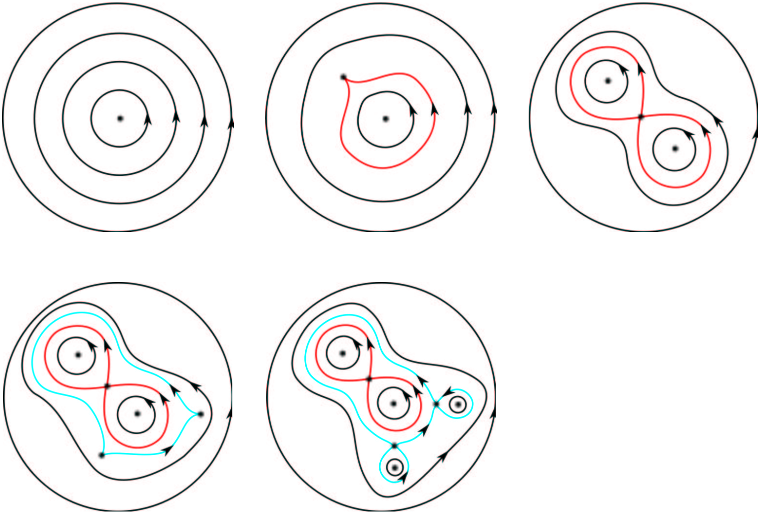
Boundary bifurcations, 1

- Case 4D: $(\alpha, \beta) \in \mathcal{D}$ $(\Pi > 0)$
  - (i) $\overline{B}_0 < \overline{B}_2 = -\overline{B}_1 < 0$.
  - (ii) $\overline{B}_2 < \overline{B}_0 = -\overline{B}_1 < 0$.

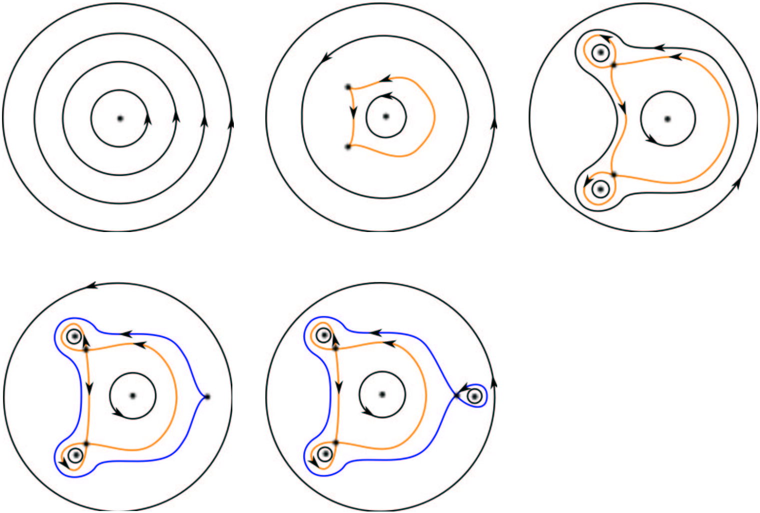
- Case 4E: $(\alpha, \beta) \in \mathcal{E}$ $(\Pi > 0)$
  - (i) $\overline{B}_0 = \overline{B}_2 < -\overline{B}_1 < 0$.
  - (ii) $\overline{B}_2 = -\overline{B}_1 < \overline{B}_0 < 0$.

- Case 4H: $(\alpha, \beta) \in \mathcal{H}$ $(\Pi > 0)$
  - $\overline{B}_0 = \overline{B}_2 = -\overline{B}_1 < 0$.

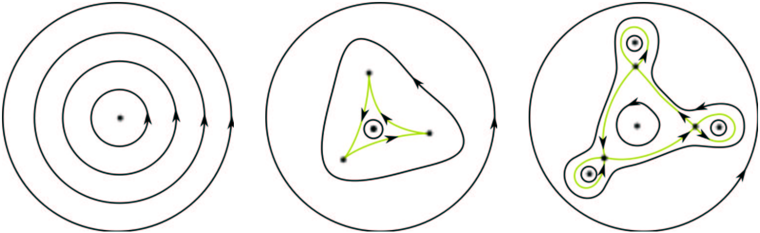
Bifurcation in Case 4D



Bifurcation in Case 4E



Bifurcation in Case 4H



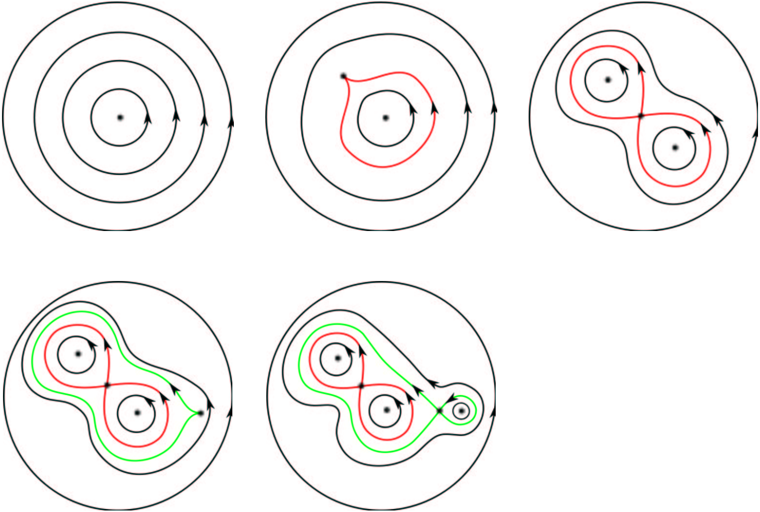
Boundary bifurcations, 2

- Case 4C: $(\alpha, \beta) \in \mathcal{C}$ $(\Pi = 0)$
  - (i) $\overline{B}_0 < \overline{B}_2 < \overline{B}_1 = 0$.
  - (ii) $\overline{B}_2 < \overline{B}_0 < \overline{B}_1 = 0$.
  - (iii) $\overline{B}_2 < -\overline{B}_1 < \overline{B}_0 = 0$.

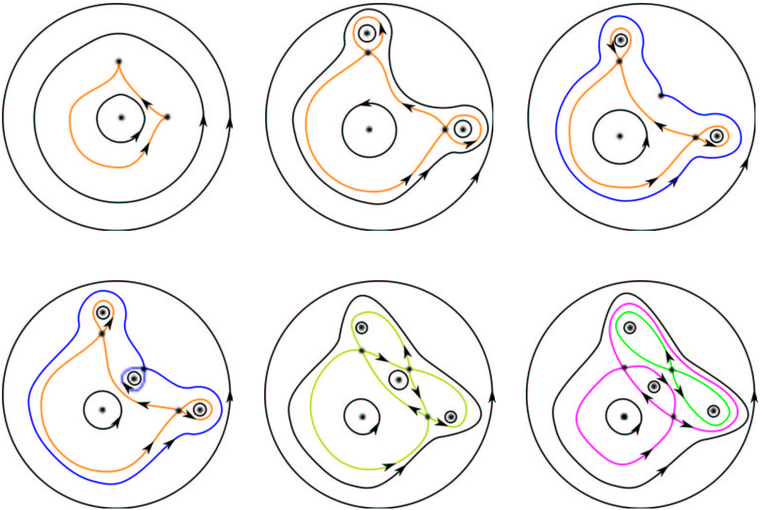
- Case 4F: $(\alpha, \beta) \in \mathcal{F}$ $(\Pi < 0)$
  - (i) $\overline{B}_0 = \overline{B}_2 < \overline{B}_1 < 0$.
  - (ii) $\overline{B}_2 = -\overline{B}_1 < -\overline{B}_0 < 0$.

- Case 4G: $(\alpha, \beta) \in \mathcal{G}$ $(\Pi = 0)$
  - (i) $\overline{B}_0 = \overline{B}_2 < \overline{B}_1 = 0$.
  - (ii) $\overline{B}_2 = -\overline{B}_1 < \overline{B}_0 = 0$.

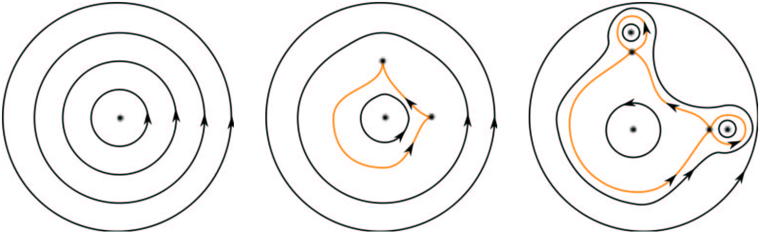
Bifurcation in Case 4C



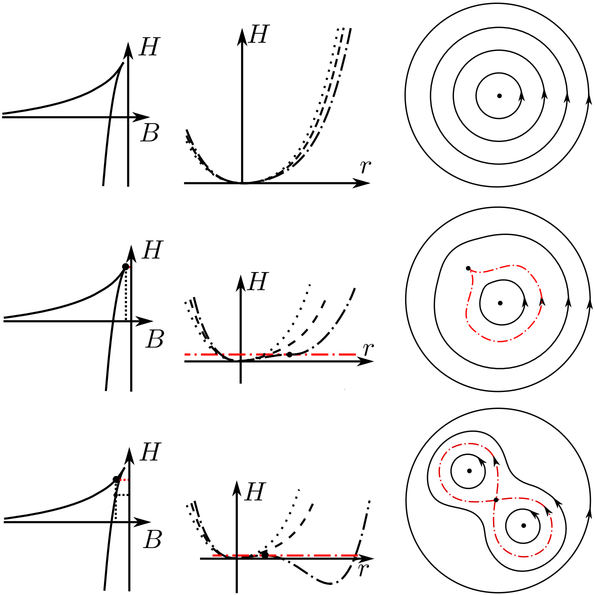
Bifurcation in Case 4F (after the global center)



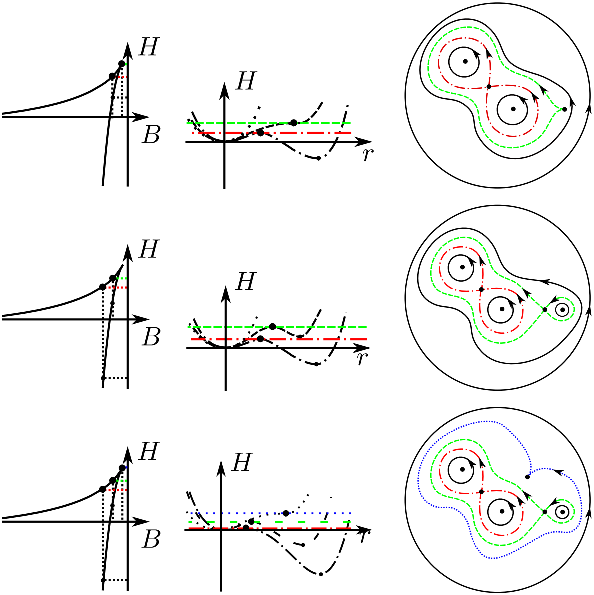
Bifurcation in Case 4G



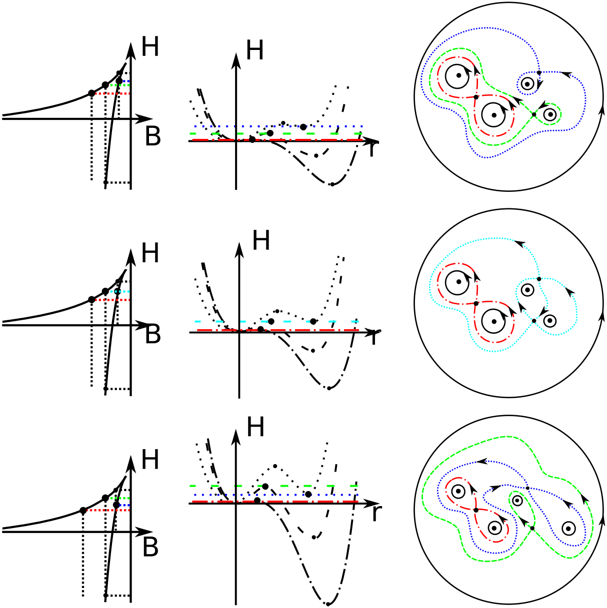
Bifurcation in case 4B(ii), 1



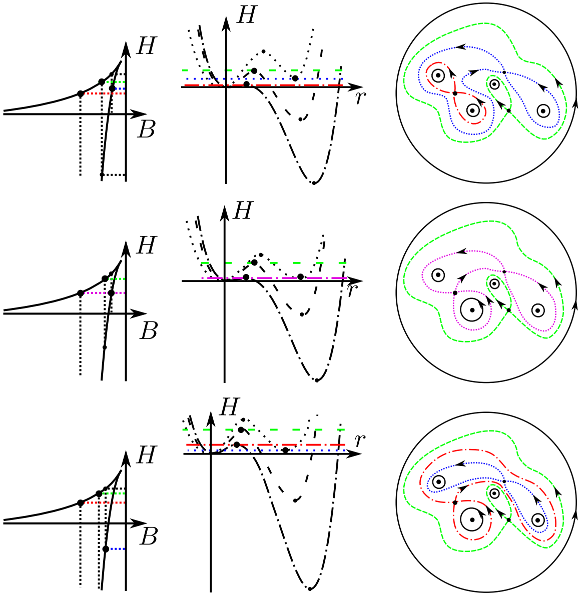
Bifurcation in case 4B(ii), 2



Bifurcation in case 4B(ii), 3



Bifurcation in case 4B(ii), 4



Hamiltonian and reversible class

After rotation

- Hamiltonian class with Hamiltonian H :

$$\begin{cases} \dot{x} = -y - 2gxy + cy^2 - y\left(x^2 + y^2\right), \\ \dot{y} = x + ex^2 + gy^2 + x\left(x^2 + y^2\right), \end{cases}$$

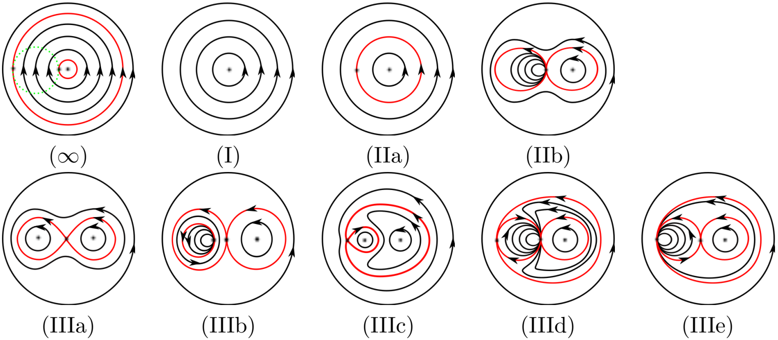
$$H(x, y) = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1}{3}ex^3 + gxy^2 - \frac{1}{3}cy^3 + \frac{1}{4}\left(x^2 + y^2\right)^2$$

- Reversible class (with respect to $(x, y, t) \to (x, -y, -t)$):

$$X_{(a,b,c)} \leftrightarrow \begin{cases} \dot{x} = -y + (a - 2b)xy - y\left(x^2 + y^2\right), \\ \dot{y} = x + cx^2 + by^2 + x\left(x^2 + y^2\right). \end{cases}$$

- Assumption $c \geq 0$ by invariance under $(x, y, t, a, b, c) \mapsto (-x, -y, t, -a, -b, -c)$

Classification of global phase portraits of $X_{(a,b,c)}$ having collinear or infinitely many singularities up to topological equivalence



Local study of singularities along horizontal ray, 1

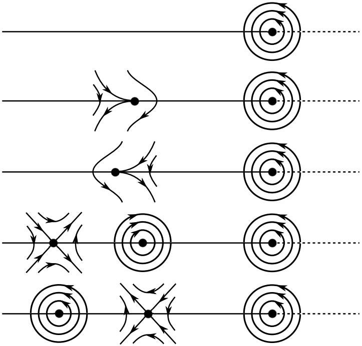
$0 \leq c < 2$

$c = 2,\ 2b - a - 2 > 0$

$c = 2,\ 2b - a - 2 < 0$

$c > 2,\ 2b - a - c > 0$

$c > 2,\ 2b - a - c < 0$



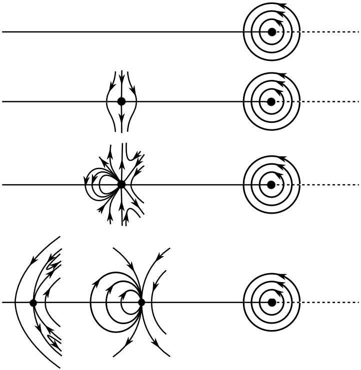
Local study of singularities along horizontal ray, 2

$0 \le c < 2$

$c = 2,\ b \le 1$

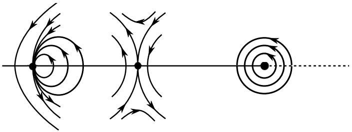
$c = 2,\ b > 1$

$c > 2,\ b < 0$

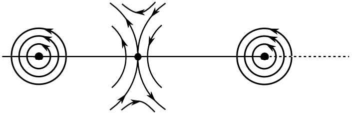


Local study of singularities along horizontal ray, 3

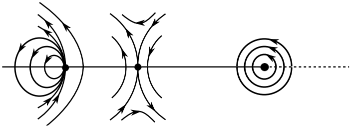
$c > 2,\ 0 < b \leq b_-(c)$



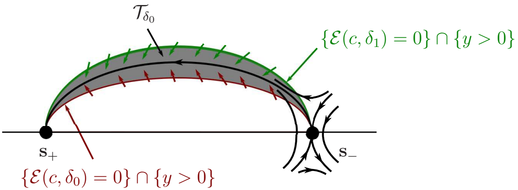
$c > 2,\ b_-(c) < b \leq b_+(c)$

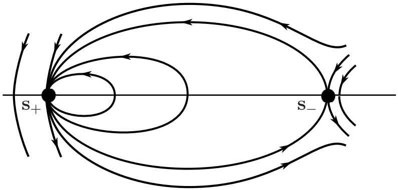


$c > 2,\ b_+(c) \leq b$



Local phase portrait determines global one





Hamiltonian and reversible class

After rotation

- Hamiltonian class with Hamiltonian H :

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$$H(x,y) = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1}{3}ex^3 + gxy^2 - \frac{1}{3}cy^3 + \frac{1}{4}\left(x^2 + y^2\right)^2$$

- Reversible class (with respect to $(x, y, t) \to (x, -y, -t)$):

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- Assumption $c \geq 0$ by invariance under $(x, y, t, a, b, c) \mapsto (-x, -y, t, -a, -b, -c)$

Reversible class $Y_{(\alpha,\gamma,\lambda)}$ for which symmetry-axis is simple

For $\alpha > 0, \gamma > 0, \lambda \in \mathbb{R}$,

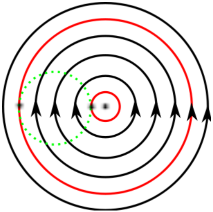
$$\dot{x} = -y - \gamma xy - y(x^2 + y^2),$$
$$\dot{y} = x + (\gamma - \lambda)x^2 + \alpha^2\lambda y^2 + x(x^2 + y^2),$$

- $\mathcal{R}_0 = \{(x, 0) : (\lambda - \gamma)x > 0\}$,
- $\mathcal{R}_{pm} = \{(x, y) : x \pm \alpha y = 0, x < 0\}$.
- $\mathcal{H} = \{(\alpha, \gamma, \lambda) \in (0, \infty)^3 : \gamma = 2\alpha^2\lambda\}$,

$\lambda = 0$

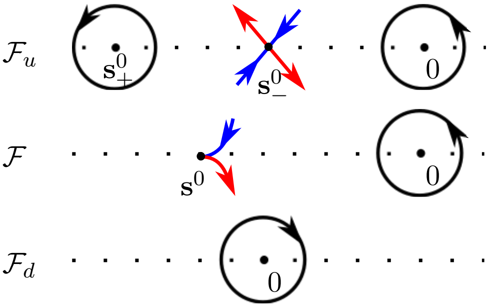
$$\dot{x} = -y(1 + \gamma x + x^2 + y^2),$$
$$\dot{y} = x(1 + \gamma x + x^2 + y^2),$$

For $|\gamma| > 2$: circle centered at $(-\gamma/2, 0)$ with radius $\gamma^2/4 - 1$:



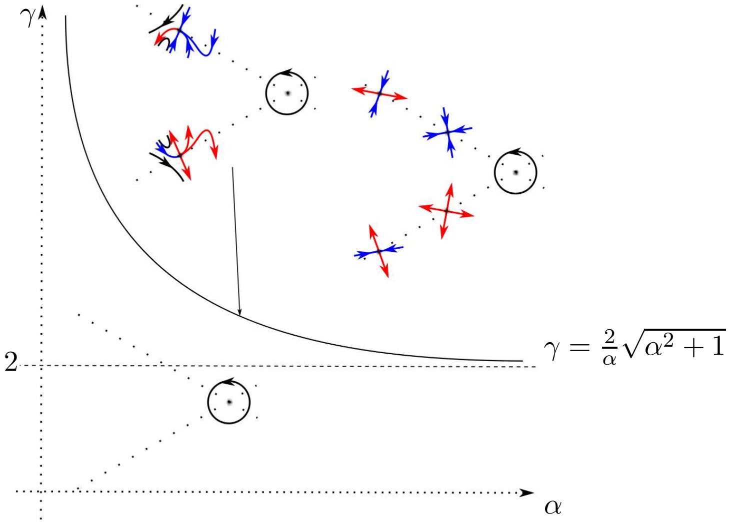
$\lambda < 0$: locally along ray $\mathcal{R}_0$

- $\mathcal{F}_u = \{(\alpha, \gamma, \lambda) : \gamma > \lambda + 2\}$,
- $\mathcal{F} = \{(\alpha, \gamma, \lambda) : \gamma = \lambda + 2\}$,
- $\mathcal{F}_d = \{(\alpha, \gamma, \lambda) : \gamma < \lambda + 2\}$.



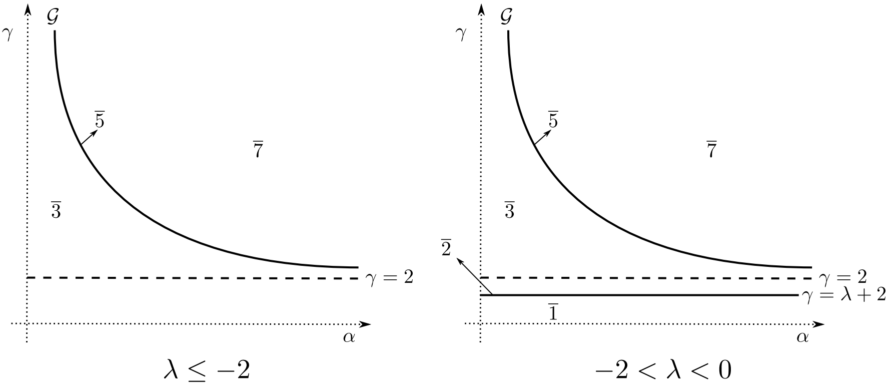
$\lambda < 0$: locally along ray $\mathcal{R}_{\pm}$

- $\mathcal{G} = \{\alpha^2\gamma^2 - 4(\alpha^2+1) = 0\}$.

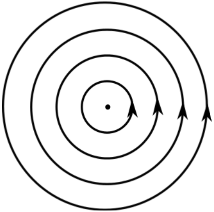


$\lambda < 0$: Global bifurcation diagram

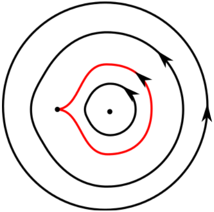
Global Bifurcation diagram = local one!



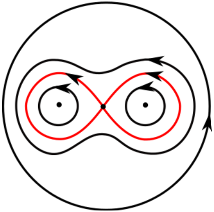
$\lambda < 0$: Global phase portraits



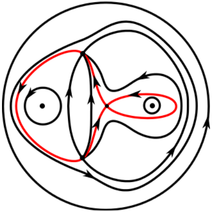
$\overline{1}$



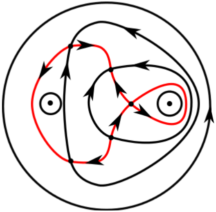
$\overline{2}$



$\overline{3}$



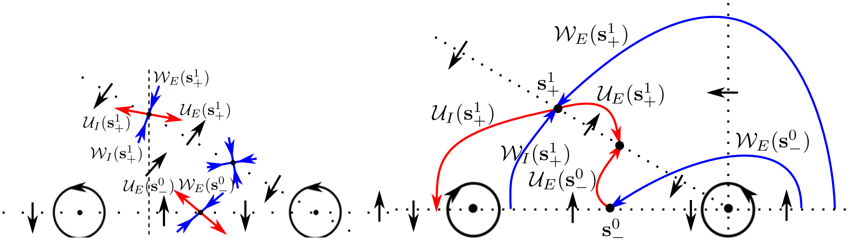
$\overline{5}$



$\overline{7}$

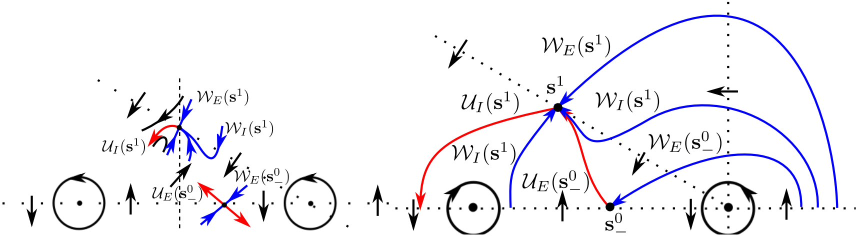
$\lambda < 0$: Proof - Global phase portrait with 7 singularities

- Poincaré-Bendixson Theorem
- Triangle bounded by $\mathcal{R}_0$, $\mathcal{R}_+$ and vertical $y = y_+^1$.



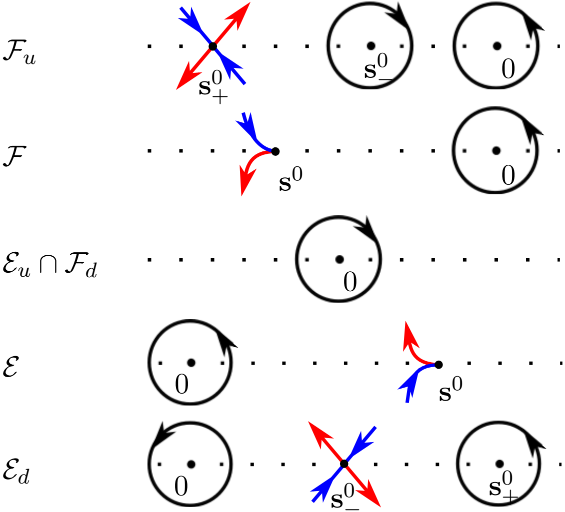
$\lambda < 0$: Proof - Global phase portraits with 5 singularities

- Poincaré-Bendixson Theorem
- Triangle bounded by $\mathcal{R}_0$, $\mathcal{R}_+$ and vertical $y = y^1$.



$\lambda > 0$: locally along ray $\mathcal{R}_0$

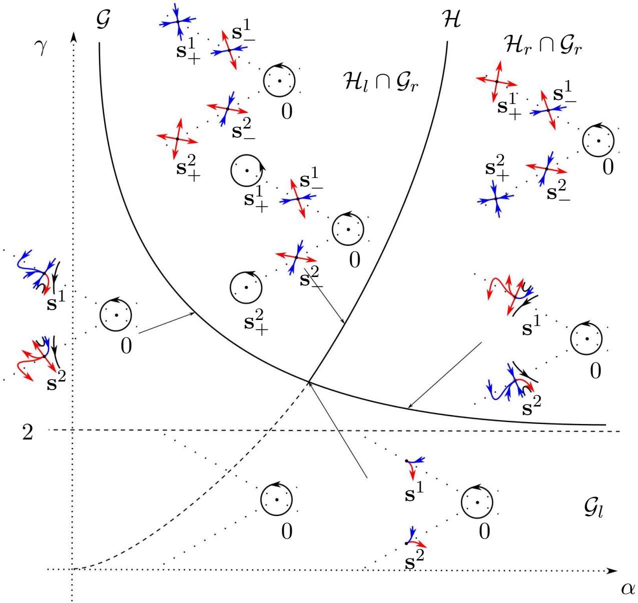
- $\mathcal{F}_u = \{\gamma > \lambda + 2\}$, $\mathcal{F} = \{\gamma = \lambda + 2\}$ and $\mathcal{F}_d = \{\gamma < \lambda + 2\}$.
- $\mathcal{E}_u = \{\gamma > \lambda - 2\}$, $\mathcal{E} = \{\gamma = \lambda - 2\}$ and $\mathcal{E}_d = \{\gamma < \lambda - 2\}$



$\lambda > 0$: Notation referring to local behavior near $\mathcal{R}_0$

h	$\cdot_h$	$\mathcal{R}_0$
u	$\zeta \in \mathcal{F} \cup \mathcal{F}_u$	one or two
c	$\zeta \in \mathcal{F}_d \cap \mathcal{E}_u$	zero
d	$\zeta \in \mathcal{E} \cup \mathcal{E}_d$	one or two

$\lambda > 0$: locally along ray $\mathcal{R}_\pm$



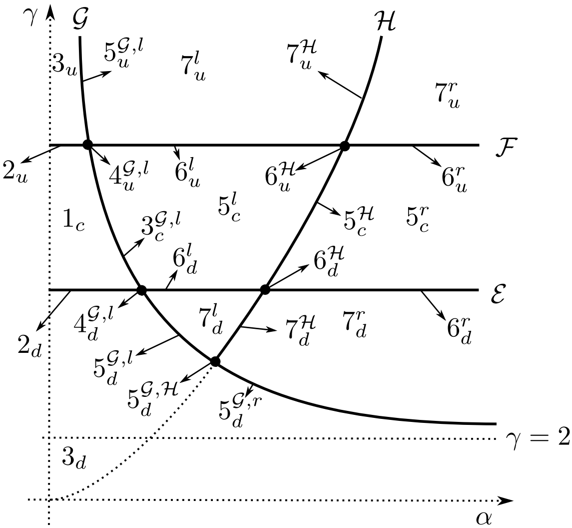
$\lambda > 0$: locally along ray $\mathcal{R}_\pm$

- $\mathcal{G}_l = \{\alpha^2\gamma^2 - 4(\alpha^2+1) < 0\}$,
- $\mathcal{G} = \{\alpha^2\gamma^2 - 4(\alpha^2+1) = 0\}$,
- $\mathcal{G}_r = \{\alpha^2\gamma^2 - 4(\alpha^2+1) > 0\}$,
- $\mathcal{H}_l = \{2\alpha^2\lambda - \gamma < 0\}$,
- $\mathcal{H} = \{2\alpha^2\lambda - \gamma = 0\}$ and
- $\mathcal{H}_r = \{2\alpha^2\lambda - \gamma > 0\}$.

$\lambda > 0$: Notation referring to local behavior near $\mathcal{R}_\pm$

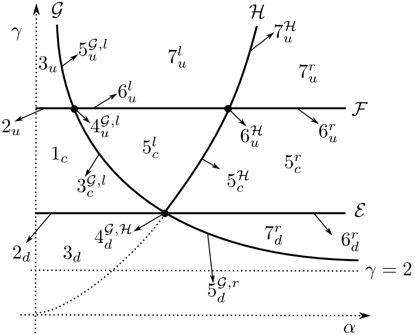
k	$.^{\mathcal{G},k}$	$\mathcal{R}_\pm$	k	$.^{k}$	$\mathcal{R}_\pm$
l	$\zeta \in \mathcal{H}_l \cap \mathcal{G}$	one	l	$\zeta \in \mathcal{H}_l \cap \mathcal{G}_r$	two
$\mathcal{H}$	$\zeta \in \mathcal{H} \cap \mathcal{G}$	one	$\mathcal{H}$	$\zeta \in \mathcal{H} \cap \mathcal{G}_r$	two
r	$\zeta \in \mathcal{H}_r \cap \mathcal{G}$	one	r	$\zeta \in \mathcal{H}_r \cap \mathcal{G}_r$	two

$\lambda > 0$: Local bifurcation diagram near $\mathcal{R}_0$ and $\mathcal{R}_\pm$

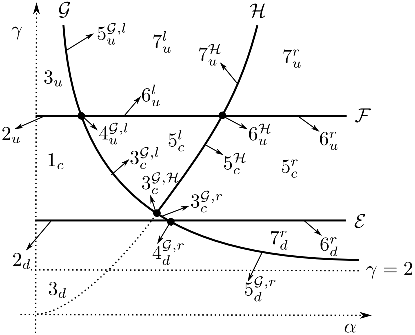


$$6 < \lambda$$

$\lambda > 0$: Local bifurcation diagram near $\mathcal{R}_0$ and $\mathcal{R}_\pm$

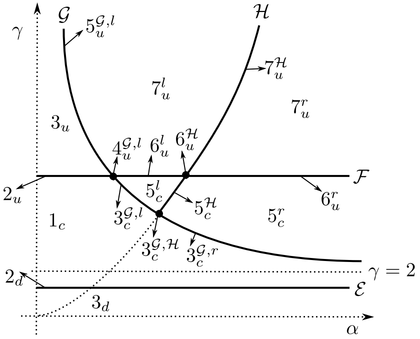


$\lambda = 6$

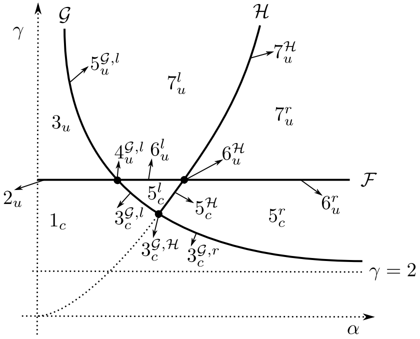


$4 < \lambda < 6$

$\lambda > 0$: Local bifurcation diagram near $\mathcal{R}_0$ and $\mathcal{R}_\pm$

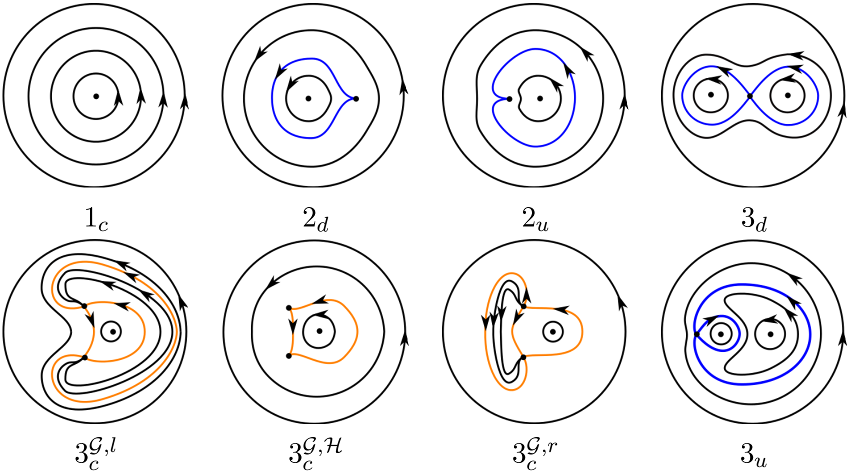


$2 < \lambda \leq 4$

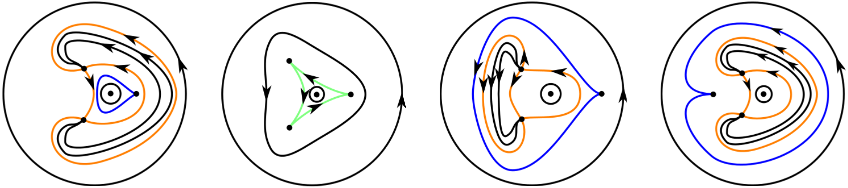


$0 < \lambda \leq 2$

$\lambda > 0$: Global phase portraits with $1 \leq n \leq 3$ singularities



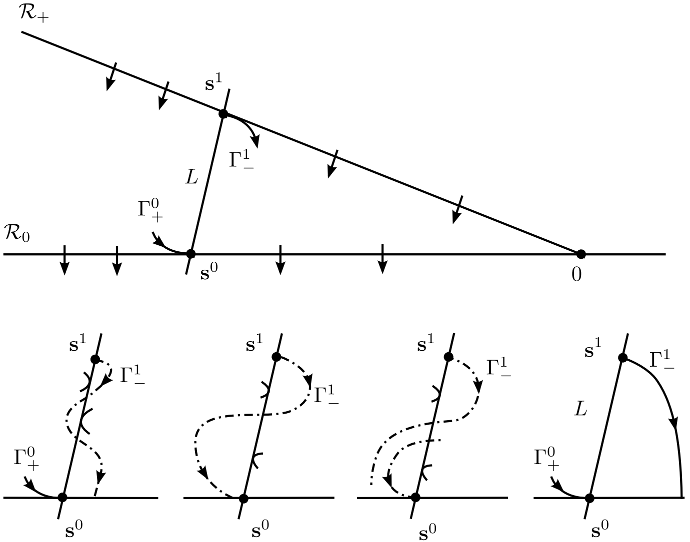
Global phase portraits with 4 singularities



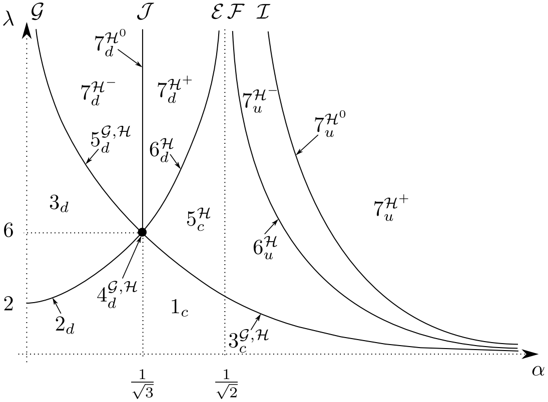
$4_d^{\mathcal{G},l}$ $4_d^{\mathcal{G},\mathcal{H}}$ $4_d^{\mathcal{G},r}$ $4_u^{\mathcal{G},l}$

Proof for uniqueness of $4_u^{\mathcal{G},l}$

- 'Cubic differential vf has ≤ 3 tangencies with straight line'

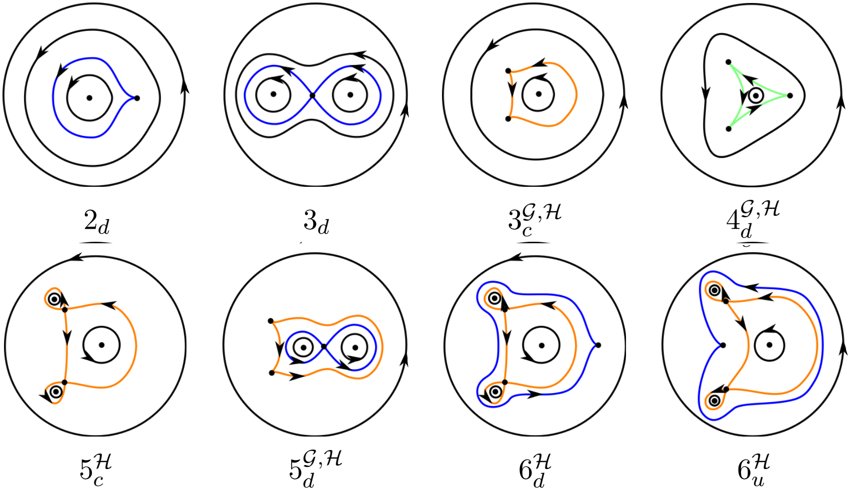


Bifurcation diagram for Hamiltonian reversible class

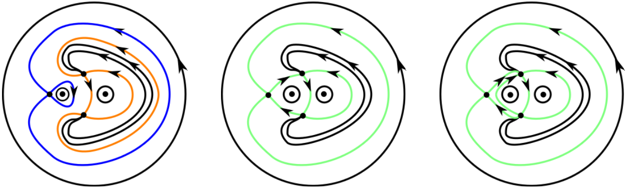


- $\mathcal{I} = \{\mathcal{W}_E(\mathbf{s}_+^0) = \mathcal{W}_I(\mathbf{s}_-^1)\}$ and $\mathcal{J} = \{\mathcal{U}_E(\mathbf{s}_-^0) = \mathcal{W}_E(\mathbf{s}_-^1)\}$

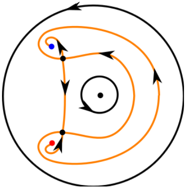
Hamiltonian reversible vfs with $2 \leq n \leq 6$ singularities



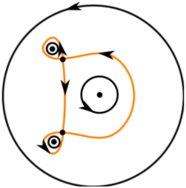
Global phase portraits for 5_u



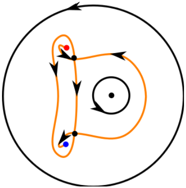
$5_u^{\mathcal{G},l^-}$ $5_u^{\mathcal{G},l^0}$ $5_u^{\mathcal{G},l^+}$



5_c^l

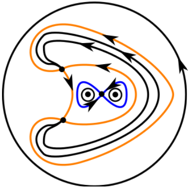


$5_c^{\mathcal{H}}$

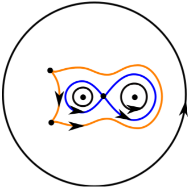


5_c^r

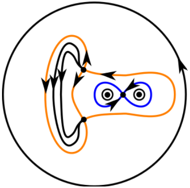
Global phase portraits for 5_d



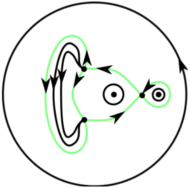
$5_d^{\mathcal{G},l}$



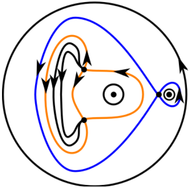
$5_d^{\mathcal{G},\mathcal{H}}$



$5_d^{\mathcal{G},r^-}$

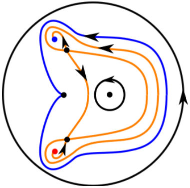


$5_d^{\mathcal{G},r^0}$

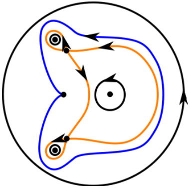


$5_d^{\mathcal{G},r^+}$

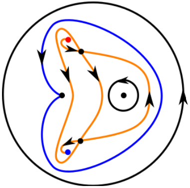
Global phase portraits for 6_u



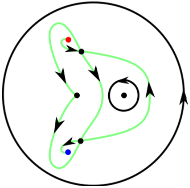
6_u^l



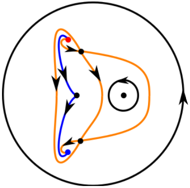
$6_u^{\mathcal{H}}$



$6_u^{r^-}$

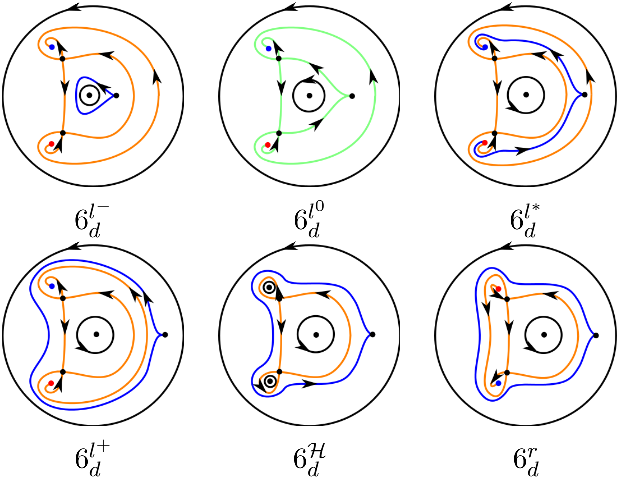


$6_u^{r^1}$

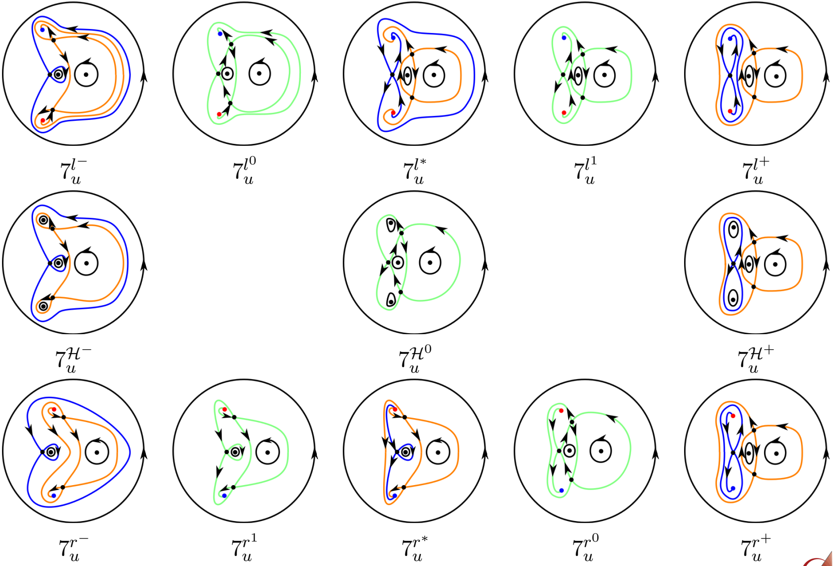


$6_u^{r^*}$

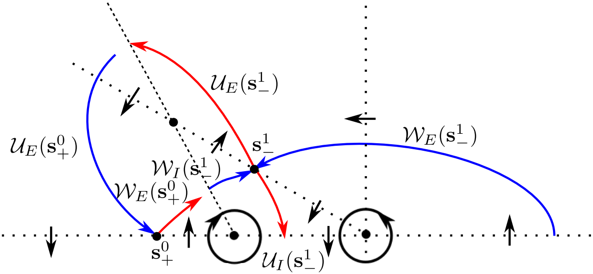
Global phase portraits for 6_d



Global phase portraits for 7_u

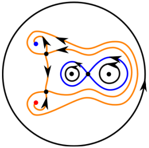


7 singularities - Movement of separatrices

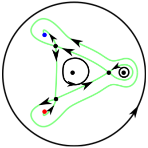


- $\mathcal{I} = \{\mathcal{W}_E(\mathbf{s}_+^0) = \mathcal{W}_I(\mathbf{s}_-^1)\}$
- $\mathcal{L} = \{\mathcal{U}_E(\mathbf{s}_+^0) = \mathcal{U}_E(\mathbf{s}_-^1)\}$

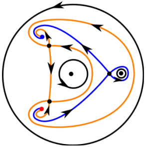
Global phase portraits for 7_d



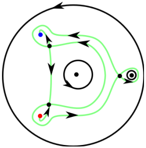
7_d^{l-}



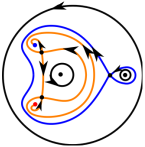
7_d^{l0}



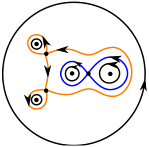
7_d^{l*}



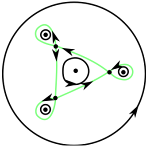
7_d^{l1}



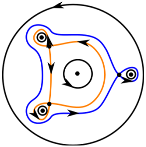
7_d^{l+}



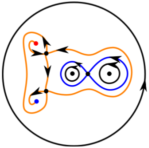
$7_d^{\mathcal{H}-}$



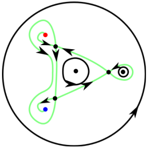
$7_d^{\mathcal{H}0}$



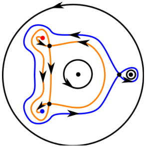
$7_d^{\mathcal{H}+}$



7_d^{r-}



7_d^{r0}



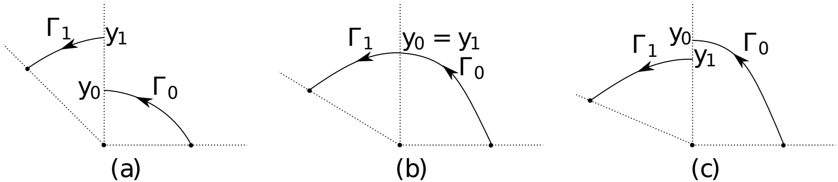
7_d^{r+}

Distribution of phase portraits for $X_{(\alpha,\gamma,\lambda)}$ for $\lambda > 0$

Let L_n be the number of topologically different phase portraits for $X_{(\alpha,\gamma,\lambda)}$ with n singularities. Then,

- $L_1 = \mathbf{1}$;
- $L_2 = \mathbf{1}$;
- $L_3 = \mathbf{4}$;
- $\mathbf{3} \leq L_4 \leq 6$;
- $\mathbf{10} \leq L_5 \leq 13$;
- $\mathbf{8} \leq L_6 \leq 9$;
- $L_7 = \mathbf{28}$.

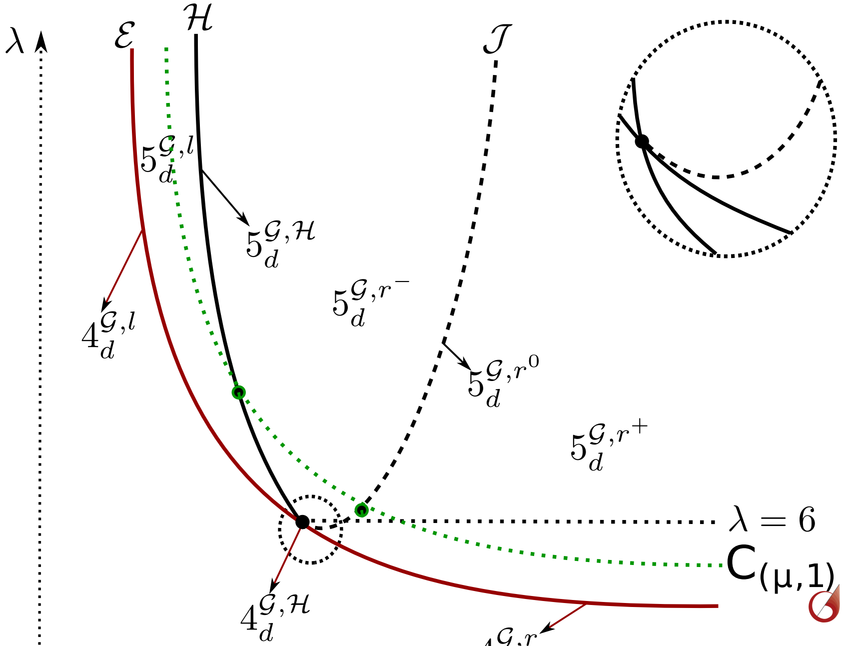
Relative movement of separatrices for $\gamma - \lambda + 2 \geq 2, \alpha^2\gamma^2 - 4(alpha^2 + 1) \geq 0$



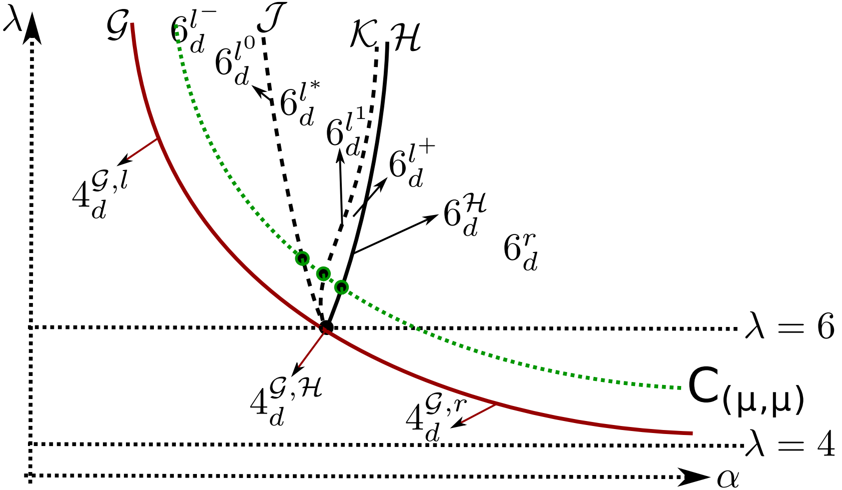
Theorem

1. $[\alpha_0, \infty) \to (0, \infty) : \alpha \mapsto y_0(\alpha)$ *is strictly increasing.*
2. $[\alpha_0, \infty) \to (0, \infty) : \alpha \mapsto y_1(\alpha)$ *is decreasing.*

Along $\mathcal{G}$ in (α, λ)-plane



For fixed $\lambda > 0$ in (α, γ)-plane



Thank you for your attention!

References (M. Caubergh, J. Llibre, J. Torregrosa)

- Global Classification of a class of Cubic Vector Fields whose canonical regions are period annuli, in International Journal of Bifurcation and Chaos (CLT2011)
- Global phase portraits of some reversible cubic centers with collinear or infinitely many singularities (CLT2012)
- Global phase portraits of some reversible cubic centers with non-collinear singularities (CT2013)
- Relative movement of connections for some reversible cubic vector fields (C2016)