# On the time of first return of a perturbed periodic orbit

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# The context

We consider some planar analytic system with a period annulus $\mathcal{P}$ (open, each orbit inside it is closed, not an equilibrium)

$$\dot{x} = X_0(x).$$

For each $q \in \mathcal{P}$ denote $\varphi_0(t,q)$ its flow and $T_0(q) > 0$ its main period. Note that we obtain an analytic function

$$T_0 : \mathcal{P} \to \mathbb{R}$$

which, if not constant, is a first integral of $\dot{x} = X_0(x)$.

In practice, one considers a transversal section $\Sigma = \{\gamma(s) \ : \ s \in I\}$ and the *period function* along it

$$\tau(s) = T_0(\gamma(s)).$$

A *critical period* is $\tau(s^*)$ where $s^* \in I$ is such that $\tau'(s^*) = 0$.

# The critical periods do not depend on $\gamma$

Note that

$$\tau'(s) = \nabla T_0(\gamma(s)) \cdot \gamma'(s)$$

We have that

(i) if $\tau'(s^*) = 0$ then $\nabla T_0(q^*) = (0,0)$ where $q^* = \gamma(s^*)$;

(ii) if $\nabla T_0(q^*) = (0,0)$ then $\tau'(s^*) = 0$ where $s^* \in I$ is such that $q^*$ and $\gamma(s^*)$ are on the same orbit.

The proofs of both (i) and (ii) relies on the facts that
- $T_0$ is a first integral, i.e.

$$\nabla T_0 \cdot X_0 = 0 \text{ and } T_0(\varphi_0(t,q)) = T_0(q) \text{ for all } t$$

- and $\gamma$ is transversal to $X_0$.

# The context

We have that $\nabla T_0(q^*) = (0,0)$ if and only if $D\varphi_0(t,q^*)$ is $T_0(q^*)$-periodic.

$$y' = DX_0(\varphi_0(t,q^*))y$$

$X_0(\varphi_0(t,q^*))$ is a solution of the variational system.

The proposition follows by taking the derivative with respect to $q$ in the equality

$$\varphi_0(T_0(q),q) = q.$$

# The context

The periodic orbits of $\dot{x} = X_0(x)$ in $\mathcal{P}$ will be perturbed, more exactly we consider a family of analytic planar vector fields

$$\dot{x} = X(x, \varepsilon) \tag{1}$$

which depends on the small parameter $\varepsilon \in (\mathbb{R}, 0)$ and such that

$$X(\cdot, 0) = X_0.$$

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Case 1. All the perturbed orbits in $\mathcal{P}$ are also periodic

Case 2. The perturbed orbits in $\mathcal{P}$ are not necessarily periodic

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Then $T(q;\varepsilon)$ is a first integral in $\mathcal{P}$ of the perturbed system and, when considering a transversal section $\Sigma = \{\gamma(s) \ : \ s \in I\}$ to $X_0$, the critical points of

$$s \mapsto \tau(s;\varepsilon) = T(\gamma(s);\varepsilon)$$

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There exists $m \geq 0$ such that

$$T(q;\varepsilon) = (T_0 + \varepsilon T_1 + \cdots + \varepsilon^{m-1} T_{m-1}) + \varepsilon^m T_m(q) + O(\varepsilon^{m+1}) \ ,$$

where $T_0, \ldots, T_{m-1}$ are constant and $T_m(q)$ is not constant.

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where $T_0, \ldots, T_{m-1}$ are constant and $T_m(q)$ is not constant.

Let $\tau_m(s) = T_m(\gamma(s))$, thus $\tau'(s;\varepsilon) = \varepsilon^m \tau_m'(s) + O(\varepsilon^{m+1})$. So, the critical points of $\tau_m$ give information on the bifurcation of critical periods.

But the critical points of $\tau_m(s)$ depend on the transversal section?

## Case 1. All the perturbed orbits are also periodic

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Indeed, since

$$\frac{1}{\varepsilon^m}[T(q,\varepsilon) - (T_0 + \cdots + \varepsilon^{m-1} T_{m-1})]$$

is also an analytic first integral of $\dot{x} = X(x,\varepsilon)$ and $T_m(q)$ is the limit as $\varepsilon \to 0$ of this function, we deduce that

$$T_m : \mathcal{P} \to \mathbb{R}$$

is a first integral of the unperturbed system.

# Case 2. The perturbed orbits are not necessarily periodic

Let $\Sigma$ be a transversal section in $\mathcal{P}$ to $X_0$, hence to $X(x,\varepsilon)$, too. For each $q \in \Sigma$, let $T^{\Sigma}(q;\varepsilon)$ be the first return time to $\Sigma$ for the flow of $\dot{x} = X(x,\varepsilon)$ in $\mathcal{P}$.

$$T^{\Sigma}(q;\varepsilon) = T_0(q) + \varepsilon T_1^{\Sigma}(q) + \cdots + \varepsilon^m T_m^{\Sigma}(q) + O^{\Sigma}(\varepsilon^{m+1}) , \quad q \in \Sigma.$$

We would like to obtain information on those critical points of $\tau(s;\varepsilon) = T^{\Sigma}(\gamma(s);\varepsilon)$ that do not depend on $\Sigma = \{\gamma(s) \ : \ s \in I\}$. Since

$$\tau'(s;\varepsilon) = \varepsilon^m \tau_m'(s) + O^{\Sigma}(\varepsilon^{m+1}), \quad \text{where} \quad \tau_m(s) = T_m(\gamma(s))$$

we need

$$T^{\Sigma}(q;\varepsilon) = (T_0 + \varepsilon T_1 + \dots) + \varepsilon^m T_m(q) + O^{\Sigma}(\varepsilon^{m+1}) , \quad q \in \Sigma$$

and $T_m$ is a first integral of $\dot{x} = X_0(x)$.

# One of the main results: Theorem 1

**Theorem**

*Assume that there exists $m \geq 1$ such that the first $m$ Melnikov functions of $\dot{x} = X(x, \varepsilon)$ vanish identically and that $T_0(q) = T_0$ is constant. Let $T_1, \dots, T_{m-1}$ be some real constants. Assume also that, for some transversal section $S$ we have*

$$T_1^S(q) = T_1, \dots, T_{m-1}^S(q) = T_{m-1} \text{ for all } q \in S.$$

*Then*

$$T_1^{\Sigma}(q) = T_1, \dots, T_{m-1}^{\Sigma}(q) = T_{m-1} \text{ for all } q \in \Sigma \text{ and for any } \Sigma$$

*and there exists an analytic function $T_m$ in $\mathcal{P}$ such that*

$$T_m^{\Sigma}(q) = T_m(q) \text{ for all } q \in \Sigma \text{ and for any } \Sigma.$$

*Moreover, if not constant, $T_m$ is a first integral in $\mathcal{P}$ of $x' = X_0(x)$.*

# Lemma 1

Let $[\cdot, \cdot]$ denote the *Lie bracket*, that is

$$[U, X_0] = DU\ X_0 - DX_0\ U$$

for two $C^1$ planar vector fields $U$ and $X_0$.

**Lemma**

*(i)* $[U, X_0] = 0$ *if and only if* $t \mapsto U(\varphi(t, q, 0))$ *is a solution of the variational system* $y' = DX_0(\varphi(t, q, 0))y$.

*(ii)* $[U, X_0] = 0$ *if and only if* $U(\varphi(t, q, 0)) = D\varphi(t, q, 0)U(q)$.

*(iii) Let* $T : \mathcal{P} \to \mathbb{R}$ *be* $C^1$ *and not locally constant. We have that* $[TX_0, X_0] = 0$ *if and only if* $T$ *is a first integral of* $x' = X_0(x)$.

# Lemma 2

Consider the notations

$$\varphi(T^{\Sigma}(q,\varepsilon),q,\varepsilon) = q + \varepsilon\varphi_1^{\Sigma}(q) + \varepsilon^2\varphi_2^{\Sigma}(q) + \cdots \ , \quad q \in \Sigma.$$

**Lemma**

*Assume that the first $m$ Melnikov functions of $\dot{x} = X(x,\varepsilon)$ vanish. Then for any transversal section $\Sigma$ we have that*

$$\varphi_k^{\Sigma}(q) = 0, \ q \in \Sigma, \ k \in \{1, ..., m\}. \tag{2}$$

*Proof.* Let $d^{\gamma}(s,\varepsilon)$ be the displacement map associated to the transversal section $\Sigma = \{\gamma(s) \ : s \in I\}$. We have

$$\varphi(T^{\Sigma}(\gamma(s),\varepsilon),\gamma(s),\varepsilon) = \gamma(s + d^{\gamma}(s,\varepsilon)).$$

Hence

$$\gamma(s+\varepsilon^{m+1}M_{m+1}^{\gamma}(s)+\cdots) = q+\varepsilon\varphi_1^{\Sigma}(\gamma(s))+\cdots+\varepsilon^m\varphi_m^{\Sigma}(\gamma(s))+\cdots .$$

Equating the coefficients of $\varepsilon^k$ in both sides of the above relation we obtain the conclusion.

# Lemma 3

**Lemma**

*Let $m \geq 1$ and $T_0, T_1, \ldots, T_{m-1}$ be some real constants. Assume that the first $m$ Melnikov functions vanish and that $T_0(q) = T_0$,*

$$T_1^{\Sigma}(q) = T_1, \ \ldots \ , T_{m-1}^{\Sigma}(q) = T_{m-1} \ \text{ for all } \ q \in \Sigma \text{ and for any } \Sigma.$$

*Let*

$$\tau(\varepsilon) = T_0 + \varepsilon T_1 + \ldots + \varepsilon^{m-1} T_{m-1}.$$

*Then there exists an analytic function $\psi_m$ such that*

$$\varphi(\tau(\varepsilon), q, \varepsilon) = q + \varepsilon^m \psi_m(q) + O(\varepsilon^{m+1}) \ \text{ for all } \ q \in \mathcal{P},$$

$$\psi_m(q) = -T_m^{\Sigma}(q) X_0(q) \ \text{ for all } \ q \in \Sigma \text{ and for any } \Sigma \ ,$$

$$[\psi_m, X_0] = 0.$$

# Lemma 3 for $m = 1$

*Assume that the first Melnikov function vanish and* $T_0(q) = T_0$. *Then the function*

$$\psi_1(q) = \frac{\partial \varphi}{\partial \varepsilon}(T_0, q, 0)$$

*is analytic in* $\mathcal{P}$ *and satisfies*

$$\varphi(T_0, q, \varepsilon) = q + \varepsilon \psi_1(q) + O(\varepsilon^2) \quad \textit{for all} \quad q \in \mathcal{P},$$

$$\psi_1(q) = -T_1^{\Sigma}(q) X_0(q) \quad \textit{for all} \quad q \in \Sigma \textit{ and for any } \Sigma$$

*and*

$$[\psi_1, X_0] = 0.$$

# Theorem 1 for $m = 1$ and its proof

*Assume that the first Melnikov function vanish and $T_0(q) = T_0$. Then there exists an analytic function $T_1$ in $\mathcal{P}$ such that*

$$T_1^{\Sigma}(q) = T_1(q) \text{ for all } q \in \Sigma \text{ and for any } \Sigma.$$

*Moreover, if not constant, $T_1$ is a first integral in $\mathcal{P}$ of $x' = X_0(x)$.*

The proof follows by Lemma 3 for $m = 1$ ($\psi_1(q) = -T_1^{\Sigma}(q)X_0(q)$ and $[T_1 X_0, X_0] = 0$) and Lemma 1(iii).

The proof of Theorem 1 for $m \geq 1$ follows by induction using Lemma 3 and Lemma 1(iii).

# Proof of Lemma 3

We have

$$\varphi(T^{\Sigma}(q,\varepsilon),q,\varepsilon) = \varphi(\varepsilon^m T^{\Sigma}_m(q) + \dots\ ,\ \varphi(\tau(\varepsilon),q,\varepsilon)\ ,\ \varepsilon) = \\ \varphi(\tau(\varepsilon),q,\varepsilon) + \varepsilon^m T^{\Sigma}_m(q)\dot{\varphi}(0,q,0) + O^{\Sigma}(\varepsilon^{m+1})$$

Using Lemma 2 we obtain that

$$\varphi(T^{\Sigma}(q,\varepsilon),q,\varepsilon) = q + O^{\Sigma}(\varepsilon^{m+1}).$$

Then, indeed,

$$\varphi(\tau(\varepsilon),q,\varepsilon) = q - \varepsilon^m T^{\Sigma}_m(q)X_0(q) + O^{\Sigma}(\varepsilon^{m+1}).$$

# Proof of Lemma 3

It remained only to prove that $\psi_m$ and $X_0$ commute. We use

$$\varphi(\tau(\varepsilon), q, \varepsilon) = q + \varepsilon^m \psi_m(q) + O(\varepsilon^{m+1})$$

to write

$$\varphi(\tau(\varepsilon), \varphi(t, q, \varepsilon), \varepsilon) = \varphi(t, q, \varepsilon) + \varepsilon^m \psi_m(\varphi(t, q, \varepsilon)) + O(\varepsilon^{m+1})$$

and in

$$\varphi(\tau(\varepsilon), \varphi(t, q, \varepsilon), \varepsilon) = \varphi(t, q + \varepsilon^m \psi_m(q) + O(\varepsilon^{m+1}), \varepsilon) = \\ \varphi(t, q, \varepsilon) + \varepsilon^m D\varphi(t, q, 0)\psi_m(q) + O(\varepsilon^{m+1}).$$

Hence

$$\psi_m(\varphi(t, q, 0)) = D\varphi(t, q, 0)\psi_m(q)$$

which, applying Lemma 1 (ii) assures that $[\psi_m, X_0] = 0$.

# The derivative of the period function

[FrGaGu] E. Freire, A. Gasull, A. Guillamon, First derivative of the period function with applications, *J. Differential Equations* 204 (2004) 139-162.

[GaYu] A. Gasull, J. Yu, On the critical periods of perturbed isochronous centers, *J. Differential Equations* 244 (2008) 696-715.

**Theorem**

*Assume that the $C^1$ vector field $X$ has a center at $p$ with period annulus $\mathcal{P}$ (open set that does not contain $p$). $\varphi(t,q)$ denotes its flow.*
*Take some $C^1$ vector field $U$ transversal to $X$ in $\mathcal{P}$.*
*Take $\alpha, \beta \in C^1(\mathcal{P})$ such that $[X,U] = \alpha X + \beta U$.*
*Take $\gamma : I \to \mathbb{R}$ a solution of $\dot{x} = U(x)$ and $\tau(s)$ be the period of $\varphi(t,\gamma(s))$. Then for each $s \in I$ we have*

$$\tau'(s) = \int_0^{\tau(s)} \alpha(\varphi(t,\gamma(s))) e^{-\int_0^t \beta(\varphi(v,\gamma(s)))dv} dt.$$

# The orbits of $X$ are not quite closed

Now we drop the assumption on $X_0$ to have a center in the previous theorem.
Still we have that $\Sigma = \{\gamma(s) \ : \ s \in I\}$ is a transversal section to the orbits of $X$ in $\mathcal{P}$.
Assume instead that for any $s \in I$, each orbit of $X$ that starts at $\gamma(s)$ returns to $\Sigma$ and let $\tau(s)$ denote the first return time to $\Sigma$ of this orbit.
Let $\pi(s) = \gamma^{-1}(\varphi(\tau(s), \gamma(s)))$ be the Poincaré return map to $\Sigma$.
Then

$$\begin{aligned} \tau'(s) &= \int_0^{\tau(s)} \alpha(\varphi(t,\gamma(s)))e^{-\int_0^t \beta(\varphi(v,\gamma(s)))dv}dt \qquad (3) \\ \pi'(s) &= e^{-\int_0^{\tau(s)} \beta(\varphi(t,\gamma(s)))dt}. \qquad (4) \end{aligned}$$

We have center if and only if $\int_0^{\tau(s)} \beta(\varphi(t,\gamma(s)))dt = 0$.

# Proof

$$\gamma(\pi(s)) = \varphi(\tau(s), \gamma(s))$$

$$\pi'(s)\,\gamma'(\pi(s)) = \tau'(s)\,\frac{d\varphi}{dt}(\tau(s), \gamma(s)) + D\varphi(\tau(s), \gamma(s)) \cdot \gamma'(s)$$

$$\pi'(s)\,U(\gamma(\pi(s))) = \tau'(s)\,X(\gamma(\pi(s))) + D\varphi(\tau(s), \gamma(s)) \cdot U(\gamma(s)).$$

$$\eta(t, q) := D\varphi(t, q)U(q)$$

$$\eta(\tau(s), \gamma(s)) = \pi'(s)\,U(\gamma(\pi(s))) - \tau'(s)\,X(\gamma(\pi(s))).$$

$\dot{\eta} = DX(\varphi(t, q))\eta$, $\eta(0) = U(q)$ and $[X, U] = \alpha X + \beta U$ imply [GaYu]

$$\begin{aligned}\eta(\tau(s), \gamma(s)) = \quad & e^{-\int_0^{\tau(s)} \beta(\varphi(v,\gamma(s)))dv}\,U(\gamma(\pi(s))) \\ & - \int_0^{\tau(s)} \alpha(\varphi(u, \gamma(s)))e^{-\int_0^u \beta(\varphi(v,\gamma(s)))dv}du\,X(\gamma(\pi(s)))\end{aligned}$$

# The case of a perturbed system [GaYu]

In the theorem from [GaYu] take $X = X_\varepsilon = X_0 + \varepsilon X_1$ where $X_0$ has an isochronous center of period $T_0$ at $p$ and choose $U_0$ to be transversal to $X_0$ and such that

$$[X_0, U_0] = 0.$$

Take $\lambda_1, \mu_1$ such that

$$X_1 = \lambda_1 X_0 + \mu_1 U_0.$$

Denote the period of $\varphi(t, \gamma(s), \varepsilon)$ by (where $\gamma(s)$ is some solution of $\dot{x} = U_0(x)$)

$$\tau(s; \varepsilon) = T_0 + \varepsilon \tau_1(s) + O(\varepsilon^2).$$

Then

$$\tau_1'(s) = -\int_0^{T_0} \nabla \lambda_1(\varphi_0(t, \gamma(s))) \cdot U_0(\varphi_0(t, \gamma(s)))dt.$$

# The case of a perturbed system [GrVi]

[GrVi] M. Grau, J. Villadelprat, Bifurcation of critical periods from Pleshkan's isochrones, *J. London Math. Soc.* 81 (2010) 142-160.

Take $m \geq 1$,

$$X = X_\varepsilon = X_0 + \varepsilon X_1 + \cdots + \varepsilon^m X_m + O(\varepsilon^{m+1}),$$

$$X_m = \lambda_m X_0 + \mu_m U_0,$$

$$\tau(s;\varepsilon) = T_0 + \varepsilon\tau_1(s) + \cdots + \varepsilon^m \tau_m(s) + O(\varepsilon^{m+1}).$$

New assumption: *there exists an analytic family of diffeomorphisms* $\{\Phi_\varepsilon\}$*, in a neighborhood of* $(0,0)$*, such that* $\Phi_\varepsilon$ *linearizes* $j^m(X_\varepsilon)$*.*
Then $\tau_1, \ldots, \tau_{m-1}$ are constant and

$$\tau_m'(s) = -\int_0^{T_0} \nabla\lambda_m(\varphi_0(t,\gamma(s))) \cdot U_0(\varphi_0(t,\gamma(s)))dt.$$

# Theorem 2

$m \geq 1$ and the first $m$ Melnikov functions vanish identically.

$$\dot{x} = X(x, \varepsilon) = X_0 + \varepsilon X_1 + \cdots + \varepsilon^m X_m + O(\varepsilon^{m+1}).$$

Choose $U_0$ to be transversal to $X_0$ and such that $[X_0, U_0] = 0$.

$$X_m = \lambda_m X_0 + \mu_m U_0.$$

New assumption: *there exists an analytic family of diffeomorphisms $\{\Phi_\varepsilon\}$ in $\mathcal{P}$, such that*

$$(\Psi)^*(j^m(X_\varepsilon)) = X_0 + o(\varepsilon^{m+1}).$$

Then $\tau^\gamma(s; \varepsilon) = T_0 + \varepsilon T_1 + \cdots + \varepsilon^m \tau_m(s) + O^\gamma(\varepsilon^{m+1})$ and

$$\tau_m'(s) = -\int_0^{T_0} \nabla \lambda_m(\varphi_0(t, \gamma(s))) \cdot U_0(\varphi_0(t, \gamma(s)))dt.$$

Thank you for your attention.